Let $A$ be a local ring (commutative with $1$) with maximal ideal $m$. The projection maps $A \to A/m^n$ for $n = 1, 2, \ldots$, induce a local homomorphism of local rings

$$c_A : A \to \widehat{A} = \lim_{\leftarrow n} A/m^n.$$ 

We say that $A$ is complete if $c_A$ is an isomorphism. We denote the maximal ideal of $\widehat{A}$ by $\widehat{m}$. Let $M_i$ be the kernel of the surjective projection map $\widehat{A} \to A/m^i$.

**Proposition 1** The local ring $\widehat{A}$ is complete if and only if $M_n = \widehat{m}^n$ for each $n \geq 1$.

**Proof.** For each $n$ the projection map $\widehat{A} \to A/m^n$ gives rise to a short exact sequence

$$0 \to (M_n/\widehat{m}^n) \to (\widehat{A}/\widehat{m}^n) \to (A/m^n) \to 0.$$ 

If we let $n$ run this becomes a short exact sequence of projective systems. The projection map $\widehat{A} \to A/m^{n+1}$ sends $\widehat{m}^n$ onto $m^n/m^{n+1}$, so $\widehat{m}^n + M_{n+1} = M_n$. This implies that the system on the left has surjective transition maps. By Mittag-Leffler we get a short exact sequence of projective limits:

$$0 \to \lim_{\leftarrow n} M_n/\widehat{m}^n \to \widehat{A} \xrightarrow{g_A} A \to 0.$$ 

It follows that $g_A$ is an isomorphism if and only if all $M_n/\widehat{m}^n$ are zero. But we have $g_Ac_A = \text{id}_\widehat{A}$, so $g_A$ is an isomorphism if and only if $c_A$ is an isomorphism. □

**Proposition 2** For any local ring $A$ for which $\widehat{A}$ is not complete, the ring $\widehat{A}$ is not complete either.

**Proof.** For each $n \geq 1$ we consider the diagram

$$
\begin{array}{ccccccc}
0 & \to & \lim_{\leftarrow n} M_n/\widehat{m}^n & \to & \widehat{A} & \xrightarrow{g_A} & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_n/\widehat{m}^n & \to & A/\widehat{m}^n & \to & A/m^n & \to & 0 \\
\end{array}
$$
in which the rows are exact and the vertical maps are surjective. By the snake lemma the kernel $K_n$ of the middle vertical map maps surjectively to $M_n$. Thus, $K_n/K_1^n$ surjects to $M_n/M_1^n$, and by the previous criterion, $\hat{A}$ is not complete. \(\square\)

The next result is exercise 12 in Bourbaki, Commutative Algebra, Ch. II sec. 2 (p. 235/236).

**Proposition 3** There is a local ring $A$ so that $\hat{A}$ is not complete.

**Proof.** Let $K$ be a field. Let $A_d$ be the localisation of $K[X_1, \ldots, X_d]$ at the maximal ideal $(X_1, \ldots, X_d)$, and let $A$ be the union of all $A_d$. Let $m$ be the maximal ideal of $A$ and let $m_d$ be the maximal ideal of $A_d$. Note that $\hat{A}$ is the power series ring $K[[X_1, X_2, \ldots]]$ in which all elements have only finitely many terms of each total degree. In particular, for any monomial in the variables $X_1, X_2, \ldots$ we can consider the coefficient of an element of $\hat{A}$ at that monomial.

We will show that $M_2 \neq \hat{m}^2$.

First note that for $a < b$ a polynomial map from affine $a$-space over $K$ to affine $b$-space over $K$ is not surjective on $K$-rational points (finite $K$: cardinality; infinite $K$: algebraic geometry).

We will construct a sufficient condition for an element of $M_2$ not being an $n$-term sum of products of two elements of $\hat{m}$. For each $d$ and $n \geq 2$ we look at the image of a polynomial map of affine spaces over $K$:

$$f_d : (m_d/m_d^n \times m_d/m_d^n)^n \longrightarrow m_d^n/m_d^{n+1},$$

that sends $(x_i, y_i)_{i=1}^n$ to the degree $n$ part of $\sum x_i y_i$. For fixed $n$ the $K$-dimension of the affine space on the right is a polynomial of degree $n$ in $d$ with a positive leading coefficient and on the left it is a polynomial of degree $n - 1$ in $d$. (In fact, on the left it is $2n\binom{d+n-1}{n-1} - 1$, and on the right it is $\binom{d+n-1}{n}$. For given $n$ this implies that for sufficiently large $d$ there is a homogeneous polynomial $s_n$ of degree $n$ in $d$ variables which is not in the image of $f_d$. By considering the map $\hat{A} \rightarrow A_d/m_d^{n+1}$ that sends all variables $X_i$ with $i > d$ to zero we see that any power series in $M_2$ which has $s_n$ as its degree-$n$-part is not a sum of $n$ products of pairs of elements of $\hat{m}$. By summing the elements $s_n$ over all $n \geq 2$ we find an element of $M_2$ that is not in $\hat{m}^2$. \(\square\)