Incomplete completions of local rings

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Let A be a local ring (commutative with 1) with maximal ideal $\mathfrak{m}$. The projection maps $A \to A/\mathfrak{m}^n$ for $n = 1, 2, \ldots,$ induce a local homomorphism of local rings

$$c_A : A \longrightarrow \widehat{A} = \varprojlim_n A/\mathfrak{m}^n.$$

We say that A is complete if c_A is an isomorphism. We denote the maximal ideal of $\widehat{A}$ by $\widehat{\mathfrak{m}}$. Let M_i be the kernel of the surjective projection map $\widehat{A} \to A/\mathfrak{m}^i$.

Proposition 1 *The local ring $\widehat{A}$ is complete if and only if $M_n = \widehat{\mathfrak{m}}^n$ for each $n \geq 1$.*

Proof. For each n the projection map $\widehat{A} \to A/\mathfrak{m}^n$ gives rise to a short exact sequence

$$0 \longrightarrow (M_n/\widehat{\mathfrak{m}}^n) \longrightarrow (\widehat{A}/\widehat{\mathfrak{m}}^n) \longrightarrow (A/\mathfrak{m}^n) \longrightarrow 0.$$

If we let n run this becomes a short exact sequence of projective systems. The projection map $\widehat{A} \to A/\mathfrak{m}^{n+1}$ sends $\widehat{\mathfrak{m}}^n$ onto $\mathfrak{m}^n/\mathfrak{m}^{n+1}$, so $\widehat{\mathfrak{m}}^n + M_{n+1} = M_n$. This implies that the system on the left has surjective transition maps. By Mittag-Lef¤er we get a short exact sequence of projective limits:

$$0 \longrightarrow \varprojlim_n M_n/\widehat{\mathfrak{m}}^n \longrightarrow \widehat{\widehat{A}} \xrightarrow{g_A} \widehat{A} \longrightarrow 0.$$

It follows that g_A is an isomorphism if and only if all $M_n/\widehat{\mathfrak{m}}^n$ are zero. But we have $g_A c_{\widehat{A}} = \mathrm{id}_{\widehat{A}}$, so g_A is an isomorphism if and only if $c_{\widehat{A}}$ is an isomorphism. $\square$

Proposition 2 *For any local ring A for which $\widehat{A}$ is not complete, the ring $\widehat{\widehat{A}}$ is not complete either.*

Proof. For each $n \geq 1$ we consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varprojlim_n M_n/\widehat{\mathfrak{m}}^n & \longrightarrow & \widehat{\widehat{A}} & \xrightarrow{g_A} & \widehat{A} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_n/\widehat{\mathfrak{m}}^n & \longrightarrow & \widehat{A}/\widehat{\mathfrak{m}}^n & \longrightarrow & A/\mathfrak{m}^n & \longrightarrow & 0
\end{array}$$

in which the rows are exact and the vertical maps are surjective. By the snake lemma the kernel K_n of the middle vertical map maps surjectively to M_n. Thus, K_n/K_1^n surjects to M_n/M_1^n, and by the previous criterion, $\widehat{\widehat{A}}$ is not complete. $\square$

The next result is exercise 12 in Bourbaki, Commutative Algebra, Ch. II sec. 2 (p. 235/236).

Proposition 3 *There is a local ring A so that $\widehat{A}$ is not complete.*

Proof. Let K be a £eld. Let A_d be the localisation of $K[X_1, \ldots, X_d]$ at the maximal ideal $(X_1, \ldots, X_d)$, and let A be the union of all A_d. Let $\mathfrak{m}$ be the maximal ideal of A and let $\mathfrak{m}_d$ be the maximal ideal of A_d. Note that $\widehat{A}$ is the power series ring $K[[X_1, X_2, \ldots]]$ in which all elements have only £nitely many terms of each total degree. In particular, for any monomial in the variables $X_1, X_2, \ldots$ we can consider the coef£cient of an element of $\widehat{A}$ at that monomial.

We will show that $M_2 \neq \widehat{\mathfrak{m}}^2$.

First note that for $a < b$ a polynomial map from af£ne a-space over K to af£ne b-space over K is not surjective on K-rational points (£nite K: cardinality; in£nite K: algebraic geometry).

We will construct a suf£cient condition for an element of M_2 not being an n-term sum of products of two elements of $\widehat{\mathfrak{m}}$. For each d and $n \geq 2$ we look at the image of a polynomial map of af£ne spaces over K:

$$f_d: \ (\mathfrak{m}_d/\mathfrak{m}_d^n \times \mathfrak{m}_d/\mathfrak{m}_d^n)^n \longrightarrow \mathfrak{m}_d^n/\mathfrak{m}_d^{n+1},$$

that sends $(x_i, y_i)_{i=1}^n$ to the degree n part of $\sum_i x_i y_i$. For £xed n the K-dimension of the af£ne space on the right is a polynomial of degree n in d with a positive leading coef£cient and on the left it is a polynomial of degree $n-1$ in d. (In fact, on the left it is $2n(\binom{d+n-1}{n-1} - 1)$, and on the right it is $\binom{d+n-1}{n}$.) For given n this implies that for suf£ciently large d there is a homogeneous polynomial s_n of degree n in d variables which is not in the image of f_d. By considering the map $\widehat{A} \to A_d/\mathfrak{m}_d^{n+1}$ that sends all variables X_i with $i > d$ to zero we see that any power series in M_2 which has s_n as its degree-n-part is not a sum of n products of pairs of elements of $\widehat{\mathfrak{m}}$. By summing the elements s_n over all $n \geq 2$ we £nd an element of M_2 that is not in $\widehat{\mathfrak{m}}^2$. $\square$

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