Multilinear algebra

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In these notes we set up the theory of multilinear algebra in a fairly broad setting. We begin by investigating the notion of free objects in a category.

Forgetful functors and free objects

Many 'well-known' categories have objects that consist of a set with some additional structure. Think, for example, of Abelian groups, rings or topological spaces. The maps between the objects are just maps of sets, possibly satisfying some additional properties.

If we have such a category, there is a functor to the category of sets, sending an object to its underlying set and a morphism to the maps of sets it induces. Such a functor is called a forgetful functor. One should think of such a functor as 'forgetting' part of the structure of the objects.

Let us look at the category of Abelian groups as a motivating example. Any Abelian group is a set with some additional properties, and the maps are just maps between the sets with some extra properties. So we have a forgetful functor U from **Ab** to **Sets**. Note that this functor is injective on morphisms (that is, a group homomorphism is completely determined by what it does with the elements of the underlying set.)

If S is a set, then the free abelian group on S, notation $\mathbf{Z}^{(S)}$, is the group

$$\bigoplus_{s\in S}\mathbf{Z}[s] = \{f : S \longrightarrow \mathbf{Z} \mid f(s) = 0 \text{ for almost all } s \text{ in } S\}\,.$$

Note that there is an inclusion $S \longrightarrow U(\mathbf{Z}^{(S)})$ sending s in S to the function e_s satisfying $e_s(s) = 1$ and $e_s(t) = 0$ for all t not equal to s.

If we have two sets S and T and a map f between them, then this map induces a morphism $\mathbf{Z}^{(S)} \longrightarrow \mathbf{Z}^{(T)}$ sending e_s to $e_{f(S)}$ for all s in S. One checks that with these definitions the process of creating the free Abelian group on a set S is a functor. Let's call it F.

Fact. The functor F is the left adjoint of U. That is, there are natural isomorphisms

$$\mathrm{Hom}(F(S), A) \cong \mathrm{Map}(S, U(A))$$

for every set S and every Abelian group A, functorial in both arguments.

We write that F is *the* left adjoint of U, as this property determines F up to isomorphism of functors.

Definition. Given a category $\mathcal{C}$ with a forgetful functor to sets, the *free functor* is the left adjoint of the forgetful functor, if it exists.

Example. We consider the category of commutative rings with its underlying set functor U. Let S be a set. What should we take for $F(S)$, the free ring on S If it exists, we must have for every ring R a natural isomorphism

$$\mathrm{Hom}(F(S), R) \cong \mathrm{Map}(S, U(R)).$$

In particular this holds for the ring $R = F(S)$. In this case there is a distinguished element on the left hand side, namely, the identity morphism. It corresponds to a map $i_S : S \longrightarrow U(F(S))$. We see that again there should be a natural way to see the elements of S as elements of $F(S)$. One checks that the good definition for $F(S)$ is $\mathbf{Z}[S]$, the polynomial ring on the elements of S.

Lemma. If a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ has a left (right) adjoint, then it preserves left (right) limits.

Example. Does the forgetful functor U on **Ab** have a right adjoint? If it does we see from the lemma that U should preserve right limits, so, for example, co-products. Let A and B be two Abelian groups. Then apparently we have

$$U(A) \amalg U(B) = U(A \amalg B) = U(A \oplus B) = U(A) \times U(B),$$

which is clearly not true.

Graded rings

A *graded ring* is a ring A written as a direct sum $\bigoplus_{i \in \mathbf{Z}} A_i$ of additive subgroups. Moreover, for any two integers i and j the multiplication map on $A_i \times A_j$ should land inside A_{i+j}. Lastly we demand that 1 is in A_0, which is therefore a subring. An element x of A is called homogeneous if there is an integer i for which x is in A_i. This i is uniquely determined by x if it exists and is called the degree of the homogeneous element.

Example. Let S be a set. Then the free ring on S, $\mathbf{Z}[S]$, has a natural grading, which is completely determined by specifying $\deg(s) = 1$ for all s in S. The homogeneous part of degree i is formed by all homogeneous polynomials of degree i, together with 0. Note that in particular the homogeneous part of any negative degree is the trivial group.

Example. Let k be a field and look at $A = k[x, y]/(y^2 - x^3)$. We can put a grading on $k[x, y]$ by specifying $\deg(x) = 2$ and $\deg(y) = 3$. For this grading, $y^2 - x^3$ is a homogeneous element and it is an easy exercise to see that the quotient of a graded ring by an ideal with homogeneous generators is again graded. In fact, we get the decomposition

$$A = k \cdot 1 + k \cdot x + k \cdot y + k \cdot x^2 + k \cdot xy + k \cdot x^3 \ldots = A_0 + A_2 + A_3 + A_4 + A_5 + A_6 \ldots$$

Another way to verify this is to note that A is isomorphic to the subring $k[T^2, T^3]$ of $k[T]$, with the usual grading.

Let A be a graded ring, then a *graded A-module* is an A-module M that is written as a direct sum $\bigoplus_{i \in \mathbf{Z}} M_i$ of Abelian groups such that for all integers i and j the product of an element of A_i with an element of M_j lands inside M_{i+j}. An A-linear map f between two graded A-modules M and N is a graded module homomorphism of degree i if and only if for all integers j we have $f(N_j) \subset M_{i+j}$. We see that the set of graded A-module homomorphisms also has a grading. If A is a commutative ring, then it is a graded A-module.

The tensor product of two graded rings is again a graded ring. Let A and B be graded rings, then we have

$$A \otimes B = \bigoplus_{i,j} A_i \otimes_{\mathbf{Z}} B_j = \bigoplus_k \left(\bigoplus_{i+j=k} A_i \otimes_{\mathbf{Z}} B_j \right) = \bigoplus_k (A \otimes B)_k$$

as $\mathbf{Z}$-modules. The multiplication is given by $(x \otimes y) \cdot (x' \otimes y') = (xx') \otimes (yy')$.

The tensor algebra

Let A be a commutative ring. We look at that category $\mathbf{GrAlg}_A$ of graded algebras over A, that is graded rings $R = \bigoplus_{i \in \mathbf{Z}} R_i$ together with a ring homomorphism $A \longrightarrow R$ that lands in the center of R, such that the R_i become sub-A-modules of R. Morphisms of graded A-algebras are A-linear homomorphisms that preserve the grading.

We consider the category $\mathbf{GrAlg}_A$ together with the forgetful functor to A-modules, sending a graded A-algebra R to R_1. The left adjoint of this functor exists and is called T, the tensor algebra. If M is an A-module, then $T(M)$ is the graded A-algebra given by

$$T(M)_i = M^{\otimes i} = \underbrace{M \otimes_A M \otimes_A \cdots \otimes_A M}_{i \text{ times}}$$

for i positive, $T(M)_0 = A$ and $T(M)_i = 0$ for i negative. The multiplication map is given by

$$\begin{array}{ccccc} M^{\otimes i} & \times & M^{\otimes j} & \longrightarrow & M^{\otimes i+j} \\ (x_1 \otimes \cdots \otimes x_i & , & y_1 \otimes \cdots \otimes y_j) & \mapsto & x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j. \end{array}$$

The tensor algebra satisfies several universal properties.

1. Let M be an A-module and R a graded A-algebra, then we have a canonical isomorphism
$$\mathrm{Hom}(T(M), R) \xrightarrow{\sim} \mathrm{Hom}_A(M, R_1).$$

2. Let M be an A-module and R an A-algebra, then we have a canonical isomorphism
$$\mathrm{Hom}(T(M), R) \xrightarrow{\sim} \mathrm{Hom}_A(M, R).$$

3. Let M and N be A-modules, then we have for every positive integer n a canonical isomorphism
$$\mathrm{Hom}_A(T(M)_n, N) \xrightarrow{\sim} \left\{ A\text{-multilinear maps } \underbrace{M \times \cdots \times M}_{n \text{ times}} \longrightarrow N \right\}.$$

Note that T preserves right limits, so in particular the tensor algebra of a direct sum is the coproduct of the tensor algebras of the summands. If M is a projective A-module of rank n, then $T(M)_i$ is also projective and is of rank n^i.

The symmetric algebra

Inside the category of graded A-algebras we have just considered there is a full subcategory formed by the commutative graded A-algebras. If we restrict the forgetful functor from before to this subcategory we again get a functor which has a left adjoint. We call this functor S, the symmetric algebra. It is defined by putting

$$S(M) = T(M)/(x \otimes y - y \otimes x : \ x, y \in M).$$

This is a quotient of a graded ring by a homogeneous ideal, so it is again a graded ring. Moreover, one checks that it is commutative.

To describe the homogeneous parts of $S(M)$, we note that for every positive integer i, the module $M^{\otimes i}$ has an action of $\mathrm{Sym}(i)$, the symmetric group on i symbols. The module $S(M)_i$ is the module of co-invariants of $M^{\otimes i}$ under this action.

Let G be a finite group that acts on an Abelian group A. Then the subgroup of invariants is defined by

$$A^G = \{x \in A : gx = x \text{ for all } g \text{ in } G\}.$$

The quotient module of co-invariants is defined by

$$A_G = A/\left(x - gx \mid g \in G, x \in A\right).$$

There is a natural map from A_G to A^G sending the class of an element x to $\sum_{g \in G} gx$. There is also a map in the other direction, simply taking an invariant element x to its class in A_G. The composition of these two maps in either order results in a multiplication by $\#G$.

For the above we conclude that if $i!$ is an invertible element of A (for example if A contains $\mathbf{Q}$), then the co-invariants and the invariants of $M^{\otimes i}$ under the action $\mathrm{Sym}(i)$ are naturally isomorphic, so we may in fact take $S(M)_i$ to be the invariants in this case.

The symmetric algebra satisfies several universal properties.

1. Let M be an A-module and R a graded commutative A-algebra, then we have a canonical isomorphism

$$\mathrm{Hom}(S(M), R) \xrightarrow{\sim} \mathrm{Hom}_A(M, R_1).$$

2. Let M be an A-module and R a commutative A-algebra, then we have a canonical isomorphism

$$\mathrm{Hom}(S(M), R) \xrightarrow{\sim} \mathrm{Hom}_A(M, R).$$

3. Let M and N be A-modules, then we have for every positive integer n a canonical isomorphism

$$\mathrm{Hom}_A(S(M)_n, N) \xrightarrow{\sim} \{\text{symmetric } A\text{-multilinear maps } M^n \longrightarrow N\}.$$

Since S has a right adjoint, it preserves right limits. So in particular we have for A-modules M and N that

$$S(M \oplus N) = S(M) \amalg S(N) = S(M) \otimes_A S(N),$$

as the tensor product is the coproduct in the category of commutative A-algebras. For the grading we have

$$S(M \oplus N)_n = \bigoplus_{i+j=n} S(M)_i \otimes_A S(N)_j$$

for every integer n.

Example. Put M equal to A. Then $M^{\otimes n}$ is naturally isomorphic to A and if we label $T(A)_i$ by t^i we get

$$T(A) = S(A) = A[t].$$

From the above remarks on the symmetric algebra and direct sums we see that for any positive integer n we get

$$S(A^n) = A[t_1] \otimes_A \cdots \otimes_A A[t_n] = A[t_1, \ldots, t_n],$$

with the grading coming from the total degree of monomials.

As a consequence of this we see that if M is a finitely generated projective module of rank r, then $S(M)_n$ is also finitely generated and projective for any integer n. Its rank is $\binom{n+r-1}{n}$.

The exterior algebra

The exterior algebra is another quotient of the tensor algebra. This time it is given by

$$\bigwedge(M) = T(M)/(x \otimes x : x \in M).$$

We call a graded A-algebra R an alternating algebra if for all x in R_i and y in R_j we have $xy = (-1)^{ij} yx$ and moreover for all odd i and all x in R_i we have $x^2 = 0$. If 2 is invertible, the latter is a consequence of the former.

Let M and N be A modules and n a positive integer. An A-multilinear map $f : M^n \longrightarrow N$ is called alternating if $f(x_1, \ldots, x_n) = 0$ whenever $x_i = x_j$ holds for some $i \neq j$. One checks that for every permutation σ in $\mathrm{Sym}(n)$ we have

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \epsilon(\sigma) f(x_1, \ldots, x_n).$$

Just as in the previous cases, the exterior algebra satisfies some universal properties, but only two this time.

1. Let M be an A-module and R an alternating A-algebra, then we have a canonical isomorphism
$$\mathrm{Hom}(\bigwedge(M), R) \xrightarrow{\sim} \mathrm{Hom}_A(M, R_1).$$

2. Let M and N be A-modules, then we have for every positive integer n a canonical isomorphism
$$\mathrm{Hom}_A(\bigwedge(M)_n, N) \xrightarrow{\sim} \{\text{alternating } A\text{-multilinear maps } M^n \longrightarrow N\}.$$

It is customary to write the multiplication in $\bigwedge(M)$ using the $\wedge$ symbol. Also, we write $x_1 \wedge \cdots \wedge x_n$ for the class in $\bigwedge(M)_n$ of $x_1 \otimes \cdots \otimes x_n$ in $T(M)_n$. Lastly, one often writes $\bigwedge^n(M)$ for $\bigwedge(M)_n$.

As $\bigwedge$ is a functor with a right adjoint, it preserves right limits, so we have a formula for the $\bigwedge$ of a direct sum. Just as before we have for M and N two A-modules that
$$\bigwedge(M \oplus N) = \bigwedge(M) \amalg \bigwedge(N) = \bigwedge(M) \otimes \bigwedge(N),$$
only this time the multiplication on the tensor product has a little twist. For x, x' in $\bigwedge(X)$ and y, y' in $\bigwedge(Y)$ such that y is homogeneous of degree i and x' is homogeneous of degree j we have
$$(x \otimes y) \cdot (x' \otimes y') = (-1)^{ij}(xx' \otimes yy').$$
Just as before we have for the homogeneous parts that
$$\bigwedge^n(M \oplus N) \cong \bigoplus_{i+j=n} \bigwedge^i(M) \otimes_A \bigwedge^j(N)$$
holds for every integer n.

Example. We again take $M = A$. Writing $\bigwedge^1(A) = At$, we have
$$\bigwedge(A) = A \oplus At,$$
as $t \wedge t = 0$. Using the previous result, we therefore have
$$\bigwedge(Ax \oplus Ay) = A \oplus (Ax \oplus Ay) \oplus A(x \wedge y).$$

From this example we can see that for M a finitely generated projective A-module of rank r and n a positive integer, $\bigwedge^n(M)$ is also finitely generated and projective and its rank is $\binom{r}{n}$. From this we see that $\bigwedge(M)$ itself is also finitely generated and projective, of rank 2^r.

If A is a Dedekind domain and M is a projective A-module of rank r, then we know that $M \cong A^{r-1} \oplus P$, where P is a projective A module of rank 1. Using the exterior powers we can easily determine P in a canonical way, as we have
$$\bigwedge^r(M) \cong \bigwedge^r(A^{r-1} \oplus P) \cong \bigwedge^{r-1}(A^{r-1}) \otimes \bigwedge^1(P) \cong A \otimes P \cong P.$$

Consider a short exact sequence of finitely generated projective A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

and suppose that M, M' and M'' have rank r, r' and r'' respectively. Then there is a natural isomorphism

$$\bigwedge^{r'}(M') \otimes \bigwedge^{r''}(M'') \xrightarrow{\sim} \bigwedge^{r}(M).$$

For all $m'_1, \dots m'_{r'}$ in M' and $m''_1, \dots, m''_{r''}$ in M'', the image of $m'_1 \wedge \cdots \wedge m'_{r'} \otimes m''_1 \wedge \cdots \wedge m''_{r''}$ is obtained by picking lifts $\tilde{m}''_1, \dots, \tilde{m}''_{r''}$ and considering the wedge product $m'_1 \wedge \cdots \wedge m'_{r'} \wedge \tilde{m}''_1 \wedge \cdots \wedge \tilde{m}''_{r''}$ in $\bigwedge^r(M)$. This element is independent of the chosen lifts, as the difference between lifts is an element of M' and so the difference between the two possible images is a wedge product containing at least $r'+1$ elements of M', which must be 0.

Dual modules

Let A be a ring and M an A-module. Then the dual module M^* is the module $\mathrm{Hom}_A(M, A)$. For A-modules M and N a pairing is a bilinear map $M \times N \longrightarrow A$. Such a pairing $< \cdot, \cdot >$ induces a map $M \longrightarrow N^*$ sending $m \in M$ to the A-linear map $x \mapsto < m, x >$. The pairing is called perfect if this map is an isomorphism.

Let M be an A-module. Then there is a pairing

$$\begin{array}{ccccc} T^n(M^*) & \times & T^n(M) & \longrightarrow & A \\ (f_1 \otimes \cdots \otimes f_n & , & m_1 \otimes \cdots \otimes m_n) & \mapsto & \prod_i f_i(m_i). \end{array}$$

If M is a finitely generated projective module, this is a perfect pairing. In this case we have a natural isomorphism $T^n(M^*) \cong T^n(M)^*$.

We have a similar pairing for the exterior powers.

$$\begin{array}{cccccc} \bigwedge^n(M^*) & \times & \bigwedge^n(M) & \longrightarrow & A & \\ (f_1 \wedge \cdots \wedge f_n & , & m_1 \wedge \cdots \wedge m_n) & \mapsto & \det(f_i(m_j))_{ij} & = \sum_\sigma \epsilon(\sigma) \prod_i f_i(m_{\sigma(i)}), \end{array}$$

where the sum runs over all σ in the symmetric group on n letters. Again if M is finitely generated projective, then this pairing is perfect.

For the symmetric powers things are slightly less nice. There still is a pairing, given by

$$\begin{array}{ccccc} S^n(M^*) & \times & S^n(M) & \longrightarrow & A \\ (f_1 \otimes \cdots \otimes f_n & , & m_1 \otimes \cdots \otimes m_n) & \mapsto & \sum_\sigma \prod_i f_i(m_{\sigma(i)}), \end{array}$$

where the sum runs over all σ in the symmetric group on n letters. For M a finitely generated projective A-module, this pairing is perfect if $n! \in A^\times$. That is, there is a canonical map $S^n(M^*) \longrightarrow S^n(M)^*$, which is an isomorphism if $n! \in A^\times$.