THE PROBABILISTIC ZETA FUNCTION OF FINITE AND PROFINITE GROUPS

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To a finitely generated profinite group G one may associate the numerical sequence $a_n(G)$ defined by

$$a_n(G) = \sum_{|G:\, H|=n} \mu_G(H),$$

where the Möbius function μ_G is defined recursively by

$$\mu_G(G) = 1 \text{ and } \sum_{K \geq H} \mu_G(K) = 0$$

for each open subgroup H of G. The formal inverse of the formal Dirichlet series $P_G(s) = \sum_{n \geq 1} a_n(G)n^{-s}$ is called the probabilistic zeta function of G (when $G = \widehat{\mathbb{Z}}$ this is the Riemann zeta function, and μ_G is the usual Möbius function; when G is finite and t is a natural number, $P_G(t)$ is equal to the probability that a random t-tuple generates G). The probabilistic zeta function encodes information about the lattice generated by the maximal subgroups of G, just as the Riemann zeta function encodes information about the primes. The series and the complex function that it represents, when it converges, have been studied by several authors. In the first part of the talk we will give a general introduction to the property of the probabilistic zeta function, with a survey of the results that have been obtained in the last ten years.

Another Dirichlet series that can be associated to a finitely generated profinite group and that has been intensively studied can be obtained considering the generating function $Z_G(s) = \sum_{n \geq 1} b_n(G)n^{-s}$, where $b_n(G)$ is the number of subgroups of G with index n (the subgroup zeta function). In the second part of the talk we will compare the properties of the probabilistic zeta function with those of the subgroup zeta function. In particular we shall examine finitely generated profinite groups in which the subgroup zeta function is the formal inverse of the probabilistic zeta function; we call these groups zeta-reversible. After showing that it is reasonable to restrict to prosolvable groups of finite rank we show some sufficient conditions for this property to hold and we produce a structural characterisation of torsion-free prosolvable groups of rank two which are zeta-reversible. Several examples of non obvious groups with this property will be produced. This problem turned out to be surprisingly intriguing and several open problems concerning the probabilistic zeta function or the subgroup zeta function hover overhead.