The ABC-conjecture

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ABC-hits

- The product of the distinct primes in a number is called the *radical* of that number. Notation: $\mathrm{rad}()$. For example,

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- Among all 15×10^6 ABC-triples with $C < 10000$ we have 120 ABC-hits.
- Among all 380×10^6 ABC-triples with $C < 50000$ we have 276 hits.

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- Notice

$$\begin{aligned} 3^{64} - 1 &= (3^{32}+1)(3^{32}-1) \\ &= (3^{32}+1)(3^{16}+1)(3^{16}-1) \\ &\ldots \\ &= (3^{32}+1)(3^{16}+1)(3^{8}+1)\cdots(3+1)(3-1) \end{aligned}$$

So $3^{64} - 1$ is divisible by $2 \cdot 2^8$.

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- In general $3^{2^k} - 1$ is divisible by 2^{k+2}. So

  $$\mathrm{rad}(B) = \mathrm{rad}(3^{2^k} - 1) \leq (3^{2^k} - 1)/2^k < C/2^{k+1}$$

  We conclude

  $$\mathrm{rad}(ABC) = 3 \cdot \mathrm{rad}(B) < 3C/2^{k+1}.$$

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- In other words, $C > \mathrm{rad}(ABC) \cdot 2^{k+1}/3$. So when $k \geq 1$ we have an *ABC*-hit.

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- In other words, $C > \mathrm{rad}(ABC) \cdot 2^{k+1}/3$. So when $k \geq 1$ we have an ABC-hit.
- But we have shown more. For any number $M > 1$ there exist infinitely many ABC-triples such that $C > M \cdot \mathrm{rad}(ABC)$.

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- Consequence: Let x, y, z, n be positive integers such that $\gcd(x, y, z) = 1$ and $x^n + y^n = z^n$. Then the *hypothesis* implies $n < 6$.

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- Consequence: Let x, y, z, n be positive integers such that $\gcd(x, y, z) = 1$ and $x^n + y^n = z^n$. Then the *hypothesis* implies $n < 6$.
- Proof: Apply the *hypothesis* to the triple $A = x^n, B = y^n, C = z^n$. Notice that $\mathrm{rad}(x^n y^n z^n) \leq xyz < z^3$. So, $z^n < (z^3)^2 = z^6$. Hence $n < 6$.
  Fermat's Last Theorem for $n \geq 6$ follows!

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- We expect at most finitely many instances.
- ABC-Conjecture (Masser-Oesterlé, 1985): *Let $\kappa > 1$. Then, with finitely many exceptions we have $C < \mathrm{rad}(ABC)^{\kappa}$.*

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- 2) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Again it is an exercise to show that (p, q, r) is a permutation of one of $(2, 4, 4), (2, 3, 6), (3, 3, 3)$. There are finitely many solutions.

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- 2) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Again it is an exercise to show that (p, q, r) is a permutation of one of $(2, 4, 4), (2, 3, 6), (3, 3, 3)$. There are finitely many solutions.
- 3) $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. There are infinitely many possible triples (p, q, r). For any such triple the number of solutions is at most finite (Darmon-Granville, 1995).

Numeric results

$$1^k + 2^3 = 3^2 \ (k > 6), \qquad 13^2 + 7^3 = 2^9, \qquad 2^7 + 17^3 = 71^2$$

$$2^5 + 7^2 = 3^4, \qquad 3^5 + 11^4 = 122^2, \qquad 17^7 + 76271^3 = 21063928^2$$

$$1414^3 + 2213459^2 = 65^7, \qquad 33^8 + 1549034^2 = 15613^3$$

$$43^8 + 96222^3 = 30042907^2, \qquad 9262^3 + 15312283^2 = 113^7.$$

Fermat-Catalan conjecture

Consequence of ABC-conjecture:
The set of triples x^p, y^q, z^r with x, y, z coprime positive integers such that $x^p + y^q = z^r$ and $1/p + 1/q + 1/r < 1$, is finite.

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- Apply ABC with $\kappa = 1.01$ to $A = x^p, B = y^q, C = z^r$. Notice that $\mathrm{rad}(x^r y^q z^r) \leq xyz < z^{r/p} z^{r/q} z$.

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- This implies $r < \kappa(r/p + r/q + 1)$ and hence $1 < \kappa(1/p + 1/q + 1/r)$. But this is impossible because $\kappa = 1.01$ and $1/p + 1/q + 1/r \leq 1 - 1/42$.

Catalan

As a special case we see that $x^p - y^q = 1$ has finitely many solutions.
But this was shown in 1974 by Tijdeman and completely solved in 2002 by Michailescu.

Mordell's conjecture

Consider a diophantine equation $P(x,y)=0$ in the unknown rational numbers x,y.
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Previously known as Mordell's conjecture (1923) and Faltings' theorem (1983).

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- Example: $P(x) = x^3 + 17$. We have $2^3 + 17 = 5^2,\ 4^3 + 17 = 9^2,\ 8^3 + 17 = 23^2,\ 43^3 + 17 = 282^2$ $52^3 + 17 = 375^2,\ 5234^3 + 17 = 378661^2$.

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Stewart, Kunrui Yu (1996): *For any* $\epsilon > 0$*:*

$$C < \exp\left(\gamma \mathrm{rad}(ABC)^{1/3+\epsilon}\right)$$

where γ *depends on the choice of* ϵ*.*

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A polynomial $F(x)$ with rational coefficients and leading coefficient 1 is called *prime* if it cannot be factored into polynomials with rational coefficients and lower degree.

Theorem: *Any polynomial with rational numbers as coefficient can be written in a unique way as a constant times a product of prime polynomials.*

Factors of polynomials

For example: x^2+1, whereas x^2-1 is reducible. Example of a factorisation:

$$\begin{aligned} x^{21}-1 &= (x^6+x^5+x^4+x^3+x^2+x+1)\times \\ &\quad (x-1)(x^2+x+1)\times \\ &\quad (x^{12}-x^{11}+x^9-x^8+x^6-x^4+x^3-x+1). \end{aligned}$$

Degree of a polynomial F: $\deg(F)$.

The *radical* of a polynomial $F(x)$ is the product of the prime polynomials dividing $F(x)$. Notation $\mathrm{rad}(F)$.

PQR-Theorem

PQR-Theorem (Hurwitz, Stothers, Mason): Let P, Q, R be coprime polynomials, not all constant, such that $P + Q = R$. Suppose that $\deg(R) \geq \deg(P), \deg(Q)$. Then

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Translation to *ABC*: Replace P, Q, R by A, B, C and deg by log. Note the analogy: $\deg(PQ) = \deg(P) + \deg(Q)$ for polynomials and $\log(ab) = \log(a) + \log(b)$ for numbers. We get:

$$\log(C) < \log(\mathrm{rad}(ABC)).$$

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Muliply first equality by P', second equality by P and subtract,

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If $P'Q - PQ' = 0$, then P/Q is constant and all of P, Q, R are constant.

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Happy hunting, or fishing!