EXERCISES ALGEBRAIC GEOMETRY - 14/04/2009

(1) Let k be a field, A a k-algebra and M an A-module. Show that $\mathrm{Der}_k(A, M)$ is an A-module for the addition and multiplication defined by $(D_1 + D_2)g = D_1 g + D_2 g$, $(fD)g = f(Dg)$.

(2) Show that if $\phi\colon A \to B$ is a morphism of k-algebras and $D \in \mathrm{Der}_k(B, M)$, then $D \circ \phi$ is in $\mathrm{Der}_k(A, M)$ (what is the A-module structure on M?).

(3) Let k be a field, A a k-algebra and $m \subset A$ a maximal ideal such that the morphism $k \to A \to A/m = k$ is an isomorphism.

  (a) Let $D \in \mathrm{Der}_k(A, A/m)$. Show that D is zero on m^2, and hence factors through a derivation $\overline{D}\colon A/m^2 \to k$.

  (b) Show that the map $\mathrm{Der}_k(A, A/m) \to (m/m^2)^\vee$, $D \mapsto \overline{D}|_{m/m^2}$ is an isomorphism of A-modules.

(4) Let k be a field, $A = k[x_1, \ldots, x_n]$. Show that $(dx_1, \ldots, dx_n)$ is an A-basis of Ω^1_A, and give a formula for df, where $f \in A$.

(5) Let k and A be as in the previous exercise. Let $I = (f_1, \ldots, f_r)$ be an ideal in A, and let $q\colon A \to B := A/I$ be the quotient map.

  (a) Show that, for any B-module M, $q^*\colon \mathrm{Der}_k(B, M) \to \mathrm{Der}_k(A, M)$ is injective and has image the set of those D such that for all i one has $D(f_i) = 0$.

  (b) Use the universal property of Ω^1_A to show that $d\colon B \to \Omega^1_A/(A{\cdot}df_1 + \cdots A{\cdot}df_r)$ is a universal derivation.

Homework

(1) Consider the rational 1-form $x^{-1}dx$ on $\mathbf{P}^1$. Compute its order and residue at all $P \in \mathbf{P}^1$.

(2) Prove that for all rational 1-forms ω on $\mathbf{P}^1$ we have $\sum_P \mathrm{res}_P(\omega) = 0$, where the sum is over all $P \in \mathbf{P}^1$. Hint: write $\omega = f{\cdot}dx$, with $f \in k(x)$, and use a suitable k-basis of $k(x)$.

(3) Let $n \in \mathbf{Z}_{\geq 2}$, $X = Z(-x_1^n + x_0^{n-1}x_2 - x_2^n) \subset \mathbf{P}^2$. Assume that $n(n-1)$ is in $k^\times$. We have already seen that X is smooth. You may now use without proof that X is irreducible (in fact, Bezout's theorem implies that reducible plane projective curves are singular). Let $U := X \cap \mathbf{A}^2$. Then $U = Z(f)$ with $f = -y^n + x^{n-1} - 1$.

  (a) Show that in $\Omega^1(U)$ we have $(n-1)x^{n-2}dx = ny^{n-1}dy$.

  (b) We define a rational 1-form ω_0 by:

$$\omega_0 = \frac{dx}{ny^{n-1}} = \frac{dy}{(n-1)x^{n-2}}.$$

  Show that ω_0 has no poles on U. Hint: $U = (U \cap D(x)) \cup (U \cap D(y))$.

  (c) Show that ω_0 has no zeros on U. Hence (you do not need to prove this) $\Omega^1(U)$ is free as $\mathcal{O}(U)$-module with basis ω_0. Hint: both dx and dy are multiples of ω_0, and, for each $P \in U$, at least one of dx and dy is a generator of $\Omega^1(U)/m_P\Omega^1(U)$.

  (d) Let $P = X \cap Z(x_2)$ be the point at infinity of X. Compute $v_P(\omega_0)$.

  (e) For $n \in \{2, 3, 4\}$, give a basis (and hence the dimension) of $\Omega^1(X)$ (hint: use computations from Lecture 2; do not do these computations again, just give the result).