Expander Families of Coverings

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Introduction

A graph $\Gamma$ is a collection of dots and connections between them (with possibly loops and multiple connections). The set of dots is called vertex set $V(\Gamma)$ and the set of connections is called edge set $E(\Gamma)$.

Let $c, k > 0$. A finite graph $\Gamma$ is a $(c, k)$-expander if every vertex has at most $k$ connections and for every partition $V(\Gamma) = A \cup B$ we have

$$|\{\text{edges of } \Gamma \text{ from } A \text{ to } B\}| \geq c \cdot \frac{|A||B|}{|V(\Gamma)|}.$$  \hspace{1cm} (*)

A family $\{\Gamma_i\}_{i \geq 0}$ of finite graphs is called an expander family if $|V(\Gamma_i)| \to \infty$ and there are $c, k > 0$ such that for every $i \geq 0$ the graph $\Gamma_i$ is a $(c, k)$-expander.

The existence of $k$ gives a kind of sparsity condition (we want few edges) and the inequality $(*)$ is a kind of super-connectivity (we need to cut many edges to split $\Gamma$ in two subgraphs of roughly the same size).

In 1973, Grigory Margulis used algebraic techniques to give the first explicit construction of an expander family ([1]). In [2] Alexander Lubotzky gives a variant of Margulis' construction considering Cayley graphs of finite quotients of Kazhdan groups (also called groups with property (T)). An example of Kazhdan group is given by the special linear group $\text{SL}(3, \mathbb{Z})$. The Cayley graph $C(G, W)$ of a group $G$ with respect to a subset $W \subseteq G$ is the graph whose vertices are the elements of $G$ and $g \in G$ is connected to $gw, gw^{-1} \in G$ for every $w \in W$. Lubotzky proved the following result:

**Theorem** (Lubotzky). Let $G$ be a finitely generated Kazhdan group and $W$ be a finite generating set for $G$. Then there exist $c, k > 0$ such that for every normal subgroup $N \triangleleft G$ of finite index the Cayley graph $C(G/N, W)$ is a $(c, k)$-expander.

The aim of this thesis is to generalize Lubotzky’s theorem.

In section 1 we give the precise definition of a graph $\Gamma$ and of the fundamental group $\pi(\Gamma, u)$ based at a vertex $u$, studying some of its properties.

In section 2 we define coverings of a graph and the universal covering of a graph in order to construct for every finite connected graph $\Gamma$ a specific equivalence of categories between the category of coverings of $\Gamma$ and the category of right $\pi(\Gamma, u)$-sets. We call $\Gamma_S$ the covering of $\Gamma$ corresponding to the right $\pi(\Gamma, u)$-set $S$.

Finally in section 3 we define Kazhdan groups and we prove the following result

**Theorem 1.** Let $\Gamma$ be a connected finite graph, $u$ be a vertex and let $h: \pi(\Gamma, u) \to G$ be a surjective morphism onto a Kazhdan group $G$. Then there exist $c, k > 0$ such that for every transitive finite right $G$-set $S$ the covering $\Gamma_S$ of $\Gamma$ is a $(c, k)$-expander.
If the graph $\Gamma$ has a fundamental group $\pi(\Gamma, u)$ free on at least two generators then we can consider a surjective morphism onto the group $\text{SL}(3, \mathbb{Z})$, which is Kazhdan, and we use Theorem 1 to construct an expander family of coverings of $\Gamma$.

In fact Theorem 1 recovers Lubotzky's theorem. Let $G$ be a finitely generated Kazhdan group and $W \subseteq G$ be a finite generating set for $G$. We consider the graph $\Gamma$ consisting of one vertex $u$ and a loop for every generator $w \in W$, called bouquet of $|W|$ loops. For every normal subgroup $N \triangleleft G$ of finite index, the set $G/N$ is a finite transitive right $G$-set and we show that the Cayley graph $C(G/N, W)$ is isomorphic to the covering $\Gamma_{G/N}$ of $\Gamma$. Therefore we get Lubotzky's Theorem from Theorem 1. Our construction generalizes Lubotzky's in two ways. Firstly we consider coverings associated to any transitive finite right $G$-set and not only those of the form $G/N$ with $N$ a normal subgroup of finite index. Secondly we allow $\Gamma$ to be any finite connected graph, and not just the bouquet of loops.
1 The fundamental group of a graph

In this section we give the definitions of graph and graph morphism. After that we define paths on a graph and we introduce an equivalence relation between them, necessary to define the fundamental group of a graph based at a vertex. Finally we show that the fundamental group is a free group.

Definition 1.1 (Graph). A graph $\Gamma$ consists of two sets $V = V(\Gamma)$ and $E = E(\Gamma)$ and two maps

$$\text{ep}: E \to V \times V, \quad x \mapsto (o(x), t(x))$$

and

$$\overline{\cdot}: E \to E, \quad x \mapsto \overline{x}$$

such that for every $x \in E$ we have $\overline{\overline{x}} = x$ and $\overline{x} = x$ and $o(x) = t(\overline{x})$. An element $u \in V$ is called a vertex of $\Gamma$, an element $x \in E$ is called an (oriented) edge and $\overline{x}$ is called the inverse edge of $x$. The map $\text{ep}$ is called the endpoints map. The vertices $o(x)$ and $t(x)$ are called respectively the origin and the terminus of $x$. If $o(x) = t(x)$ then $x$ is called a loop.

Remark 1.2. A graph is often represented by a diagram, using the following conventions: a point marked on the diagram corresponds to a vertex of the graph and a line joining two marked points corresponds to a pair of edges of the form $\{x, \overline{x}\}$.

Example 1.3. Consider the graph $\Gamma$ with $V(\Gamma) = \{u\}$ and $E(\Gamma) = \{x, \overline{x}\}$. It is a graph with one vertex and two edges with maps $\text{ep}: x \mapsto (u, u)$ and $\overline{\cdot}: x \mapsto \overline{x}$.

This graph is called simple loop.

Sometimes we need to distinguish an edge and its inverse in the picture, hence we introduce some arrows as follows:

$\text{ep}(y) = (v, u)$

$\text{ep}(y) = (u, v)$
The following are other examples of graphs:

\[ \emptyset \quad \bullet \quad \circ \quad \circ \quad \circ \quad \circ \]

empty graph no edges two edges four edges six edges

**Definition 1.4 (Path_{n}).** Let \( n \geq 0 \) be an integer. The path of length \( n \), denoted \( \text{Path}_{n} \), is the graph with set of vertices \( V = \{0, \ldots, n\} \), set of edges \( E = \{(i, i+1), (i+1, i) : 0 \leq i \leq n-1\} \) and maps

\[ \text{ep} = \text{id} \quad \text{and} \quad (i, i+1) = (i+1, i). \]

The path of length 0 is the graph with a single vertex and no edges.

**Example 1.5.** A path of length four:

\[ (0, 1) \quad (1, 2) \quad (2, 3) \quad (3, 4) \]

\[ 0 \quad 1 \quad 2 \quad 3 \quad 4 \]

**Definition 1.6 (morphism of graphs).** Let \( \Gamma \) and \( \Gamma' \) be two graphs. A morphism \( f \) from \( \Gamma \) to \( \Gamma' \) is a pair \( (f, f_\ast) \) where

\[ f : V(\Gamma) \to V(\Gamma') \]

is a map between the vertex sets and

\[ f_\ast : E(\Gamma) \to E(\Gamma') \]

is a map between the edge sets such that for every \( x \in E(\Gamma) \) we have

\[ \text{ep}(f_\ast(x)) = (f(o(x)), f(t(x)) \quad \text{and} \quad f_\ast(x) = f_\ast(x) \]

**Definition 1.7 (Path).** Let \( \Gamma \) be a graph and let \( u, v \in V(\Gamma) \) be vertices. A path of length \( n \) from \( u \) to \( v \) in \( \Gamma \) is a graph morphism \( \gamma : \text{Path}_{n} \to \Gamma \) with \( \gamma(0) = u \) and \( \gamma(n) = v \). The vertices \( u \) and \( v \) are called respectively the initial vertex and the final vertex of the path. For every \( u, v \in V(\Gamma) \) we denote with \( P(u, v) \) the set of all paths of \( \Gamma \) from \( u \) to \( v \). If \( \gamma \in P(u, u) \) we say that it is a closed path based at \( u \). A path of length zero based at \( u \) is called trivial path and denoted \( 1_u \).
Lemma 1.8. Let $u$ and $v$ be vertices of $\Gamma$. Let $P_n(u,v)$ be the set of paths from $u$ to $v$ of length $n$ and $\sigma_n = \{(x_1,\ldots,x_n) \in E(\Gamma)^n : o(x_1) = u, t(x_n) = v, o(x_{i+1}) = t(x_i)\}$. Then for every $n \geq 1$ the map

$$P_n(u,v) \rightarrow \sigma_n \quad \gamma \mapsto (\gamma(i,i+1))_{0 \leq i \leq n-1}$$

is a bijection with inverse given by

$$(x_1,\ldots,x_n) \mapsto \gamma : \begin{cases} \gamma : i \mapsto t(x_i) \text{ for every } 1 \leq i \leq n \\ \gamma_i : (i,i+1) \mapsto x_{i+1} \text{ for every } 0 \leq i \leq n - 1 \end{cases}$$

Definition 1.9. We write $\gamma = x_1 \cdots x_n$ and we put $o(\gamma) := o(x_1)$ and $t(\gamma) := t(x_n)$. In this notation the inverse path of $\gamma$ is defined as $\gamma^{-1} = \bar{x}_n \cdots \bar{x}_1$. Notice that if $\gamma \in P(u,v)$ then $\gamma^{-1} \in P(v,u)$.

Definition 1.10 (connected graph). A graph is connected if it is nonempty and for every couple of vertices $u$ and $v$ we have $P(u,v) \neq \emptyset$.

Definition 1.11 (Composition of paths). Let $u, v, w$ be vertices of $\Gamma$. We define the composition of paths as the map

$$P(u,v) \times P(v,w) \rightarrow P(u,w)$$

$$(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$$

where if $\gamma_1 = x_1 \cdots x_n$ and $\gamma_2 = y_1 \cdots y_m$ then $\gamma_1 \cdot \gamma_2 = x_1 \cdots x_n \cdot y_1 \cdots y_m$.

The notation $\gamma_1 \cdot \gamma_2 = x_1 \cdots x_n \cdot y_1 \cdots y_m$ is unambiguous thanks to the following lemma:

Lemma 1.12. The composition of paths is associative, i.e. we have

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 = \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$$

for all paths $\gamma_1, \gamma_2, \gamma_3$ where the composition makes sense.

Lemma 1.13. Let $\Gamma$ be a graph and $u$ be a vertex of $\Gamma$. Then the set $P(u,u)$ with the composition of paths is a semigroup with identity element the trivial path $1_u$.

Definition 1.14 (path equivalence). An elementary transformation of a path $\gamma$ is a path obtained from $\gamma$ by inserting or deleting a sequence of edges of the form $x \cdot \bar{x}$. Two paths $\gamma_1$ and $\gamma_2$ are equivalent if there is a sequence of paths $(\gamma_1 = \psi_1, \psi_2, \ldots, \psi_n = \gamma_2)$ where each $\psi_{i+1}$ is an elementary transformation of $\psi_i$. In this case we write $\gamma_1 \sim \gamma_2$ and we call this relation path equivalence.
Lemma 1.15. Path equivalence is an equivalence relation.

Definition 1.16 (reduced path). We say that a path is reduced if it does not contain sequences of edges of the form $x \cdot \bar{x}$.

Lemma 1.17. There is a unique reduced path in every path equivalence class. We denote with $\text{red}(\gamma)$ the reduced path in the class of $\gamma$.

Proof. For sure there exists a reduced path in every class. Assume by contradiction that $\gamma_1$ and $\gamma_2$ are distinct reduced paths in $[\gamma]$. In particular we have $\gamma_1 \sim \gamma_2$ so by definition there exists a sequence of paths $\sigma = (\gamma_1 = \psi_1, \psi_2, \ldots, \psi_n = \gamma_2)$ where each $\psi_{i+1}$ is an elementary transformation of $\psi_i$ for every $1 \leq i \leq n - 1$. Define $N(\sigma) := \sum_i |\psi_i|$, where $|\psi_i|$ denotes the length of $\psi_i$, and consider a sequence $\sigma$ from $\gamma_1$ to $\gamma_2$ with $N(\sigma)$ minimal. Notice that $|\psi_1| < |\psi_2|$ and $|\psi_{n-1}| > |\psi_n|$ since $\psi_1$ and $\psi_n$ are reduced. Hence there exists $i$ such that $|\psi_{i-1}| < |\psi_i|$ and $|\psi_i| > |\psi_{i+1}|$. It means that if $\psi_{i-1} = x_1 \cdots x_n$ and $\psi_{i+1} = y_1 \cdots y_m$ then we have

$$\psi_i = x_1 \cdots x_t \cdot z \cdot \bar{z} \cdot x_{t+1} \cdots x_n = y_1 \cdots y_r \cdot s \cdot \bar{s} \cdot y_{r+1} \cdots y_m$$

for some edges $z$ and $s$.

If $r = t$ then $\psi_{i-1} = \psi_{i+1}$ and we can remove $\psi_i$ and $\psi_{i+1}$ from the sequence $\sigma$, getting a sequence $\sigma'$ with $N(\sigma') < N(\sigma)$, contradicting the minimality of $N(\sigma)$.

Assume without loss of generality $r < t$.

If $r = t - 1$ then we have $x_t = s$ and $z = \bar{s}$.

$$\cdots z \bar{z} \cdots$$

$$\cdots s \bar{s} \cdots$$

Notice that $y_{r+1} = \bar{z} = \bar{s} = s = x_t$. Hence $\psi_{i-1} = \psi_{i+1}$ and we get a contradiction as in the previous case.

Finally if $r < t - 1$ then we have

$$\psi_i = x_1 \cdots x_t \cdot s \cdot \bar{s} \cdot x_{t+3} \cdots x_t \cdot z \cdot \bar{z} \cdot x_{t+1} \cdots x_n$$

$$= y_1 \cdots y_r \cdot s \cdot \bar{s} \cdot y_{r+1} \cdots y_t \cdot z \cdot \bar{z} \cdot y_{t+3} \cdots y_n.$$ 

Hence we can replace $\psi_i$ with the path $\psi'_i := x_1 \cdots x_r \cdot x_{r+3} \cdots x_n$ and we get a new sequence $\sigma'$ with $N(\sigma') = N - 4$, which contradicts the minimality of $N(\sigma)$.

In all cases we get a contradiction. Hence $\gamma_1 = \gamma_2$. □

Lemma 1.18. Path equivalence satisfies the following properties

Compatibility with the composition:

$$\gamma_1 \sim \gamma_2 \Rightarrow \gamma_1 \cdot \gamma_3 \sim \gamma_2 \cdot \gamma_3$$

$$\gamma_1 \sim \gamma_2 \Rightarrow \gamma_3 \cdot \gamma_1 \sim \gamma_3 \cdot \gamma_2$$
Compatibility with the inverse:
\[ \gamma_1 \sim \gamma_2 \Rightarrow \gamma_1^{-1} \sim \gamma_2^{-1} \]

For all paths \( \gamma_1, \gamma_2, \gamma_3 \) where the composition makes sense.

**Definition 1.19 (Fundamental group).**
Let \( \Gamma \) be a graph and \( u \) be a vertex. We define the fundamental group of \( \Gamma \) with respect to \( u \) as the set
\[ \pi(\Gamma, u) = P(u, u)/\sim \]
equipped with the operation of composition of paths.

**Proposition 1.20.** The fundamental group \( \pi(\Gamma, u) \) is a group.

**Proof.** By lemma 1.18 the operation is well defined and by lemma 1.13 we have that \( \pi(\Gamma, u) \) is a semigroup with identity element \([1_u] \). Moreover for every path \( \gamma \in P(u, u) \) we have \( \gamma \cdot \gamma^{-1} \sim 1_u \).

**Example 1.21.** Let \( \Gamma \) be the simple loop.

![Simple Loop](image)

Then the fundamental group \( \pi(\Gamma, u) \) is generated by the class of the edge \( x \) and we have
\[ \pi(\Gamma, u) = \langle [x] \rangle \cong \mathbb{Z} \]
In fact for every path \( \gamma \) in \( P(u, u) \) we have \( \text{red}(\gamma) = x^n \) for some \( n \in \mathbb{Z} \) where \( x^0 := 1_u \) and for every \( n \in \mathbb{Z}_{>0} \) we have \( x^n := \prod_{i=1}^{n} x \) and \( x^{-n} := \prod_{i=1}^{n} \bar{x} \).

**Example 1.22.** Consider the graph \( \Gamma \) with \( V(\Gamma) = \{u\} \) and \( E(\Gamma) = \{x, \bar{x}, y, \bar{y}\} \).

![Graph Example](image)

This graph is called the double loop. The fundamental group \( \pi(\Gamma, u) \) is generated by the classes of the edges \( x \) and \( y \) and we have
\[ \pi(\Gamma, u) = \langle [x], [y] \rangle \cong F_2 \]
It is a free group of rank two. In fact \( 1_u = x^0 \) and every non trivial reduced path in \( P(u, u) \) can be written in a unique way in the form \( \prod_{i=1}^{n} x^{a_i} y^{b_i} \) for some \( a_i, b_i \in \mathbb{Z} \) with \( a_i \neq 0 \) for \( i > 1 \) and \( b_i \neq 0 \) for \( i < n \).
Example 1.23. More generally, let $\Gamma$ be a graph with one vertex $u$ and $n$ couples of edges $\{x_i, \overline{x}_i\}_{1 \leq i \leq n}$.

This graph is called the bouquet of $n$ loops.

The fundamental group $\pi(\Gamma, u)$ is given by the semigroup generated by the set $\{x_i, \overline{x}_i : 1 \leq i \leq n\}$ modulo the relations $x_i \cdot \overline{x}_i = 1_u$ for every $1 \leq i \leq n$. Hence we have $\pi(\Gamma, u) \cong F_n$, the free group of rank $n$.

At the end of this section we will show that the fundamental group based on a vertex of a connected graph is always a free group.

Definition 1.24 (Tree). A tree is a connected graph whose unique reduced closed path is the trivial path.

Definition 1.25. A subgraph of a graph $\Gamma$ is a graph $\Gamma'$ such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma') \subseteq E(\Gamma)$.

Definition 1.26. Let $\Gamma$ be a graph and $u$ be a vertex. The connected component of $\Gamma$ containing $u$ is the subgraph $\Gamma'$ of $\Gamma$ with sets $V(\Gamma') := \{v \in V(\Gamma) : P(u, v) \neq \emptyset\}$ and $E(\Gamma') := \{x \in E(\Gamma) : o(x), t(x) \in V(\Gamma')\}$.

Remark 1.27. Every connected component of a graph is a connected graph. If $\Gamma$ is a connected graph then for every vertex $u \in V(\Gamma)$ the connected component of $\Gamma$ containing $u$ is equal to $\Gamma$.

Definition 1.28 (Maximal tree). A maximal tree in a graph $\Gamma$ is a subgraph $T$ which is a tree and such that $V(T) = V(\Gamma)$.

Lemma 1.29. Every connected component of a graph contains a maximal tree.

Theorem 1.30. For every vertex $u$ of a graph $\Gamma$ the fundamental group $\pi(\Gamma, u)$ is a free group. Moreover if $\Gamma$ is finite then $\pi(\Gamma, u)$ is finitely generated.

Proof. Consider the connected component $\Gamma'$ of $\Gamma$ containing the vertex $u$ and let $T$ be a maximal tree inside $\Gamma'$. For every vertex $v$ of $\Gamma'$ we define $\gamma_v$ as the unique reduced path in $T$ from $u$ to $v$. Notice that $\gamma_u = 1_u$. For every edge $x \in E(\Gamma')$ we define $\psi_x := \gamma_{o(x)}x\gamma_{t(x)}^{-1} \in P(u, u)$. If $x \in E(T)$ then $\gamma_{t(x)}^{-1} = x^{-1}\gamma_{o(x)}^{-1}$,
so $\psi_x = \gamma_0(x)x^{-1}\gamma_0^{-1}(x)$ and $[\psi_x] = [1_u]$. Let $\gamma = x_1 \cdots x_n$ be a path in $P(u,u)$. We have

$$[\gamma] = [x_1 \cdots x_n] = [\psi_{x_1} \cdots \psi_{x_n}] = \prod_{x_i \notin E(T)} [\psi_{x_i}].$$

Thus $\pi(\Gamma, u)$ is generated by $\Psi := \{[\psi_x] : x \in E(\Gamma') \setminus E(T)\}$.

Moreover once we have chosen the maximal tree, the paths $\gamma_v$ are uniquely defined for every vertex $v$. Hence we can find a bijection between $\pi(\Gamma, u)$ and reduced words in $\Psi$ and we conclude that the fundamental group is a free group. 

Remark 1.31. Notice that for every $x \in E(\Gamma')$ the paths $\gamma_0(x)$ and $\gamma_{1(x)}^{-1}$ do not have edges in common. Hence for every $x \in E(\Gamma') \setminus E(T)$ we have $\psi_x = x_1 \cdots x_n$ with $x_i \neq x_j$ if $i \neq j$.

Lemma 1.32. Let $\Gamma$ be a connected graph and $u, v \in V(\Gamma)$ be vertices. Let $\gamma_v \in P(u,v)$ be a reduced path of $\Gamma$. Then the map $\pi(\Gamma, u) \to \pi(\Gamma, v)$ given by $[\psi] \mapsto [\gamma_v^{-1}\psi\gamma_v]$ is an isomorphism of groups.

Proof. The map is well defined and has as inverse the map $\pi(\Gamma, v) \to \pi(\Gamma, u)$ given by $[\psi] \mapsto [\gamma_v\psi\gamma_v^{-1}]$. Hence it is a bijection. Moreover

$$[\psi_1\psi_2] \mapsto [\gamma_v^{-1}\psi_1\psi_2\gamma_v] = [\gamma_v^{-1}\psi_1\gamma_v\gamma_v^{-1}\psi_2\gamma_v] = [\gamma_v^{-1}\psi_1\gamma_v][\gamma_v^{-1}\psi_2\gamma_v]$$

so it is an isomorphism of groups. 

Remark 1.33. Lemma 1.32 shows that if $\Gamma$ is a connected graph then the fundamental group $\pi(\Gamma, u)$ is independent of $u$, up to a non-canonical isomorphism.
2 Coverings of a graph and action of the fundamental group

In this section we define coverings of a graph and in particular the universal covering of a graph. Then we construct a special covering starting from the universal covering and a set provided with a right action of the fundamental group $\pi$. Finally we prove that the category of coverings of a connected graph is equivalent to the category of right $\pi$-sets.

**Definition 2.1** (Neighborhood of a vertex). Let $\Gamma$ be a graph and $u \in V(\Gamma)$ be a vertex. The neighborhood of $u$ is the set $N(u) := \{x \in E : o(x) = u\}$. We call valency of $u$ the cardinality of $N(u)$. If all the vertices of $\Gamma$ have the same valency $n$ then we say that $\Gamma$ is a regular graph of degree $n$.

**Example 2.2.** Consider the following graph:

```
\begin{figure}[h]
\centering
\begin{tikzpicture}
\node (u) at (0,0) [circle, inner sep=0.5pt, fill=red] {$u$};
\node (v) at (1,1) [circle] {$v$};
\node (x) at (0,1) [circle] {$x$};
\node (y) at (1,0) [circle] {$y$};
\node (z) at (0.5,0.5) [circle] {$z$};
\node (w) at (1,0.5) [circle] {$w$};
\node (t) at (-1,0) [circle] {$t$};
\node (s) at (0,-1) [circle] {$s$};
\draw[->] (u) to (x);
\draw[->] (u) to (y);
\draw[->] (u) to (z);
\draw[->] (u) to (w);
\draw[->] (u) to (t);
\draw[->] (u) to (s);
\end{tikzpicture}
\end{figure}
```

Then $N(u) = \{x, \bar{x}, y, \bar{z}, s\}$ so $u$ has valency 5. However $N(o) = \{t, \bar{t}, s\}$ so $o$ has valency 3 and the graph is not regular.

**Lemma 2.3.** Let $f: \Gamma \rightarrow \Gamma'$ be a morphism of graphs and let $u$ be a vertex of $\Gamma$. Then $f_*$ maps $N(u)$ to $N(f(u)) \subseteq E(\Gamma')$.

**Proof.** Let $x \in N(u)$ be an edge of $\Gamma$. Then $o(x) = u$ and $o(f_*(x)) = f(o(x)) = f(u)$ since $f$ is a morphism. Hence $f_*(x) \in N(f(u))$. \qed

**Definition 2.4** (Covering). A covering of a graph $\Gamma$ is a graph morphism

$$f: X \rightarrow \Gamma$$

such that for every $v \in V(X)$ we have that

$$f_*: N(v) \rightarrow N(f(v))$$

is a bijection.

**Example 2.5.** The empty morphism $\emptyset: \emptyset \rightarrow \Gamma$ is a covering for every graph $\Gamma$. 

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Example 2.6. Consider the following graphs.

\[
\begin{array}{c}
X \\
\begin{array}{c}
\vdots \\
y \\
w \\
z \\
v \\
x \\
u \\
v \\
w \\
x \\
z \\
y \\
z
\end{array}
\end{array}
\quad \Gamma
\begin{array}{c}
\vdots \\
x \\
\end{array}
\]

Notice that \(N(v) = \{y, z\}\) and \(N(u) = \{x, \bar{x}\}\).

All morphisms from \(X\) to \(\Gamma\) have to map the vertices \(v\) and \(w\) to \(u\).

Let \(f, g : X \to \Gamma\) be the morphisms given by

\[
\begin{align*}
f_* : & \quad y \mapsto x, \quad z \mapsto x \\
g_* : & \quad y \mapsto x, \quad z \mapsto \bar{x}
\end{align*}
\]

Then only \(g_*\) is a covering of \(\Gamma\).

Definition 2.7. Let \(\Gamma\) be a graph. The category of coverings of \(\Gamma\), denoted \(\text{Cov}(\Gamma)\), is the category whose objects are the coverings \(f : X \to \Gamma\) and where a morphism from \(f : X \to \Gamma\) to \(g : Y \to \Gamma\) is a graph morphism \(\rho : X \to Y\) such that \(g \circ \rho = f\).
2.1 Properties of coverings

**Proposition 2.8** (path lifting). Let \( f: X \to \Gamma \) be a covering of \( \Gamma \). Let \( u \) and \( v \) be vertices of \( \Gamma \) and let \( \gamma \) be a path from \( u \) to \( v \). Let \( \tilde{u} \in f^{-1}(u) \) be a vertex of \( X \). Then there exists a unique vertex \( \tilde{v} \in f^{-1}(v) \) and a unique path \( \tilde{\gamma} \) from \( \tilde{u} \) to \( \tilde{v} \) such that \( f \circ \tilde{\gamma} = \gamma \). Such a path is called the lift of \( \gamma \) in \( X \) with initial vertex \( \tilde{u} \).

**Proof.** We will prove it by induction on the length \( n \) of the path.

Let \( n = 0 \). Then \( \gamma = 1_\bar{u} \) is the trivial path based at \( u \). So we have \( \tilde{v} = \tilde{u} \) and the unique lift is the trivial path \( 1_{\bar{u}} \).

Assume \( n \geq 1 \). Let \( \gamma = x_1 \cdots x_{n-1} \cdot x_n = \psi \cdot x_n \). By induction there exists a unique vertex \( \tilde{w} \in f^{-1}(t(\psi)) = f^{-1}(o(x_n)) \) and a unique path \( \tilde{\psi} \) from \( \tilde{u} \) to \( \tilde{w} \) such that \( f \circ \tilde{\psi} = \psi \). Since \( N(o(x_n)) \) is in bijection with \( N(\tilde{w}) \), there exists a unique edge \( \tilde{x}_n \in N(\tilde{w}) \) such that \( f \circ \tilde{x}_n = x_n \). Let \( \tilde{v} := t(\tilde{x}_n) \) and \( \tilde{\gamma} := \tilde{\psi} \cdot \tilde{x}_n \). Then \( \tilde{\gamma} \) is a path from \( \tilde{u} \) to \( \tilde{v} \) such that \( f \circ \tilde{\gamma} = \gamma \) and it is unique by construction.

**Proposition 2.9** (homotopy lifting). Let \( f: X \to \Gamma \) be a covering of \( \Gamma \). Let \( u \) and \( v \) be vertices of \( \Gamma \) and let \( \gamma_1 \) and \( \gamma_2 \) be paths in \( \Gamma \) from \( u \) to \( v \). Let \( \tilde{u} \in f^{-1}(u) \) and let \( \tilde{v} \) be the final vertex of the unique lift \( \tilde{\gamma}_i \) of \( \gamma_i \) in \( X \) with initial vertex \( \tilde{u} \). If \( \gamma_1 \sim \gamma_2 \) then \( \tilde{v}_1 = \tilde{v}_2 \).

**Proof.** It is enough to prove it for two paths that are elementarily equivalent. Let \( \gamma_1 = \alpha \cdot \beta \) and \( \gamma_2 = \alpha \cdot z \cdot \tilde{z} \cdot \beta \). By proposition 2.8 there exists a unique vertex \( \tilde{w} \) in \( X \) and a unique lift \( \tilde{\alpha} \) of \( \alpha \) from \( \tilde{u} \) to \( \tilde{w} \). Similarly there exists a unique lift \( \tilde{z} \) of \( z \) with initial vertex \( \tilde{w} \). Hence we have \( \tilde{\gamma}_1 = \tilde{\alpha} \cdot \tilde{\beta}_1 \) and \( \tilde{\gamma}_2 = \tilde{\alpha} \cdot \tilde{z} \cdot \tilde{z} \cdot \tilde{\beta}_2 \) where \( \tilde{\beta}_1 \) and \( \tilde{\beta}_2 \) are lifts of \( \beta \) with \( o(\tilde{\beta}_1) = \tilde{w} = o(\tilde{z}) = o(\tilde{\beta}_2) \). Hence by uniqueness of the lift \( \tilde{\beta}_1 = \tilde{\beta}_2 \) and \( \tilde{v}_1 = t(\tilde{\beta}_1) = t(\tilde{\beta}_2) = \tilde{v}_2 \).

The assumption that the paths \( \gamma_1 \) and \( \gamma_2 \) are equivalent is essential.

**Example 2.10.** Consider the following covering of graphs

![Diagram showing coverings and paths](image)

Let \( \gamma_1 = xy \) and \( \gamma_2 = y \). Then the lifts with initial vertex \( \tilde{u}_1 \) are \( \tilde{\gamma}_1 = x_1y_2 \) and \( \tilde{\gamma}_2 = y_1 \) and \( t(\tilde{\gamma}_1) = \tilde{v}_2 \neq \tilde{v}_1 = t(\tilde{\gamma}_2) \).
**Definition 2.11.** Let $\Gamma$ be a graph, $u, v \in V(\Gamma)$ be vertices and $f: X \to \Gamma$ be a covering. Let $\gamma \in P(u, v)$ be a path in $\Gamma$. Then for every $\tilde{u} \in f^{-1}(u)$ we define

$$\tilde{u} \cdot \gamma := t(\tilde{\gamma})$$

where $\tilde{\gamma}$ is the unique lift of $\gamma$ in $X$ with initial vertex $\tilde{u}$.

**Lemma 2.12.** Let $f: X \to \Gamma$ be a covering of the graph $\Gamma$. Let $u, v \in V(\Gamma)$ be vertices and let $\gamma \in P(u, v)$ be a path in $\Gamma$. Then the map

$$(\cdot \gamma): f^{-1}(u) \to f^{-1}(v) \quad \text{given by} \quad \tilde{u} \mapsto \tilde{u} \cdot \gamma$$

is a bijection with inverse the map $(\cdot \gamma^{-1})$.

**2.2 From coverings to right $\pi(\Gamma, u)$-sets**

Given a covering $f: X \to \Gamma$ and a vertex $u$ of $\Gamma$, we can now define an action of $\pi(\Gamma, u)$ on the fiber $f^{-1}(u)$.

**Proposition 2.13 (Group action).** Let $f: X \to \Gamma$ be a covering of $\Gamma$ and $u \in V(\Gamma)$ be a vertex. Then the fundamental group $\pi(\Gamma, u)$ acts on the set $f^{-1}(u)$ with the following (right) action:

$$f^{-1}(u) \times \pi(\Gamma, u) \to f^{-1}(u)$$

$$(\tilde{u}, [\gamma]) \mapsto \tilde{u} \cdot [\gamma] := \tilde{u} \cdot \gamma$$

**Proof.** The map is well defined by propositions 2.8 and 2.9. The identity element of $\pi(\Gamma, u)$ acts trivially on $f^{-1}(u)$. Let $[\gamma], [\psi] \in \pi(\Gamma, u)$. Then $(\tilde{u} \cdot [\gamma]) \cdot [\psi] = t(\tilde{\psi})$, where $\tilde{\psi}$ is the unique lift of $\psi$ with $o(\psi) = \tilde{u} \cdot [\gamma]$, and $\tilde{u} \cdot ([\gamma][\psi]) = \tilde{u} \cdot [\gamma \cdot \psi]$. Notice that the unique lift of $\gamma \cdot \psi$ with initial vertex $\tilde{u}$ is the concatenation of the unique lift $\tilde{\gamma}$ of $\gamma$ with initial vertex $\tilde{u}$ and $\tilde{\psi}$, which is the unique lift of $\psi$ with initial vertex $t(\tilde{\gamma}) = \tilde{u} \cdot [\gamma]$. Hence $\tilde{u} \cdot [\gamma \cdot \psi] = t(\tilde{\gamma} \cdot \tilde{\psi}) = t(\tilde{\psi}) = (\tilde{u} \cdot [\gamma]) \cdot [\psi]$. $\square$

**Example 2.14.** Let $f: X \to \Gamma$ be the following covering of graphs

Recall that $\pi(\Gamma, u) = \langle [x] \rangle$, hence its action on $f^{-1}(u)$ is determined by the action of $[x]$. We have:

$$u_1 \cdot [x] = t(x_1) = u_2, \quad u_2 \cdot [x] = t(x_2) = u_3, \quad u_3 \cdot [x] = t(x_3) = u_1,$$

So $[x]$ permutes the vertices of $X$ cyclically.
Example 2.15. Let $f: X \to \Gamma$ be the following covering of graphs

Recall that $\pi(\Gamma, u) = ([x], [y])$, hence its action on $f^{-1}(u)$ is determined by the action of $[x]$ and $[y]$. Notice that

$$u_1 \cdot [x] = u_2, \quad u_2 \cdot [x] = u_1, \quad u_3 \cdot [x] = u_3,$$

$$u_1 \cdot [y] = u_2, \quad u_2 \cdot [y] = u_3, \quad u_3 \cdot [y] = u_1,$$

so we have an $S_3$-action where $[x]$ acts as the 2-cycle $(1, 2)$ and $[y]$ acts as the 3-cycle $(1, 2, 3)$.

Definition 2.16. Let $G$ be a group. The category $\textbf{Set}-G$ is the category whose objects are sets equipped with a right action of $G$, called right $G$-sets, and a morphism between two $G$-sets $A$ and $B$ is a map $h: A \to B$ such that $h(a \cdot g) = h(a) \cdot g$ for every $a \in A$ and $g \in G$.

Proposition 2.17. Let $f: X \to \Gamma$ and $g: Y \to \Gamma$ be coverings. Let $\rho: X \to Y$ be a morphism of coverings and $u$ be a vertex of $\Gamma$. Then

$$\rho_*: f^{-1}(u) \to g^{-1}(u)$$

is a morphism of right $\pi(\Gamma, u)$-sets.

Proof. First notice that $f^{-1}(u)$ and $g^{-1}(u)$ are $\pi(\Gamma, u)$-sets by proposition 2.13 and $\rho$ maps $f^{-1}(u)$ to $g^{-1}(u)$ since $g \circ \rho = f$. So the map $\rho_*$ is well-defined. Let $[\gamma] \in \pi(\Gamma, u)$ and $\tilde{u} \in f^{-1}(u)$. We want to prove that $\rho(\tilde{u} \cdot [\gamma]) = \rho(\tilde{u}) \cdot [\gamma]$.

Let $\tilde{\gamma}$ be the unique lift of $\gamma$ in $X$ with initial vertex $\tilde{u}$. Then $\rho(\tilde{u} \cdot [\gamma]) = \rho(t(\tilde{\gamma})) = t(\rho \circ \tilde{\gamma})$ since $\rho$ is a graph morphism. Notice that $g \circ (\rho \circ \tilde{\gamma}) = f \circ \tilde{\gamma} = \gamma$. So $\rho \circ \tilde{\gamma}$ is a lift of $\gamma$ in $Y$ and $o(\rho \circ \tilde{\gamma}) = \rho(o(\tilde{\gamma})) = \rho(\tilde{u})$. Hence $\rho(\tilde{u}) \cdot [\gamma] = t(\rho \circ \tilde{\gamma}) = \rho(\tilde{u} \cdot [\gamma])$. □
We can now define a functor from the category of coverings of $\Gamma$ to the category of right $\pi(\Gamma, u)$-sets.

**Proposition 2.18.** Let $\Gamma$ be a graph and let $u \in V(\Gamma)$. Then

$$F: \text{Cov}(\Gamma) \to \text{Set}-\pi(\Gamma, u)$$

$$(f: X \to \Gamma) \mapsto f^{-1}(u)$$

$$(\rho: X \to Y) \mapsto (\rho_*: f^{-1}(u) \to g^{-1}(u))$$

is a functor.

In section 2.4 we show that the functor $F$ is essentially surjective and in section 2.5 that if $\Gamma$ is a connected graph then $F$ is an equivalence of categories. To do it we first need to introduce the universal covering of a graph.

### 2.3 The universal covering of a graph

**Definition 2.19** (universal covering). Let $\Gamma$ be a graph and $u$ be a vertex. We define the universal covering of $\Gamma$ based at $u$ as the graph morphism $f: \tilde{\Gamma}_u \to \Gamma$ where $\tilde{\Gamma}_u$ is a graph with sets

$$V(\tilde{\Gamma}_u) := \{\text{reduced paths in } \Gamma \text{ with initial vertex } u\}$$

$$E(\tilde{\Gamma}_u) := \{ (\gamma, x) : \gamma \in V(\tilde{\Gamma}_u) \text{ and } x \text{ is an edge of } \Gamma \text{ with } o(x) = t(\gamma) \}$$

and maps $\text{ep}: (\gamma, x) \mapsto (\gamma, \text{red}(\gamma x))$ and $\overline{(\gamma, x)} = (\text{red}(\gamma x), \bar{x})$, and $f: \tilde{\Gamma}_u \to \Gamma$ is given by $f: V(\tilde{\Gamma}_u) \to V(\Gamma)$, $\gamma \mapsto t(\gamma)$ and $f_*: E(\tilde{\Gamma}_u) \to E(\Gamma)$, $(\gamma, x) \mapsto x$.

**Proposition 2.20.** The universal covering of a graph is a covering.

**Proof.** Note that $\tilde{\Gamma}_u$ is a graph because $\overline{(\gamma, x)} = (\text{red}(\gamma x), \bar{x}) = (\text{red}(\text{red}(\gamma x)\bar{x}), \bar{x})$ and $\text{red}(\text{red}(\gamma x)\bar{x}) = \text{red}(\gamma x\bar{x}) = \text{red}(\gamma) = \gamma$ since $\gamma \in V(\tilde{\Gamma}_u)$, so $\overline{(\gamma, x)} = (\gamma, x)$. Moreover for every edge $(\gamma, x)$ we have

$$(f(o(\gamma, x)), f(t(\gamma, x))) = (f(\gamma), f(\text{red}(\gamma x))) = (t(\gamma), t(x)) = \text{ep}(x) = \text{ep}(f_*(\gamma, x))$$

and $\overline{f_*(\gamma, x)} = \bar{x} = f(\text{red}(\gamma x), \bar{x}) = f(\overline{(\gamma, x)})$ so $f$ is a morphism of graphs. Finally the map $f_*: N(\gamma) \to N(t(\gamma))$, $(\gamma, x) \mapsto x$ is clearly a bijection. \qed

**Example 2.21.** If $\Gamma$ is a graph with one vertex $u$ and no edges then $\tilde{\Gamma}_u = \Gamma$. 

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Example 2.22. Some examples of universal coverings:

- Let $\Gamma$ be the simple loop with vertex $u$. Then $\tilde{\Gamma}_u$ is a regular tree of degree 2:

$$
\begin{array}{c}
\begin{array}{c}
\tilde{\Gamma}_u \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Gamma \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 u \quad x \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 \tilde{x} \bar{x} \bar{x} 1_u x \bar{x} xx \\
\end{array}
\end{array}
\end{array}
$$

- Let $\Gamma$ be the double loop with vertex $u$. Then $\tilde{\Gamma}_u$ is a regular tree of degree 4:

$$
\begin{array}{c}
\begin{array}{c}
\tilde{\Gamma}_u \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Gamma \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 x \quad y \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 u \quad y \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 \tilde{y} \bar{y} \bar{y} yx \bar{y} xy \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
 \tilde{x} \bar{x} \bar{x} 1_u x \bar{x} xx \\
\end{array}
\end{array}
\end{array}
$$

- More generally if $\Gamma$ is the bouquet of $n$ loops then $\tilde{\Gamma}_u$ is a regular tree of degree $2n$.

Remark 2.23. Let $f : \tilde{\Gamma}_u \rightarrow \Gamma$ be the universal covering of $\Gamma$ based at $u$. If $v$ is a vertex of $\Gamma$ such that $P(u, v) = \emptyset$ then $f^{-1}(v) = \emptyset$.

Proposition 2.24. Let $f : \tilde{\Gamma}_u \rightarrow \Gamma$ be the universal covering of $\Gamma$ based at $u$. Then $\tilde{\Gamma}_u$ is a connected graph.

Proof. It is enough to prove that for every non trivial $\gamma \in V(\tilde{\Gamma}_u)$ we have $P(1_u, \gamma) \neq \emptyset$. Let $\gamma = x_1 \cdots x_n$ for some $x_j$ edges of $\Gamma$. Define $e_1 = (1_u, x_1)$ and $e_i = (x_1 \cdots x_{i-1}, x_i)$ for $2 \leq i \leq n$. Notice that all the $e_i$ are edges of $\tilde{\Gamma}_u$ since $\gamma$ is a reduced path of $\Gamma$. Consider $\psi := e_1 \cdots e_n$. It is a path in $\tilde{\Gamma}_u$ and we have $o(\psi) = o(e_1) = 1_u$ and $t(\psi) = t(e_n) = \text{red}(x_1 \cdots x_n) = \gamma$. Hence $\psi \in P(1_u, \gamma)$. $\blacksquare$
Definition 2.25 (Action on a graph). Let $G$ be a group and $\Gamma$ be a graph. A (left) action of $G$ on $\Gamma$ is a (left) action of $G$ on $V(\Gamma)$ and $E(\Gamma)$ such that for every $g \in G$ the map $g : \Gamma \to \Gamma$ given by
\[
g : u \mapsto g \cdot u \text{ for every } u \in V(\Gamma)
\]
given by
\[
g_x : x \mapsto g \cdot x \text{ for every } x \in E(\Gamma)
\]
is a morphism of graphs.

Proposition 2.26. Let $f : \tilde{\Gamma}_u \to \Gamma$ be the universal covering of the graph $\Gamma$ based at $u \in V(\Gamma)$. Then the maps
\[
\pi(\Gamma,u) \times V(\tilde{\Gamma}_u) \to V(\tilde{\Gamma}_u), \quad ([\gamma], \alpha) \mapsto \text{red}(\gamma \alpha)
\]
\[
\pi(\Gamma,u) \times E(\tilde{\Gamma}_u) \to E(\tilde{\Gamma}_u), \quad ([\gamma],(\alpha,x)) \mapsto (\text{red}(\gamma \alpha),x)
\]
define a left action of $\pi(\Gamma,u)$ on $\tilde{\Gamma}_u$.

Proof. Both the maps are well defined. In fact if $[\gamma] = [\psi]$ then $\text{red}(\gamma) = \text{red}(\psi)$ by Lemma 1.17 and $\text{red}(\gamma \alpha) = \text{red}(\text{red}(\gamma) \alpha) = \text{red}(\psi \alpha)$. The identity element $[1_u]$ acts trivially. Moreover for every $\alpha \in V(\tilde{\Gamma}_u)$, every edge $x \in E(\Gamma)$ and every $[\gamma], [\psi] \in \pi(\Gamma,u)$ it is easy to check that $[\gamma][\psi] \cdot \alpha = ([\gamma][\psi]) \cdot \alpha$ and $[\gamma][\psi] \cdot (\alpha,x) = ([\gamma][\psi]) \cdot (\alpha,x)$. So $\pi(\Gamma,u)$ acts on $V(\tilde{\Gamma}_u)$ and $E(\tilde{\Gamma}_u)$.

Now we want to show that every $[\gamma] \in \pi(\Gamma,u)$ induces a graph morphism $\tilde{\Gamma}_u \to \tilde{\Gamma}_u$. Let $(\alpha,x) \in E(\tilde{\Gamma}_u)$ be an edge, then
\[
\text{ep}([\gamma] \cdot (\alpha,x)) = \text{ep}(\text{red}(\gamma \alpha),x) = (\text{red}(\gamma \alpha),\text{red}(\gamma \alpha x)) = ([\gamma] \cdot o(\alpha,x),[\gamma] \cdot t(\alpha,x))
\]
and $[\gamma] \cdot (\alpha,x) = (\text{red}(\gamma \alpha),x) = (\text{red}(\gamma \alpha x),\bar{x}) = [\gamma] \cdot (\text{red}(\alpha x),\bar{x}) = [\gamma] \cdot (\alpha,x)$.

Remark 2.27. Since the universal covering is a covering we have also the usual right action of the fundamental group $\pi(\Gamma,u)$ on the fiber $f^{-1}(u)$. More precisely, let $\alpha \in f^{-1}(u)$ and $[\gamma] \in \pi(\Gamma,u)$. Then $\alpha \cdot [\gamma] = t(\tilde{\gamma})$ where $\tilde{\gamma}$ is the unique lift of $\gamma$ in $\tilde{\Gamma}$ with initial vertex $\alpha$. If $\gamma = x_1 \cdots x_n$ then $\tilde{\gamma} = e_1 \cdots e_n$ where $e_1 = (\alpha,x_1)$ and $e_i = (\text{red}(\alpha x_1 \cdots x_{i-1}),x_i)$. In particular $t(\tilde{\gamma}) = t(e_n) = \text{red}(\alpha x_1 \cdots x_n) = \text{red}(\alpha \gamma)$. So $\alpha \cdot [\gamma] = \text{red}(\alpha \gamma)$.

Definition 2.28 (G-biset). Let $G$ be a group. A $G$-biset is a set $A$ with a left and a right action of $G$ such that for every $a \in A$ and every $g,h \in G$ we have $(ga)h = g(ah)$.

Proposition 2.29. Let $f : \tilde{\Gamma}_u \to \Gamma$ be the universal covering of the graph $\Gamma$ based at $u$. The set $f^{-1}(u)$ is a $\pi(\Gamma,u)$-biset via the left action $[\gamma] \cdot \alpha := \text{red}(\gamma \alpha)$ and the right action $\alpha \cdot [\gamma] := \text{red}(\alpha \gamma)$. Moreover $f^{-1}(u)$ is isomorphic to $\pi(\Gamma,u)$ seen as $\pi(\Gamma,u)$-biset via left and right multiplication.
Proof. The fundamental group \( \pi(\Gamma, u) \) acts on \( f^{-1}(u) \) on the left and on the right by propositions 2.13 and 2.26. If \( \alpha \in f^{-1}(u) \) and \( [\gamma], [\psi] \in \pi(\Gamma, u) \) then

\[
([\gamma]\alpha)[\psi] = \text{red}(\gamma\alpha)[\psi] = \text{red}(\gamma\alpha\psi) = [\gamma] \text{red}(\alpha\psi) = [\gamma](\alpha[\psi]).
\]

So \( f^{-1}(u) \) is a \( \pi(\Gamma, u) \)-biset.

Since \( f^{-1}(u) \) is the set of reduced paths in \( P(u, u) \) we have that the map \( h: f^{-1}(u) \to \pi(\Gamma, u) \) defined by \( \alpha \mapsto [\alpha] \) is a bijection by Lemma 1.17. Moreover for every \( [\gamma] \in \pi(\Gamma, u) \) we have:

\[
h([\gamma] \cdot \alpha) = h(\text{red}(\gamma\alpha)) = [\gamma\alpha] = [\gamma] \cdot [\alpha] = [\gamma] : h(\alpha)
\]

\[
h(\alpha \cdot [\gamma]) = h(\text{red}(\alpha\gamma)) = [\alpha\gamma] = [\alpha] \cdot [\gamma] = h(\alpha) : [\gamma]
\]

so \( h \) is an isomorphism of \( \pi(\Gamma, u) \)-bisets. \qed

2.4 Construction of a covering from a right \( \pi(\Gamma, u) \)-set

In this section we define the quotient graph of a graph \( \Gamma \) under a group action and we use it to construct a covering of \( \Gamma \) starting from a right \( \pi(\Gamma, u) \)-set. Hence we prove that the functor defined in proposition 2.18 is essentially surjective.

Definition 2.30 (Action without inversion). Let \( G \) be a group acting on a graph \( \Gamma \). An inversion is a pair consisting of an element \( g \in G \) and an edge \( x \in E(\Gamma) \) such that \( gx = \bar{x} \). If there is no such pair we say \( G \) acts without inversion on \( \Gamma \).

Lemma 2.31. Let \( G \) be a group that acts without inversion on the graph \( \Gamma \). Then the sets \( G \setminus V(\Gamma) \) and \( G \setminus E(\Gamma) \) of the classes of vertices and edges modulo the (left) action of \( G \) and the maps \( \text{ep}: [x] \mapsto ([o(x)], [t(x)]) \) and \( [x] = [\bar{x}] \) define a graph, denoted \( G \setminus \Gamma \).

Proof. First notice that \( y \in [x] \) if there exists \( g \in G \) such that \( y = gx \). So \( \text{ep}(y) = ([o(y)], [t(y)]) = ([o(gx)], [t(gx)]) = ([g(o(x)]), [g(t(x)]) = ([o(x)], [t(x)]) = \text{ep}(x) \) and \( [y] = [\bar{y}] = [gx] = [\bar{g}\bar{x}] = \bar{x} = \bar{x} = [x] \). Hence the maps are well defined.

Moreover \( [\bar{x}] = [\bar{x}] \). Finally \( \bar{x} = [\bar{x}] \) if and only if there exists \( g \in G \) such that \( \bar{x} = gx \), which is not possible because \( G \) acts without inversion on \( \Gamma \). So \( [\bar{x}] \neq [x] \) for every \( x \in E(\Gamma) \). \qed

Definition 2.32 (Quotient Graph). If \( G \) is a group that acts without inversion on the graph \( \Gamma \), then the graph \( G \setminus \Gamma \) is called the quotient graph of \( \Gamma \) under the (left) action of the group \( G \).

Proposition 2.33. If \( G \) acts without inversion on \( \Gamma \) and freely on \( V(\Gamma) \) then \( \Gamma \to G \setminus \Gamma \) is a covering.
Proof. By definition $\ep([x]) = ([o(x)],[t(x)])$ and $[x] = [\bar{x}]$, so $\Gamma \to G \backslash \Gamma$ is a morphism. Let $u \in V(\Gamma)$ and consider the map $N(u) \to N([u])$. It is trivially surjective. Moreover let $x,y \in N(u)$, so $o(x) = o(y) = u$. If $[x] = [y]$ then there exists $g \in G$ such that $y = gx$, so $u = o(y) = (gx) = g \cdot o(x) = gu$. Since the action is free, it implies $g = 1$. So $x = y$ and the map is injective. \hfill$\square$

**Proposition 2.34.** Let $S$ be a right $G$-set and $A$ be a $G$-biset. Then

1. $S \times A$ is a left $G$-set via the action $g \cdot (s,a) = (s \cdot g^{-1}, g \cdot a)$.

2. The set $S \times_G A := G \backslash (S \times A)$ is a right $G$-set via the action $[s,a] \cdot g = [s,a \cdot g]$.

3. The map $\rho: S \times_G G \to S$ defined by $[s,g] \mapsto s \cdot g$ is an isomorphism of right $G$-sets.

**Proof.** We prove the third statement. The map $\rho$ is well defined. In fact if $[z,h] = [s,g]$ then $(z,h) = (s \cdot k^{-1}, k \cdot g)$ for some $k \in G$. So $\rho([z,h]) = z \cdot h = (s \cdot k^{-1}) \cdot (k \cdot g) = s \cdot g = \rho([s,g])$. Assume $\rho([s,g]) = \rho([z,h])$ for some $[s,g],[z,h] \in S \times_G G$. Then $s \cdot g = z \cdot h$ so $z = s \cdot gh^{-1}$. Hence $[z,h] = [s \cdot gh^{-1}, h] = [s,gh^{-1}h] = [s,g]$. Thus the map $\rho$ is injective. Let $s \in S$. Then $\rho([s,1_G]) = s$ so the map $\rho$ is also surjective, hence a bijection. Finally we have $\rho([s,g] \cdot h) = \rho([s,gh]) = s \cdot gh = (s \cdot g) \cdot h = \rho([s,g]) \cdot h$, so $\rho$ is an isomorphism of right $G$-sets. \hfill$\square$

**Remark 2.35.** Note the analogy with modules: if $R$ is a ring, $M$ is a right $R$-module and $N$ is an $R$-bimodule then the tensor product $M \otimes_R N$ is a right $R$-module and $M \otimes_R R$ is isomorphic to $M$ as right $R$-module.

**Corollary 2.36.** Let $f: \tilde{\Gamma}_u \to \Gamma$ be the universal covering of the graph $\Gamma$ based at $u$ and let $S$ be a right $\pi(\Gamma,u)$-set. Then $S \times f^{-1}(u)$ is a right $\pi(\Gamma,u)$-set isomorphic to $S$.

**Proof.** By proposition 2.29 we have that $f^{-1}(u)$ is isomorphic to $\pi(\Gamma,u)$ as $\pi(\Gamma,u)$-biset, hence we can conclude thanks to proposition 2.34. \hfill$\square$

From now we assume that $f: \tilde{\Gamma}_u \to \Gamma$ is the universal covering of the graph $\Gamma$ based at $u$ and $S$ is a right $\pi(\Gamma,u)$-set.

**Definition 2.37.** We write $S \times \tilde{\Gamma}_u$ for the graph with vertex set $S \times V(\tilde{\Gamma}_u)$, edge set $S \times E(\tilde{\Gamma}_u)$ and maps $\ep: (s,e) \mapsto ((s,o(e)),(s,t(e)))$ and $(s,e) = (s,e)$.

**Remark 2.38.** The graph $S \times \tilde{\Gamma}_u$ is a disjoint union of copies of $\tilde{\Gamma}$ indexed by $S$.

**Proposition 2.39.** The fundamental group $\pi(\Gamma,u)$ acts without inversion on the graph $S \times \tilde{\Gamma}_u$ via the actions

$$
\pi(\Gamma,u) \times (S \times V(\tilde{\Gamma}_u)) \to S \times V(\tilde{\Gamma}_u), \quad ([\gamma],[s,\alpha]) \mapsto (s \cdot [\gamma^{-1}],[\gamma] \cdot \alpha)
$$

$$
\pi(\Gamma,u) \times (S \times E(\tilde{\Gamma}_u)) \to S \times E(\tilde{\Gamma}_u), \quad ([\gamma],[s,(\alpha,x)]) \mapsto (s \cdot [\gamma^{-1}],[\gamma] \cdot (\alpha,x))
$$
Proof. Notice that $V(\tilde{\Gamma}_u)$ and $E(\tilde{\Gamma}_u)$ are left $\pi(\Gamma, u)$-sets by proposition 2.26, so by proposition 2.34 we have that $S \times V(\tilde{\Gamma}_u)$ and $S \times E(\tilde{\Gamma}_u)$ are left $\pi(\Gamma, u)$-sets with the actions described in the statement. Moreover we have an action on the graph $S \times \tilde{\Gamma}_u$ since all the maps are induced by the maps of $\tilde{\Gamma}_u$.

For every $[\gamma] \in \pi(\Gamma, u)$ and every $(s, e) \in S \times E(\tilde{\Gamma}_u)$ we have $[\gamma] \cdot (s, e) = (\bar{s}, \bar{e})$ if and only if $(s \cdot [\gamma^{-1}], [\gamma] \cdot e) = (s, \bar{e})$. However if $e = (\alpha, x)$ then $[\gamma] \cdot e = (\text{red}(\gamma \alpha), x)$ and $\bar{e} = (\text{red}(\alpha x), \bar{x})$, so they cannot be equal since $x \neq \bar{x}$. So $\pi(\Gamma, u)$ acts on $S \times \tilde{\Gamma}_u$ without inversion. \qed

**Definition 2.40.** We denote with $\Gamma_{S,u}$ the quotient graph of $S \times \tilde{\Gamma}_u$ under the (left) action of the group $\pi(\Gamma, u)$.

**Example 2.41.** Let $\Gamma$ be the graph with vertex set $V(\Gamma) = \{u, v\}$ and edge set $E(\Gamma) = \{x, x, y, y\}$ and maps $\text{ep}(x) = (u, v)$ and $\text{ep}(y) = (v, u)$. Let $\gamma = xy \in P(u, u)$. Then $\pi(\Gamma, u) = \langle \langle \gamma \rangle \rangle$. Consider the set $S := \{0, 1\}$ endowed with the right action of $\pi(\Gamma, u)$ given by $0 \cdot [\gamma] = 1$ and $1 \cdot [\gamma] = 0$. We can now construct the graph $\Gamma_{S,u}$. Notice that $[0, \bar{y}] = [1 \cdot [\gamma], \bar{y}] = [1, \text{red}(xy\bar{y})] = [1, x]$ and similarly $[1, \bar{y}] = [0, \bar{x}]$. Hence we have:

\[
\begin{array}{c}
\tilde{\Gamma}_u & \xrightarrow{s} & y \xleftarrow{1_u} x & \xrightarrow{u} & y \\
1_u & \xleftarrow{x} & 1_u & \xrightarrow{u} & 1_u
\end{array}
\]

\[
\begin{array}{c}
\Gamma & \xrightarrow{\pi(\Gamma, u)} & \Gamma_{S,u} \\
\xrightarrow{[0,1_u]} & \xrightarrow{[1,1_u]} & \xrightarrow{[0,1]} & \xrightarrow{[1,1]} & \xrightarrow{[0,1]}
\end{array}
\]

**Definition 2.42.** Let $f : \tilde{\Gamma}_u \to \Gamma$ be the universal covering of the graph $\Gamma$ based at $u \in V(\Gamma)$ and let $S$ be a right $\pi(\Gamma, u)$-set. We define the map $g : \Gamma_{S,u} \to \Gamma$ as

\[
g : \begin{cases}
g : [s, \alpha] \to f(\alpha) = t(\alpha) & \text{for every } [s, \alpha] \in V(\Gamma_{S,u}) \\
g_s : [s, (\alpha, x)] \to f_s(\alpha, x) = x & \text{for every } [s, (\alpha, x)] \in E(\Gamma_{S,u}).
\end{cases}
\]

**Proposition 2.43.** The map $g : \Gamma_{S,u} \to \Gamma$ is a morphism of graphs.

Proof. We only need to show that $g$ is well defined. If it is true, then we can conclude that $g$ is a morphism because all the maps are induced by the maps of the universal covering $f : \tilde{\Gamma}_u \to \Gamma$.

Assume $[s, \alpha] = [z, \beta]$. Then $(z, \beta) = (s \cdot [\gamma^{-1}], [\gamma] \cdot \alpha)$ for some $[\gamma] \in \pi(\Gamma, u)$ so $g([s, \alpha]) = t(\alpha) = t(\text{red}(\gamma \alpha)) = g([z, \beta])$. Similarly $[s, (\alpha, x)] = [z, (\beta, y)]$ if and only if $(z, (\beta, y)) = (s \cdot [\gamma^{-1}], [\gamma] \cdot (\alpha, x)) = (s \cdot [\gamma^{-1}], \text{red}(\gamma \alpha, x))$ for some $[\gamma] \in \pi(\Gamma, u)$. In particular $x = y$ so $g_s([s, (\alpha, x)]) = x = y = g_s([z, (\beta, y)])$. Hence $g$ is well defined. \qed

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Proposition 2.44. The map \( g: \Gamma_{S,u} \to \Gamma \) is a covering of \( \Gamma \).

Proof. Note that every edge in a neighborhood \( N([s, \alpha]) \) is of the form \([s, (\alpha, x)]\); so \( s \) is fixed and \( g_*: N([s, \alpha]) \to N(t(\alpha)) \) is a bijection because it is induced by \( f_*: N(\alpha) \to N(t(\alpha)) \).

\[ \square \]

Theorem 2.45. Let \( \Gamma \) be a graph, \( u \) be a vertex and \( S \) be a right \( \pi(\Gamma, u) \)-set. Let \( f: \Gamma_u \to \Gamma \) be the universal covering of \( \Gamma \) based at \( u \) and \( g: \Gamma_{S,u} \to \Gamma \) be the covering of \( \Gamma \) defined in 2.42. Let \( F: \text{Cov}(\Gamma) \to \text{Set} - \pi(\Gamma, u) \) be the functor defined in proposition 2.18. Then

\[ S \cong F(g: \Gamma_{S,u} \to \Gamma) \]

as right \( \pi(\Gamma, u) \)-sets. In particular the functor \( F \) is essentially surjective.

Proof. By definition \( F(g: \Gamma_{S,u} \to \Gamma) = g^{-1}(u) \). Notice that \( g^{-1}(u) = \{ [s, \alpha] \in V(\Gamma_{S,u}) : f(\alpha) = u \} = S \times_f f^{-1}(u) \) that is a right \( \pi(\Gamma, u) \)-set with the action defined in proposition 2.13. However it is the same right action on \( S \times_f f^{-1}(u) \) defined in proposition 2.34. In fact let \([s, \alpha] \in S \times_f f^{-1}(u) \) and \([\gamma] \in \pi(\Gamma, u)\). With respect to the action defined in proposition 2.13 we have \([s, \alpha] \cdot [\gamma] = t(\gamma) \) where \( \gamma \) is the unique lift of \( \gamma \) in \( \Gamma_{S,u} \) with initial vertex \([s, \alpha]\). If \( \gamma = x_1 \cdots x_n \) then \( \tilde{\gamma} = e_1 \cdots e_n \) where \( e_1 = [s, (\alpha, x_1)] \) and \( e_1 = [s, \text{red}(\alpha x_1 \cdots x_{i-1}, x_i)] \). In particular \( t(\gamma) = t(\gamma) = t(e_n) = [s, \text{red}(\alpha x_1 \cdots x_n)] = [s, \text{red}(\alpha \gamma)] \). So \([s, \alpha] \cdot [\gamma] = [s, \text{red}(\alpha \gamma)] = [s, \alpha \cdot [\gamma]] \) that is the action defined in proposition 2.34.

Finally by corollary 2.36 we have that \( S \times_f f^{-1}(u) \) is isomorphic to \( S \) as right \( G \)-set.

\[ \square \]

If \( \Gamma \) is a finite connected graph, we now have a formula for \( |V(\Gamma_{S,u})| \).

Lemma 2.46. Let \( \Gamma \) be a finite connected graph and \( u \) be a vertex. Then for every transitive finite right \( \pi(\Gamma, u) \)-set \( S \) we have

\[ |V(\Gamma_{S,u})| = |V(\Gamma)||S| \]

Proof. Notice that \( V(\Gamma_{S,u}) = \bigcup_{v \in V(\Gamma)} g^{-1}(v) \) so \( |V(\Gamma_{S,u})| = \sum_{v \in V(\Gamma)} |g^{-1}(v)| \). By lemma 2.12 for every \( v \in V(\Gamma) \) the set \( g^{-1}(v) \) is in bijection with \( g^{-1}(u) \) and by theorem 2.45 we have \( g^{-1}(u) \cong S \). Thus \( |V(\Gamma_{S,u})| = |V(\Gamma)||S| \).

\[ \square \]
2.5 Equivalence of categories for connected graphs

**Theorem 2.47.** Let $\Gamma$ be a connected graph and let $u$ be a vertex. Then the functor 
\[ F: \text{Cov}(\Gamma) \to \text{Set} - \pi(\Gamma, u) \]
defined in proposition 2.18 is an equivalence of categories.

We have to show that $F$ is full and faithful (since we know from theorem 2.45 that it is essentially surjective). In other words we need to prove the following proposition.

**Proposition 2.48.** Let $\Gamma$ be a connected graph and let $f: X \to \Gamma$ and $g: Y \to \Gamma$ be two coverings. Then the map $\text{Hom}(f, g) \to \text{Hom}(f^{-1}(u), g^{-1}(u))$ is a bijection.

To prove it we need a useful lemma.

**Lemma 2.49.** Let $f: X \to \Gamma$ be a covering of the connected graph $\Gamma$. Let $u \in V(\Gamma)$ and $\tilde{v} \in V(X)$. Then there exists a path $\tilde{\gamma}$ in $X$ from $\tilde{v}$ to a vertex in $f^{-1}(u)$.

**Proof.** Let $v = f(\tilde{v})$. Since $\Gamma$ is connected there exists a path $\gamma \in P(v, u)$ in $\Gamma$. By path lifting there exists a unique path $\tilde{\gamma}$ in $X$ with initial vertex $\tilde{v}$ such that $f \circ \tilde{\gamma} = \gamma$. We have $f(t(\tilde{\gamma})) = t(f \circ \tilde{\gamma}) = t(\gamma) = u$ so $t(\tilde{\gamma}) \in f^{-1}(u)$. \hfill \Box

We can now prove proposition 2.48

**Proof.** The map is injective. Let $\rho, \eta \in \text{Hom}(f, g)$ such that $\rho(\tilde{u}) = \eta(\tilde{u})$ for every $\tilde{u} \in f^{-1}(u)$. We want to show that $\rho = \eta$. Let $\tilde{v} \in V(X)$. By lemma 2.49 there exist $\tilde{u} \in f^{-1}(u)$ and a path $\tilde{\gamma} \in P(\tilde{u}, \tilde{v})$. Let $\gamma = f \circ \tilde{\gamma}$. Notice that $\rho \circ \tilde{\gamma}$ and $\eta \circ \tilde{\gamma}$ are lifts of $\gamma$ in $Y$ with the same initial vertex $\rho(\tilde{u}) = \eta(\tilde{u})$, so by uniqueness of the lift they must be equal. In particular $\rho(\tilde{v}) = t(\rho \circ \tilde{\gamma}) = t(\eta \circ \tilde{\gamma}) = \eta(\tilde{v})$, so $\rho$ and $\eta$ agree on $V(\Gamma)$. Let $x \in E(X)$ be an edge. Then $\rho_*(x) = \eta_*(x)$ since they are both lifts of $f_* (x)$ in $Y$ with the same initial vertex (because $\rho$ and $\eta$ agree on vertices). We can conclude that $\rho$ and $\eta$ agree also on $E(X)$, hence they are equal.

The map is surjective. Let $h: f^{-1}(u) \to g^{-1}(u)$ be a $\pi(\Gamma, u)$-morphism. We construct a graph morphism $\rho$ such that $g \circ \rho = f$ and $\rho|_{f^{-1}(u)} = h$.

Let $\tilde{v} \in V(X)$. By lemma 2.49 there exist $\tilde{u} \in f^{-1}(u)$ and a path $\tilde{\gamma} \in P(\tilde{u}, \tilde{v})$. Let $\gamma = f \circ \tilde{\gamma}$. Notice that $o(\gamma) = o(\gamma_i) = u$ so $\gamma \cdot \gamma_i^{-1} \in P(u, u)$. Since $h$ is a $\pi(\Gamma, u)$-morphism we get

\[ h(\tilde{u} \cdot [\gamma \gamma_i^{-1}]) = h(\tilde{u}) \cdot [\gamma \gamma_i^{-1}] \quad (\ast) \]
Note $\tilde{u} \cdot [\gamma \gamma_1^{-1}] = t(\tilde{\gamma} \gamma_1^{-1})$ by uniqueness of the lift, so $h(\tilde{u} \cdot [\gamma \gamma_1^{-1}]) = h(\tilde{u}_1)$. On the other hand $h(\tilde{u}) \cdot [\gamma \gamma_1^{-1}] = t(\psi \psi_2^{-1})$ where $\psi_2^{-1}$ is the unique lift of $\gamma_1^{-1}$ in $Y$ with initial vertex $t(\psi)$. By $(\ast)$ we have $o(\psi_2) = t(\psi_2^{-1}) = h(\tilde{u}_1)$, so by uniqueness of the lift $\psi_2 = \psi_1$ and $t(\psi_1) = o(\psi_2^{-1}) = t(\psi)$.

It remains to define $\rho$ on $E(X)$. Let $x \in E(X)$ with $\text{ep}(x) = (\tilde{v}, \tilde{w})$ and let $y$ be the unique lift of $f_*(x)$ in $Y$ with initial vertex $\rho(\tilde{v})$. We define $\rho_*(x) := y$. Notice that $g_*(\rho_*(x)) = f_*(x)$ and $t(y) = \rho(\tilde{w})$ (by path lift, lemma 2.49 and definition of $\rho$ on vertices).
3 Expander graphs

In this section we define expander graphs and Kazhdan groups in order to generalize Lubotzky’s construction of expander families.

**Definition 3.1.** Let $\Gamma$ be a graph. For every couple $(A,B)$ of subsets of $V(\Gamma)$ we define the set of edges from $A$ to $B$ as the set

$$E(A,B) := \{ x \in E(\Gamma) : o(x) \in A \text{ and } t(x) \in B \}.$$ 

**Remark 3.2.** Note that $E(A,B) \rightarrow E(B,A)$, $x \mapsto \overline{x}$ is a bijection. In particular $|E(A,B)| = |E(B,A)|$.

**Example 3.3.** Consider the following graph

![Graph Image]

If $A = \{u_1, u_2\}$ and $B = V(\Gamma) \setminus A$ then $E(A,B) = \{\overline{x_4}, \overline{x_6}, x_7\}$.

If $A = \{u_1\}$ and $B = V(\Gamma) \setminus A$ then $E(A,B) = \{x_2, \overline{x_3}, \overline{x_4}\} = N(u_1) \setminus \{x_1, \overline{x_1}\}$.

In general for every graph $\Gamma$ and every vertex $u \in V(\Gamma)$ we have

$$E(\{u\}, V(\Gamma) \setminus \{u\}) = N(u) \setminus \{x : o(x) = t(x) = u\}.$$ 

**Definition 3.4.** Let $\Gamma$ be a finite graph. We say that $(A,B)$ is a partition of $V(\Gamma)$ if $V(\Gamma) = A \cup B$ and $A \cap B = \emptyset$.

**Definition 3.5 ((c,k)-expander).** Let $c, k > 0$ be real numbers. A finite graph $\Gamma$ is a $(c,k)$-expander if every vertex has valency at most $k$ and for every partition $(A,B)$ of $V(\Gamma)$ we have

$$|E(A,B)| \geq c \cdot \frac{|A||B|}{|V(\Gamma)|}.$$
Definition 3.6 (Expander family). A family \((\Gamma_i)_{i \in I}\) of finite graphs is an expander family if it satisfies the following properties:

i. \(I\) is infinite

ii. for every \(N \geq 1\) there are only finitely many \(i \in I\) such that \(|V(\Gamma_i)| < N\)

iii. there exist \(c, k > 0\) such that for every \(i \in I\) the graph \(\Gamma_i\) is a \((c, k)\)-expander.

For every finite connected graph \(\Gamma\) with a fundamental group \(\pi(\Gamma, u)\) free on at least two generators we will construct an expander family consisting of covering graphs of \(\Gamma\). We first need to define a special class of groups.

3.1 Kazhdan groups

Definition 3.7. A representation of a topological group \(G\) on a vector space \(V\) over a field \(K\) is a group homomorphism \(\rho: G \to \text{GL}(V)\), where \(\text{GL}(V)\) denotes the general linear group on \(V\).

Definition 3.8. Let \(G\) be a locally compact group, \(H\) be an Hilbert space (over \(\mathbb{C}\)) and \(U(H)\) be the unitary group of \(H\). A unitary representation of \(G\) on \(H\) (over \(\mathbb{C}\)) is a group homomorphism \(\rho: G \to U(H)\) such that \(g \mapsto \rho(g)v\) is a norm continuous function for every \(v \in H\).

Definition 3.9 ((\(K, \varepsilon\))-invariant vector). Let \((H, \rho)\) be a unitary representation of a topological group \(G\) on a Hilbert space \(H\). Let \(K\) be a compact non-empty subset of \(G\) and let \(\varepsilon > 0\). A vector \(v\) of \(H\) is a \((K, \varepsilon)\)-invariant vector if

\[
\sup_{k \in K} \| \rho(k)v - v \| < \varepsilon \| v \|
\]

Remark 3.10. Every \((K, \varepsilon)\)-invariant vector is a non-zero vector and every non-zero invariant vector is a \((K, \varepsilon)\)-invariant vector for every compact \(K \subseteq G\) and \(\varepsilon > 0\).

Definition 3.11 (Kazhdan group). A locally compact group \(G\) is a Kazhdan group if there exist an \(\varepsilon > 0\) and a compact subset \(K\) of \(G\) such that every unitary representation of \(G\) with \((K, \varepsilon)\)-invariant vectors has a non-zero invariant vector. We will refer to \((K, \varepsilon)\) as the couple that satisfies the Kazhdan property.

Example 3.12. The trivial group and the groups \(\text{SL}(n, \mathbb{Z})\) with \(n \geq 3\) are Kazhdan groups. (See [2], Example 3.2.4). However \(\text{SL}(2, \mathbb{Z})\) is not Kazhdan (See [2], propositions 3.1.8 and 3.1.9) and at the end of this section we will show that for every \(n \geq 1\) the free group on \(n\) generators is not Kazhdan either.
Lemma 3.13. Let \( G \) be a finitely generated discrete group and let \( W \subseteq G \) be a symmetric finite generating set. Then for every finite subset \( K \) of \( G \) and every \( \varepsilon > 0 \) there exists an \( \varepsilon' > 0 \) such that for every unitary representation \((H, \rho)\) of \( G \), every \((W, \varepsilon')\)-invariant vector is a \((K, \varepsilon)\)-invariant vector.

Proof. Notice that there exists \( N > 0 \) such that every element of \( K \) can be written in the form \( k = w_1 \cdots w_n \) with \( w_i \in W \) for some \( n < N \). Let \( \varepsilon' := \varepsilon N \). Consider a unitary representation \((H, \rho)\) of \( G \) and a \((W, \varepsilon')\)-invariant vector \( v \). First we want to prove that if \( k = w_1 \cdots w_n \) then \( \| \rho(k)v - v \| < n \varepsilon' \| v \| \).

If \( n = 1 \) then \( \| \rho(w_1)v - v \| < \varepsilon' \| v \| \) since \( v \) is \((W, \varepsilon')\)-invariant.

Assume the statement true for \( n-1 \). Then

\[
\| \rho(k)v - v \| \leq \| \rho(k)v - \rho(w_1 \cdots w_{n-1})v \| + \| \rho(w_1 \cdots w_{n-1})v - v \|
\]

By induction \( \| \rho(w_1 \cdots w_{n-1})v - v \| < (n-1) \varepsilon' \| v \| \). Moreover \( \| \rho(k)v - \rho(w_1 \cdots w_{n-1})v \| = \| \rho(w_1 \cdots w_{n-1})(\rho(w_n)v - v) \| = \| \rho(w_n)v - v \| < \varepsilon' \| v \| \). So \( \| \rho(k)v - v \| < n \varepsilon' \| v \| \).

Hence for every \( k \in K \) we have \( \| \rho(k)v - v \| < N \varepsilon' \| v \| = \varepsilon \| v \| \) and since \( K \) is finite it implies

\[
\sup_{k \in K} \| \rho(k)v - v \| < \varepsilon \| v \| .
\]

Corollary 3.14. Let \( G \) be a finitely generated discrete Kazhdan group and let \( W \subseteq G \) be a finite generating set. Then there exists an \( \varepsilon > 0 \) such that every unitary representation \((H, \rho)\) of \( G \) that has \((W, \varepsilon)\)-invariant vectors has a non-zero invariant vector.

Example 3.15. We can use the previous corollary to show that for every \( n \geq 1 \) the free group on \( n \) generators, denoted \( F_n \), is not Kazhdan. Let \( W = \{w_1, \ldots, w_n\} \) be a generating set for \( F_n \). Fix \( \varepsilon > 0 \) and take \( m \in \mathbb{Z} \) such that \( \| e^{2\pi i/m} - 1 \| < \varepsilon \).

Consider the unitary representation \( \rho: F_n \to \mathbb{C}^* \) on the one dimensional Hilbert space \( \mathbb{C} \) given by

\[
\rho: w_1 \mapsto e^{2\pi i/m}, \quad w_j \mapsto 1 \quad \text{if } 2 \leq j \leq n
\]

The unique invariant vector for \( \rho \) is the zero vector and \( 1 \) is a \((W, \varepsilon)\)-invariant vector. Hence \( F_n \) is not a Kazhdan group.
3.2 Construction of expander families

Let $\Gamma$ be a finite connected graph and $u$ be a vertex. For every transitive finite right $\pi(\Gamma, u)$-set $S$ we will write $g: \Gamma_S \to \Gamma$ to denote the covering defined in 2.42.

We prove the following result.

**Theorem 3.16 (Main theorem).** Let $\Gamma$ be a finite connected graph. Let $u$ be a vertex and let $h : \pi(\Gamma, u) \to G$ be a surjective morphism from $\pi(\Gamma, u)$ onto a discrete Kazhdan group $G$. Then there exist $c, k > 0$ such that for every transitive finite right $G$-set $S$ the covering $\Gamma_S$ of $\Gamma$ is a $(c, k)$-expander.

**Corollary 3.17.** Let $\Gamma$ be a finite connected graph with fundamental group $\pi(\Gamma, u)$ free on at least two generators. Then there is an expander family of coverings of $\Gamma$.

**Proof.** (Corollary 3.2) We want to apply theorem 3.16 with $G = \text{SL}(3, \mathbb{Z})$. Note that $\text{SL}(3, \mathbb{Z})$ is generated by two matrices (Appendix), hence we can find a surjection from the fundamental group $\pi(\Gamma, u)$ onto $\text{SL}(3, \mathbb{Z})$. Since $\text{SL}(3, \mathbb{Z})$ is Kazhdan, by theorem 3.16 there exist $c, k > 0$ such that for every transitive finite right $\text{SL}(3, \mathbb{Z})$-set $S$ the covering $\Gamma_S$ of $\Gamma$ is a $(c, k)$-expander.

For every prime number $p$ the group $\text{SL}(3, \mathbb{Z})$ acts on the right on the finite group $\text{SL}(3, \mathbb{F}_p)$ by right multiplication modulo $p$. Since the reduction modulo $p$ is a surjective morphism from $\text{SL}(3, \mathbb{Z})$ onto $\text{SL}(3, \mathbb{F}_p)$ (Appendix), this action is transitive. Therefore the family of coverings $\{\Gamma_{\text{SL}(3, \mathbb{F}_p)}\}_p$ is an expander family. □

**Definition 3.18.** Let $g : \Gamma_S \to \Gamma$ be as before. If $(A, B)$ is a partition of $V(\Gamma_S)$ we define $A_v := A \cap g^{-1}(v)$ and $B_v = B \cap g^{-1}(v)$ for every vertex $v \in V(\Gamma)$.

The remaining part of this section is dedicated to the proof of Theorem 3.16.

The first step is the following theorem.

**Theorem 3.19.** Let $\Gamma$ be a finite connected graph, $u$ be a vertex and $h : \pi(\Gamma, u) \to G$ be a surjective morphism from $\pi(\Gamma, u)$ onto a discrete Kazhdan group $G$. Then there exists an $\varepsilon > 0$ such that for every transitive finite right $G$-set $S$ and for every partition $(A, B)$ of $V(\Gamma_S)$ we have

$$|E(A, B)| \geq \varepsilon \frac{|A_u||B_u|}{|S|}$$

**Proof.** By theorem 1.30 we have that $\pi(\Gamma, u)$ is a finitely generated group and we can find a symmetric finite generating set $\Psi$ whose elements are of the form $[\gamma] \in \pi(\Gamma, u)$ with $\gamma = x_1 \cdots x_n$ such that $x_i \neq x_j$ if $i \neq j$.

Notice that $W := h(\Psi) \subseteq G$ is a symmetric finite generating set for $G$. Since $G$ is Kazhdan, by corollary 3.14 there exists a $\delta > 0$ such that for every unitary
representation \((H, \rho)\) of \(G\) either there is a non-zero invariant vector or every vector \(v\) of \(H\) satisfies \(\|\rho(w)v - v\| \geq \delta \|v\|\) for some \(w \in W\).

Consider the Hilbert space

\[ H := \{ f : g^{-1}(u) \to \mathbb{C} \mid \sum_{\tilde{v} \in g^{-1}(u)} f(\tilde{v}) = 0 \} \]

with norm \(\|f\|^2 := \sum_{\tilde{v} \in g^{-1}(u)} |f(\tilde{v})|^2\).

Via the surjection \(h\) every right \(G\)-set \(S\) is a right \(\pi(\Gamma, u)\)-set and by theorem 2.45 the sets \(g^{-1}(u)\) and \(S\) are isomorphic as transitive right \(\pi(\Gamma, u)\)-sets and therefore as \(G\)-sets. For every \(s \in G\) we have \(\sum_{\tilde{v} \in g^{-1}(u)} f(\tilde{v}) = \sum_{\tilde{v} \in g^{-1}(u)} f(\tilde{v}s)\).

Hence \(H\) is a left \(G\)-set with the action given by \((s \cdot f)(\tilde{v}) := f(\tilde{v}s)\) and we can consider the unitary representation \(\rho : G \to U(H)\) given by \(\rho(s)(f) = s \cdot f\).

A vector \(f\) in \(H\) is invariant if and only if \(f(\tilde{v}) = f(\tilde{v}s)\) for every \(s \in G\) and every \(\tilde{v} \in g^{-1}(u)\). Since the action of \(G\) on \(g^{-1}(u)\) is transitive we can conclude that the invariant vectors are the constant functions. In particular the zero vector is the unique invariant vector in \(H\). Hence for every non-zero \(f \in H\) we have

\[ \|w \cdot f - f\| \geq \delta \|f\| \quad \text{for some } w \in W. \]

Let \((A, B)\) be a partition of \(V(\Gamma_S)\). Let \(a := |A \cap g^{-1}(u)| = |A_u|\) and \(b := |B_u|\).

Consider the function \(f : g^{-1}(u) \to \mathbb{C}\) defined as

\[ f(\tilde{v}) := \begin{cases} b & \text{if } \tilde{v} \in A_u \\ -a & \text{if } \tilde{v} \in B_u \end{cases} \]

Clearly \(f \in H\) and \(\|f\|^2 = |S|ab\).

For every \([\gamma] \in \pi(\Gamma, u)\) let \(L_\gamma(A, B)\) be the set of lifts \(\hat{\gamma}\) of \(\gamma\) with initial vertex in \(A\) and final vertex in \(B\). Let \(w \in W\) such that \(\|w \cdot f - f\| \geq \delta \|f\|\).

In particular \(w = h([\gamma])\) for some \(\gamma = x_1 \cdots x_n \in \Psi\). Notice that every vertex \(\tilde{v} \in g^{-1}(u)\) is the initial vertex of a lift \(\hat{\gamma}\) of \(\gamma\) in \(\Gamma_S\) with final vertex \(\tilde{w}\). Hence we have

\[ (w \cdot f - f)(\tilde{v}) := \begin{cases} -a - b & \text{if } \tilde{v} = o(\hat{\gamma}) \text{ for some } \hat{\gamma} \in L_\gamma(A, B) \\ b + a & \text{if } \tilde{v} = o(\hat{\gamma}) \text{ for some } \hat{\gamma} \in L_\gamma(B, A) \\ 0 & \text{otherwise} \end{cases} \]

So \(\|w \cdot f - f\|^2 = (a + b)^2(|L_\gamma(A, B)| + |L_\gamma(B, A)|)\).

We claim that \(|L_\gamma(A, B)| \leq |E(A, B)|\). In fact, let \(\alpha = e_1 \cdots e_n \in L_\gamma(A, B)\). Then at least one of the \(e_i\) is an element of \(E(A, B)\). Moreover let \(\beta = d_1 \cdots d_n \in L_\gamma(A, B)\) such that \(\alpha \neq \beta\) and assume \(e_i, d_j \in E(A, B)\). We want to show that
Let \( e_i \neq d_j \). Note that for every \( i \) the edges \( e_i \) and \( d_i \) are lifts of \( x_i \). If \( i \neq j \) then \( x_i \neq x_j \) so \( e_i \neq d_j \). Assume \( i = j \) and \( e_i = d_i \). Notice that \( t(e_{i-1}) = t(d_{i-1}) \) hence by uniqueness of the lift we have \( e_{i-1} = d_{i-1} \). We can repeat the same argument and we have \( e_j = d_j \) for every \( j \leq i \). In particular \( e_1 = d_1 \) so \( o(\alpha) = o(\beta) \) and by uniqueness of the lift it implies \( \alpha = \beta \), contradiction.

Therefore we have \(|L_\gamma(A, B)| \leq |E(A, B)| \) and similarly \(|L_\gamma(B, A)| \leq |E(B, A)| = |E(A, B)| \). Thus

\[
|E(A, B)| \geq \frac{\|w \cdot f - f\|^2}{2(a+b)^2} \geq \frac{\delta^2 \|f\|^2}{2|S|^2} = \frac{\delta^2 ab}{2|S|}.
\]

We can now conclude taking \( \varepsilon := \frac{\delta^2}{2} \).

**Corollary 3.20.** Let \( \Gamma \) be the bouquet of \( k \) loops. Let \( h: \pi(\Gamma, u) \rightarrow G \) be a surjective morphism from \( \pi(\Gamma, u) \) onto a discrete Kazhdan group \( G \). Then there exists \( \varepsilon > 0 \) such that for every transitive finite right \( G \)-set \( S \) the covering \( \Gamma_S \) of \( \Gamma \) is a \((\varepsilon, 2k)\)-expander.

**Proof.** Note that for every transitive finite right \( G \)-set \( S \) we have \(|V(\Gamma_S)| = |S| \) and we conclude by theorem 3.19.

**Definition 3.21 (Cayley graph).** Let \( G \) be a group, \( W \) be a set and \( f: W \rightarrow G \) be a map. The Cayley graph \( C(G, W) \) of \( G \) with respect to \( W \) is the graph with set of vertices \( V(C(G, W)) := G \), set of edges \( E(C(G, W)) := \{(g, w, s) \mid g \in G, w \in W, s = \pm 1\} \) and maps \( ep(g, w, s) = (g, g \cdot f(w)^s) \) and \( (g, w, s) = (g \cdot f(w)^s, w, -s) \).

**Remark 3.22.** If \( N \triangleleft G \) is a normal subgroup of \( G \) we can also define the Cayley graph of \( G/N \) with respect to \( W \) and we get a graph with \(|G : N| \) vertices that is regular of degree \( 2|W| \).

The proof of theorem 3.19 is essentially the same given by Lubotzky in [2] to prove the following result.

**Theorem 3.23.** (Lubotzky, Theorem 3.3.1) Let \( G \) be a finitely generated Kazhdan group and \( W \) be a finite generating set for \( G \). Then there exist \( c, k > 0 \) such that for every normal subgroup \( N \triangleleft G \) of finite index the Cayley graph \( C(G/N, W) \) is a \((c, k)\)-expander.

Theorem 3.19 implies Lubotzky's theorem as follows. Let \( G \) be a finitely generated Kazhdan group and \( W \) be a finite generating set for \( G \). Consider the bouquet \( \Gamma \) of \(|W| \) loops with vertex \( u \), which is a regular graph of degree \( 2|W| \) and whose fundamental group \( \pi(\Gamma, u) \) is a free group on \(|W|\) generators. We can construct a surjection from \( \pi(\Gamma, u) \) onto the Kazhdan group \( G \) and we have that every graph \( C(G/N, W) \) corresponds to the covering \( \Gamma_{G/N} \) of \( \Gamma \). Hence by theorem 3.19 there
exists \( \varepsilon > 0 \) such that for every normal subgroup \( N \trianglelefteq G \) of finite index and for every partition \((A, B)\) of \( V(C(G/N, W)) = G/N \) we have

\[
|E(A, B)| \geq \varepsilon \frac{|A_u||B_v|}{|G/N|} = \varepsilon \frac{|A||B|}{|V(C(G/N, W))|}
\]

So for every normal subgroup \( N \trianglelefteq G \) of finite index the Cayley graph \( C(G/N, W) \) is a \((\varepsilon, 2|W|)\)-expander.

Also, note that in theorem 3.19 we consider arbitrary finite right \( G \)-sets and not just \( G \)-sets of the form \( G/N \) for some \( N \trianglelefteq G \).

To prove theorem 3.16 we still need to do some work, since we want to consider also graphs with more than one vertex and the proof of Lubotzky does not immediately apply.

**Corollary 3.24.** Let \( \Gamma \) be a finite connected graph, \( u \) be a vertex and \( h: \pi(\Gamma, u) \to G \) be a surjective morphism from \( \pi(\Gamma, u) \) onto a discrete Kazhdan group \( G \). Then there exists an \( \varepsilon > 0 \) such that for every transitive finite right \( G \)-set \( S \), for every partition \((A, B)\) of \( V(\Gamma_S) \) and for every \( v \in V(\Gamma) \) we have

\[
|E(A, B)| \geq \varepsilon \frac{|A_u||B_v|}{|S|}
\]

**Proof.** For every \( v \in V(\Gamma) \) let \( \gamma_v \in P(u, v) \) be a path in \( \Gamma \). Notice that \( \gamma_v \) induces an isomorphism between \( \pi(\Gamma, u) \) and \( \pi(\Gamma, v) \) by lemma 1.32 and a bijection between \( g^{-1}(u) \) and \( g^{-1}(v) \) by lemma 2.12. Hence we can define a surjective morphism from \( \pi(\Gamma, v) \) onto \( G \) and a right action of \( G \) on \( g^{-1}(v) \) isomorphic to the action of \( G \) on \( S \). Moreover the action of \( \pi(\Gamma, v) \) on \( g^{-1}(v) \) induced by the two bijections is the ordinary action of the fundamental group based at \( v \) on the fiber above \( v \). So we can repeat the same argument done for the vertex \( u \) and for every \( v \in V(\Gamma) \) we find an \( \varepsilon_v > 0 \) such that for every partition \((A, B)\) of \( V(\Gamma_S) \) we have

\[
|E(A, B)| \geq \varepsilon_v \frac{|A_u||B_v|}{|S|}
\]

Since \( V(\Gamma) \) is finite we can conclude taking \( \varepsilon := \min_{v \in V(\Gamma)} \{ \varepsilon_v \} \). \( \square \)

We now finish the proof of theorem 3.16.

**Proof.** By corollary 3.24 there exists \( \varepsilon > 0 \) such that for every transitive finite right \( G \)-set \( S \), every vertex \( v \in V(\Gamma) \) and every partition \((A, B)\) of \( V(\Gamma_S) \) we have

\[
|E(A, B)| \geq \varepsilon \frac{|A_u||B_b|}{|S|}
\]

Let \( m := |V(\Gamma)| \). Hence by lemma 2.46 we have

\[
|V(\Gamma_S)| = m|S|
\]

Define

\[
c := \min \left\{ \frac{\varepsilon}{8m}, \frac{1}{\sqrt{2m}} \right\}
\]

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We shall prove that for every transitive finite right $G$-set $S$ and every partition $(A, B)$ of $V(\Gamma_S)$ we have

$$|E(A, B)| \geq c \frac{|A||B|}{|V(\Gamma_S)|}$$

Consider a transitive finite right $G$-set $S$ and a partition $(A, B)$ of $V(\Gamma_S)$.

Case 1. Assume that either for every $v \in V(\Gamma)$ we have $|A_v| \geq |B_v|$ or for every $v \in V(\Gamma)$ we have $|B_v| \geq |A_v|$. Without loss of generality we can assume that the first inequality holds. Let $w \in V(\Gamma)$ such that $|B_w| \geq |B_v|$ for every $v \in V(\Gamma)$ and let $\sigma_v := |B_w| - |B_v| \geq 0$. For every $v \in V(\Gamma)$ we get $|B_v| = |B_w| - \sigma_v$ and $|A_v| = |A_w| + \sigma_v$. Note that $|A| = \sum_{v \in V(\Gamma)} |A_v|$ and $|B| = \sum_{v \in V(\Gamma)} |B_v|$. Then

$$|A||B| = m^2|A_w||B_w| + m(|B_w| - |A_w|) \sum_v \sigma_v - \left( \sum_v \sigma_v \right)^2$$

Notice that $(|B_w| - |A_w|) \leq 0$ and $\sum_v \sigma_v \geq 0$, so $|A||B| \leq m^2|A_w||B_w|$. Hence

$$|E(A, B)| \geq \varepsilon \frac{|A_w||B_w|}{|S|} \geq \varepsilon \frac{|A||B|}{m |S|} = \varepsilon \frac{|A||B|}{m |V(\Gamma_S)|} \geq c \frac{|A||B|}{|V(\Gamma_S)|}$$

Case 2. Assume there exists $v \in V(\Gamma)$ such that $\frac{|A_v||B_v|}{|S|} \geq \frac{1}{8m} \frac{|A||B|}{|V(\Gamma_S)|}$. Then we find

$$|E(A, B)| \geq \varepsilon \frac{|A_v||B_v|}{|S|} \geq \varepsilon \frac{|A||B|}{8m |V(\Gamma_S)|} \geq c \frac{|A||B|}{|V(\Gamma_S)|}$$

Case 3. Assume that for every $v \in V(\Gamma)$ we have

$$\frac{|A_v||B_v|}{|S|} < \frac{1}{8m |V(\Gamma_S)|}$$

and that there exist $v, w \in V(\Gamma)$ such that $|A_v| \geq |B_v|$ and $|A_w| < |B_w|$. Since $\Gamma$ is connected we can assume that $(v, w) = ep(x)$ for some edge $x \in E(\Gamma)$. Let $|A_v| = \frac{|S|}{2} + \delta_v$ and $|B_v| = \frac{|S|}{2} - \delta_v$ for some $0 \leq \delta_v \leq \frac{|S|}{2}$. Notice that

$$\frac{|A_v||B_v|}{|S|} = \frac{|S|^2}{4|S|} - \frac{\delta_v^2}{|S|}.$$ 

By (1) we get $\delta_v^2 > \frac{|S|^2}{4} - \frac{|A||B|}{8m |S|} > \frac{1}{4} \left( |S|^2 - \frac{|V(\Gamma_S)|^2}{2m} \right) = \frac{|S|^2}{8}$. So $\delta_v > \frac{|S|}{2\sqrt{2}}$. Hence we have $|B_v| = \frac{|S|}{2} - \delta_v < \frac{|S|}{2} - \frac{|S|}{2\sqrt{2}}$.

Similarly we have $|A_w| = \frac{|S|}{2} - \delta_w$ for some $0 < \delta_w \leq \frac{|S|}{2}$ and using (1) we get $|A_w| < \frac{|S|}{2} - \frac{|S|}{2\sqrt{2}}$.

Let $E_x(A, B)$ be the set of edges of $\Gamma_S$ in $E(A, B)$ above $x$. Then

$$|E(A, B)| \geq |E_x(A, B)| \geq |S| - |B_v| - |A_w| \geq \frac{|S|}{\sqrt{2}} = \frac{|V(\Gamma_S)|}{\sqrt{2m}} \geq c \frac{|A||B|}{|V(\Gamma_S)|}$$

$\square$
Example 3.25. Let $\Gamma$ be the simple loop with vertex $u$. Then its fundamental group is isomorphic to $\mathbb{Z}$, which is not Kazhdan, and its only Kazhdan quotient is the trivial group. In particular we cannot apply corollary 3.2 to construct an expander family of coverings and in fact we show that such a family does not exist.

Note that the vertex $u$ has valency 2, so every finite connected covering of $\Gamma$ must be of the following form

\[ u \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots \rightarrow u_N \rightarrow u \]

where $N = |V(X)|$.

Let $n = \left\lfloor \frac{N}{2} \right\rfloor$ and let $A = \{u_1, \ldots, u_n\}$ and $B = \{u_{n+1}, \ldots, u_N\}$. Then $(A, B)$ is a partition of $V(X)$ and $E(A, B) = \{x_n, \bar{x}_N\}$. Assume $X$ is a $(c, 2)$-expander for some $c > 0$. Then we must have

\[ 2 \geq c \frac{|A||B|}{N} = c \frac{n(N - n)}{N} = cn \left(1 - \frac{n}{N}\right) \geq \frac{c}{2}. \]

Thus

\[ c \leq \frac{4}{n} \leq \frac{8}{N - 1}. \]

So it is not possible to find $c > 0$ for which there exists an infinite collection of finite coverings of $\Gamma$ that are $(c, 2)$-expanders and whose number of vertices is arbitrarily large.
Appendix: Some properties of \( \text{SL}(3, \mathbb{Z}) \)

**Definition.** Let \( n \geq 1 \) be an integer and let \( R \) be a commutative ring. For every \( 1 \leq i, j \leq n \) we denote with \( e_{i,j} \) the \( n \times n \) matrix whose entries are all equal to 0 except for the \((i,j)\) entry which is equal to 1. We call the matrices \( e_{i,j}(a) := I_n + ae_{i,j} \in \text{SL}(n, R) \) for some \( a \in R \) and \( i \neq j \) the \( n \times n \) elementary matrices over \( R \).

We denote with \( E(n, R) \) the subgroup of \( \text{SL}(n, R) \) generated by the \( n \times n \) elementary matrices over \( R \).

We have the following result

**Theorem.** If \( R \) is a commutative euclidean domain then \( \text{SL}(n, R) = E(n, R) \) for all \( n > 1 \).

**Proof.** See [3], pages 193–197.

**Lemma.** The group \( \text{SL}(3, \mathbb{Z}) \) is generated by the two matrices

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

**Proof.** By the previous theorem it suffices to show that \( A \) and \( B \) generate the elementary matrices over \( \mathbb{Z} \). In fact for every \( n \in \mathbb{Z} \) we have

\[
e_{1,2}(n) = A^n \quad e_{3,1}(n) = BA^nB^2 \quad e_{2,3}(n) = B^2A^nB \\
e_{1,3}(n) = e_{2,3}(-1)A^n e_{2,3}(1)A^{-n} \quad e_{2,1}(n) = B^2e_{1,3}(n)B \quad e_{3,2}(n) = Be_{1,3}(n)B^2
\]

**Lemma.** For every prime number \( p \) the reduction modulo \( p \) gives a surjective morphism from the group \( \text{SL}(3, \mathbb{Z}) \) onto the finite group \( \text{SL}(3, \mathbb{F}_p) \).

**Proof.** By the previous theorem for every prime number \( p \) the group \( \text{SL}(3, \mathbb{F}_p) \) is generated by the elementary matrices \( e_{i,j}(\bar{a}) \) with \( a \in \mathbb{Z} \) and \( 0 \leq a < p \), which are the images under the reduction modulo \( p \) of the matrices \( e_{i,j}(a) \in \text{SL}(3, \mathbb{Z}) \).
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