MASTER ALGANT

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Metric and arithmetic properties of a new class of continued fraction expansions

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Contents

Introduction

The first systematic approach to the study of continued fractions is dated at the end of the seventeenth century with the work of the english mathematician John Wallis and the Dutch mathematician and astronomer Christiaan Huygens.
Later, the theory grew with the work of Leonhard Euler, Johann Lambert and Joseph Louis Lagrange. Much of the modern theory of continued fraction is based on what Euler developed in his work of 1737, *De Fractionibous Continuis*.
In particular, due to Euler, we know the result that numbers with a infinite periodic continued fraction expansion are irrational solutions of a quadratic equation. Later, Lagrange proved the converse implication, establishing the equivalence between periodic continued fractions and quadratic irrational numbers.
Since the beginning of the twentieth century, continued fractions have become more common, finding application in solving Diophantine equations, like the Pell equation, and in computing rational approximations to real numbers.
It is a classical result, the idea was already present in Huygens' work, that the convergents of the regular continued fraction expansion of a real number x provide an approximation of x that is 'the best possible'. In this work we will not devote our attention to diophantine approximation. An introduction to diophantine approximation and continued fractions can be found in [HW].

In recent years, much work has been done on iterates of maps of the unit interval into itself. The two main concerns in this area were: the ergodic properties of the dynamical systems underlying the maps, and theorems which guarantee the existence of an invariant measure, equivalent to the Lebesgue measure.
Although many theorems about the existence of invariant measures have been proved, the problem of finding the explicit density turned out to be extremely hard. It is quite exceptional the result of Gauss [Gau76], dated in the nineteenth century, that the invariant density for the regular continued fractions is given by the formula $1/(\log 2) \cdot 1/(1+x)$.

At the end of the nineteenth century, a new kind of semi-regular continued fraction expansion was introduced by Minnigerode [Min73], the continued fraction to the *nearest integer*. In 1981, Nakada generalized in [N81] the nearest integer continued fractions with the Nakada α-expansions.
One of the major results in [N81], was that for $\alpha \in [1/2, 1]$, Nakada was able to find the

natural extension of the ergodic system underlying these α-expansions.
Successively, this result has been extended to the cases with $\alpha \in [\sqrt{2}-1, 1/2]$ by Moussa, Cassa and Marmi in [MCM], using a 'folded' version of the Nakada's α-maps.
In [K91], Kraaikamp obtained the same result as Nakada, but in a completely different way, using the singularization procedure and defining a new family of continued fraction expansions, the S-expansions, which have as a special case, among others, the α-continued fractions of Nakada.
As an indication of the difficulty of writing the explicit invariant density for a continued fraction map, note that for $\alpha \in (0, \sqrt{2}-1)$, nothing is known yet about the density of the α-invariant measure, except its existence.

In Chapter 1 of this thesis we introduce a new family of semi-regular continued fraction expansion, that we call D-expansion, obtained by iteration on the unit interval of the maps T_D. The D-map T_D is obtained from the regular continued fraction map T by flipping the latter, on a particular subset D of $[0, 1]$, around the horizontal line $y = \frac{1}{2}$.
We then give the algorithm that, under specific conditions of the set D, shows how the map T_D acts on the digits of a continued fraction expansions as a singularization or an insertion, and allows us to derive the D-expansion of a real number from its regular expansion via these two procedures. With this tool, it is also very easy to obtain as a special case of the D-expansions many well-known continued fractions, like the odd- or even-continued fractions [HK02], the backward continued fractions [AF84], the folded α-continued fractions [MCM], or a continued fraction expansions without one (or more) particular digits.
At the end of the chapter, we introduce some notions about periodicity of continued fractions, and we extend to D-expansions the theorem that states the equivalence between numbers with an (infinite) periodic continued fraction expansion and quadratic irrationals. Finally, we extend the definition of Hurwitzian numbers, and with a counter-example, we show that, differently than in the α-case or in the backward case, D-hurwitzian numbers do not coincide, for a general D, with regular Hurwitzian numbers.

In Chapter 2, we deal with metric and ergodic properties of the T-maps. First, we briefly introduce the notions and the tools we need in the chapter. Using the same ideas as in [K91], we extend to the case $\alpha \in [\sqrt{2}-1, 1/2)$ the result obtained by Kraaikamp in [K91] for $\alpha \in [1/2, 1]$, i.e. we derive the natural extensions and the invariant densities of the Nakada α-expansions when $\alpha \in [\sqrt{2}-1, 1/2)$. This case is however more complicated as the one for $\alpha \in [\sqrt{2}-1, 1/2)$, since here we have to deal with both singularizations and insertions.
After that, we study the maps T_D. From the work of Thaler [Th80, Th83], it follows that the T_D maps admit an ergodic invariant measure equivalent to the Lebesgue measure, which is finite if D does not contain a neighborhood of 1, and σ-finite, infinite in case D contains a neighborhood of 1. We then give an example with some particular sets D's.
As for the explicit formula of the invariant density, in view of the close link between α-expansions and D-expansions, we expect it to be extremely difficult to find the explicit invariant D-density. Recall, in fact, that nothing is known yet about the invariant density

for the Nakada's α-continued fractions with $\alpha \in (0, \sqrt{2}-1)$.
This difficulty seems to be confirmed by some computer simulations that we present at the end of the chapter, for particular choices of the set D.

Conclusion

This thesis is written with the aim of defining a family of maps which includes, as special cases, other continued fraction maps studied in literature. For this family then, we extended some results which hold in the other cases.
As an addition, we compute the natural extensions of the Nakada's α-continued fractions, with $\alpha = [\sqrt{2}-1, 1/2)$, via singularizations and insertions only.

Acknowledgements

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Grazie ai miei amici; to Lenny, Pantelis, the tango dancers and my 'adoptive family', Marta, Gianfranco, Eleonora and Clara, that lit up my dutch days; ed a Sandro e Alice, perché ci sono.
Infine grazie alla mia famiglia. Perché a loro devo tutto, e tutto questo é per loro.

Chapter 1

Arithmetic properties

1.1 Introduction

Consider a real number x. Such a number admits a continued fraction expansion of the form:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}}, \tag{1.1}$$

where $a_0 \in \mathbb{Z}$ is such that $x - a_0 \in [0,1)$, i.e. $a_0 = \lfloor x \rfloor$, and $a_i \in \mathbb{N}$ for $n \geq 1$. We call an expression like (1.1) the *regular continued fraction* (RCF) expansion of x.
It follows from Euclid's algorithm that x has finitely many partial quotients if and only if $x \in \mathbb{Q}$. This means that Expression (1.1) will be finite if x is rational, and infinite if x is irrational. Moreover, it is unique if and only if x is irrational.
The *partial quotients* a_i are given by

$$a_n = a_n(x) := \left\lfloor \frac{1}{T^{n-1}(x)} \right\rfloor, \qquad \text{if } T^{n-1}(x) \neq 0,$$

where $T : [0,1) \to [0,1)$ is the continued fraction map (or Gauss' map) defined by

$$T(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \qquad \text{if } x \neq 0,$$

and $T(0) = 0$; see Figure 1.1.

Given the partial quotients a_i, define the nth *regular continued fraction convergent of* x as the following rational number:

$$w_n = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = [a_0; a_1, \ldots, a_n]. \tag{1.2}$$

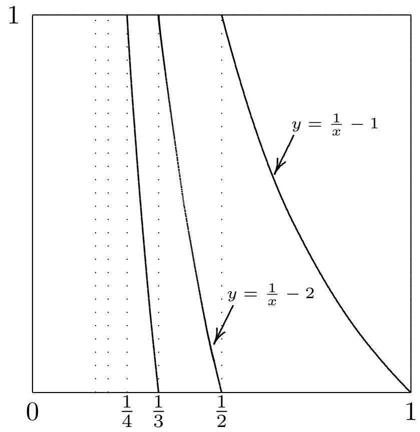

Figure 1.1: Continued fraction map.

Example 1.1. Let $x = \sqrt{2} - 1$. Then

$$\frac{1}{x} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1$$

hence

$$a_1 = \left\lfloor \frac{1}{x} \right\rfloor = 2 \qquad \text{and} \qquad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \sqrt{2} + 1 - 2 = x.$$

Thus x has the following expansion

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \dots}}$$

which shows in particular that x is an irrational number.

We state the two following well-known results ([DK00], Subsection 1.3.2).

Proposition 1.2. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$, then $\lim_{n\to\infty} w_n = x$, where w_n is obtained by finite truncation, cf. (1.2).*
Moreover, set $w_n = p_n/q_n$ with $\gcd(p_n, q_n) = 1$ and $q_n > 0$, the coefficients p_n and q_n satisfy the following recurrence relations

$$\begin{array}{lll} p_{-1} = 1; & p_0 = 0; & p_n = a_n p_{n-1} + p_{n-2}, \quad n \geq 1, \\ q_{-1} = 0; & q_0 = 1; & q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 1. \end{array} \tag{1.3}$$

Note that this proposition allows us to write

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_n + \cdots}}} = [a_0; a_1, \dots, a_n, \dots]. \tag{1.4}$$

Proposition 1.3. *Let $(a_n)_{n\geq 0}$ be a sequence of positive integers, and let the sequence of rational numbers $(w_n)_{n\geq 1)}$ be given by*

$$w_n := [a_0; a_1, \ldots, a_n], \quad n \geq 1.$$

Then there exists an irrational number x for which

$$\lim_{n\to\infty} w_n = x$$

and we moreover have that

$$x = [a_0; a_1, \ldots, a_n, \ldots].$$

In Section 1.2 a proof of the recurrence relations (1.3) is given.

Finally, from the definition of the partial quotients a_n, $n \geq 1$, of $x \in [0, 1)$ it follows that:

$$a_n = a \Longleftrightarrow t_{n-1} = T^{n-1}(x) \in \left(\frac{1}{a+1}, \frac{1}{a}\right]. \tag{1.5}$$

The interval $\left(\frac{1}{a+1}, \frac{1}{a}\right]$ is called a *fundamental interval.*

1.2 D-continued fractions

In the previous section we only considered fractions with positive numerators. Allowing the numerators of each fraction to be negative, we get a generalisation of the regular continued fraction expansion, a so-called *semi-regular* continued fraction expansion.
A *semi-regular continued fraction*, *(SRCF)*, is a finite or infinite fraction

$$a_0 + \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cfrac{\varepsilon_3}{\ddots}}} = [a_0; \varepsilon_1/a_1, \varepsilon_2/a_2, \ldots], \tag{1.6}$$

with $\varepsilon_n = \pm 1$, $a_0 \in \mathbb{Z}$, $a_n \in \mathbb{N}$, $n \geq 1$, subject to the condition

$$\varepsilon_{n+1} + a_n \geq 1, \text{ for } n \geq 1,$$

and with the restriction that in the infinite case

$$\varepsilon_{n+1} + a_n \geq 2 \text{ infinitely often.}$$

Let us now define a new family of semi-regular continued fraction maps that starts from the classical RFC-map and modifies it on a subset of $[0, 1]$.
In this particular definition, we shift the signs in the notation of semi-regular continued fractions, writing instead

$$a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cfrac{\varepsilon_2}{\ddots}}},$$

that we will abbreviate with

$$[a_0; \varepsilon_0/a_1, \varepsilon_1/a_2, \ldots]. \tag{1.7}$$

1.2.1 D-continued fractions

Fix a measurable subset $D \subseteq [0,1]$. The *D-continued fraction map corresponding to D*, $T_D : [0,1) \to [0,1)$ (or simply T, if this does not lead to ambiguities), is defined by: $T(0) = 0$, and, when $x \neq 0$,

$$T_D(x) := \begin{cases} 1 - \dfrac{1}{x} + \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \in D, \\ \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \in [0,1] \setminus D. \end{cases}$$

We then extend the map to the whole interval $[0,1]$ by setting $T_D(1) = 1$.
Note that we require D to be measurable for further observations about the metric properties of T_D, that we will investigate in Chapter 2. All the definitions and results in this chapter hold for a general subset D of $[0,1]$.

Let x be a real number and, as in the regular case, let a_0 be such that $x - a_0 \in [0,1]$. Set $\varepsilon_0 = 1$ and

$$t_0 = x - a_0,\ t_1 = T_D(x - a_0),\ t_2 = T_D(t_1), \dots.$$

The *D-continued fraction expansion* of x is the semi-regular continued fraction expansion with partial quotients $a_n = a_n(x) := a(t_{n-1}(x)),\ n \geq 1$, defined by

$$a(x) := \begin{cases} \left\lfloor \dfrac{1}{x} \right\rfloor + 1 & \text{if } x \in D, \\ \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \in [0,1] \setminus D. \end{cases}$$

and with signs $\varepsilon_n = \varepsilon_n(x) := \varepsilon(T_D^{n-1}(x))$, $n \geq 1$, given by

$$\varepsilon(x) := \begin{cases} -1 & \text{if } x \in D, \\ 1 & \text{if } x \in [0,1] \setminus D. \end{cases}$$

With these definitions we have

$$T_D(x) = \varepsilon(x) \left(\frac{1}{x} - a(x) \right),$$

yielding

$$x = \frac{1}{a_1 + \varepsilon_1 T_D(x)} = \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{a_2 + \cdots + \cfrac{\varepsilon_{n-1}}{a_n + \varepsilon_n T_D^n(x)}}} = [0; 1/a_1, \varepsilon_1/a_2, \varepsilon_2/a_3, \dots].$$

Note that in the D-case we automatically get the condition $a_n + \varepsilon_n \geq 1$ for $n \geq 1$, because if $a_n = 1$ for some n, then $T^{n-1}(x) \notin D$, thus $\varepsilon_n = 1$.
We give now some examples of continued fraction expansions already studied in literature, which can be obtain as a special case of D-expansion, for a suitable choice of D.

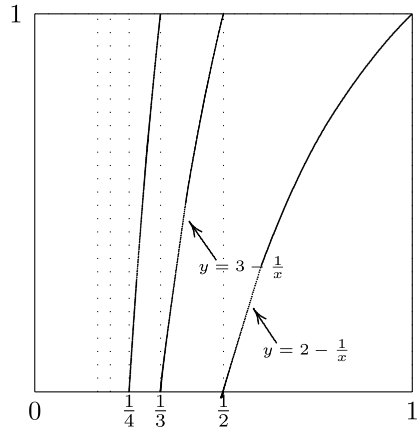

Figure 1.2: D-Backward continued fraction map.

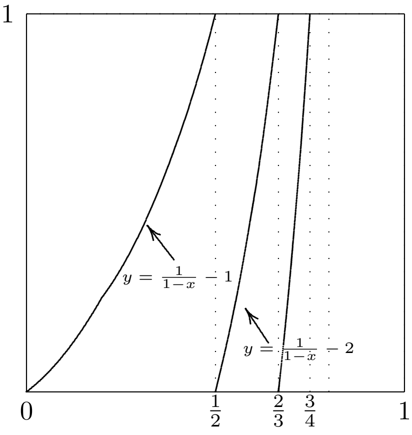

Figure 1.3: Backward continued fraction map.

Examples 1.4. *1.* (*Backward continued fractions*). Take D as the entire interval $[0,1]$, and let $x \in [0,1]$. In this case we always use the modified map $T_D = 1 + \lfloor 1/x \rfloor - 1/x$, so we will get an expansion for x of the form:

$$x = \cfrac{1}{a_1 + \cfrac{-1}{a_2 + \cfrac{-1}{a_3 + \cfrac{-1}{\ddots}}}}.$$

It is a classical result that every $x \in [0,1]$, $x \notin \mathbb{Q}$, can be expanded uniquely through its *backward* CF expansion (cf. [AF84]),

$$x = 1 - \cfrac{1}{c_1 - \cfrac{1}{c_2 - \dots}} = [0; -1/c_1, -1/c_2, \dots],$$

where the $c_i's$ are all integers greater than 1. This continued fraction is generated by the map

$$T_b(x) = \frac{1}{1-x} - \left\lfloor \frac{1}{1-x} \right\rfloor,$$

that we obtain from T_D by factoring through the map $\psi : x \mapsto 1-x$, i.e., $\psi T_b = T_D \psi$. Figure 1.2 represents the D-continued fraction map corresponding to such a D, while Figure 1.3 represents the classic backward continued fraction map.

2. (*Continued fraction expansion without a particular digit*). D continued fraction makes it also possible to eliminate very easily a particular digit from the expansion of $x \in [0,1]$.

Fix a positive integer l, and suppose that we want an expansion in which the digit l never appears, that is $a_n \neq l$ for all $n \geq 1$.
In view of Remark (1.5), we can just take $D = (\frac{1}{l+1}, \frac{1}{l}]$ in order to get an expansion with no digits equal to l. Figure 1.4 represents the map which yields the expansion without 2.

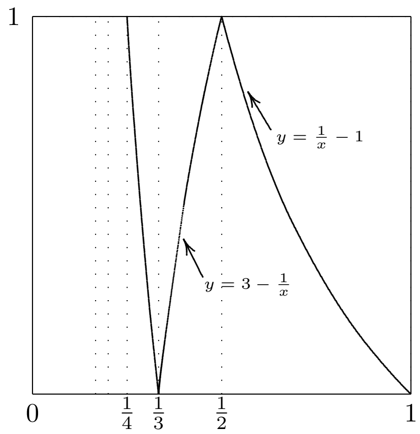

Figure 1.4: D-fraction map corresponding to $D = (1/3, 1/2]$.

3. (*Odd and even continued fractions*). In Example (1.8) of the next section, we introduce odd (resp. even) continued fractions, i.e. semi-regular continued fraction where only odd (resp. even) digits are allowed, and we show how these can be derived from a regular expansion through two algebraic procedure called singularization and insertion. In Section 1.3.1 it is then proved how the D-map works in practice as an insertion or a singularization. This means, taking a suitable set D, we can describe odd (resp. even) fractions as D-expansions. Looking at the previous example, a suitable set D for the odd case is

$$D_{\text{odd}} = \bigcup_{n \text{ even}} \left[\frac{1}{n+1}, \frac{1}{n}\right).$$

Similarly in the even case, with $D_{\text{even}} = [0,1] \setminus D_{odd}$.

4. (*Nakada's α-continued fractions*). In 1981, H.Nakada generalized in [N81] the RCF-expansion to a new class of continued fraction expansions, which is now well known as *Nakada's α-expansions.*
The generating continued fraction transformation T_α, for $\alpha \in [1/2, 1]$, is given by

$$T_\alpha = \left|\frac{1}{x}\right| - \left[\left|\frac{1}{x}\right| + 1 - \alpha\right] \qquad \text{for } \alpha - 1 < x < \alpha, \quad x \neq 0,$$

and $T_\alpha(0) = 0$.
In 1997, Marmi, Moussa and Yoccoz generalized in [MMY] the α-expansions to the

folded or *Japanese* continued fractions, with generating function

$$\widetilde{T_\alpha} = \left|\frac{1}{x} - \left[\frac{1}{x}\right]_\alpha\right|, \qquad \text{for } \alpha - 1 < x < \alpha, \quad x \neq 0; \qquad \widetilde{T_\alpha}(0) = 0,$$

where $[x]_\alpha = \min\{p \in \mathbb{Z} \ : \ x < \alpha + p\}$ and where now α may vary in the whole interval $[0, 1]$. We have that, for the values of α for which T_α is defined, $\widetilde{T_\alpha} = |T_\alpha|$. It is shown in Section 1.4 that folded continued fractions can also be described as D-expansions with $D = \bigcup_n \left(\frac{1}{n}, \frac{1}{n-1+\alpha}\right]$.

1.2.2 Elementary properties

We can describe the D-continued fraction via matrices as follows. Let $A \in SL_2(\mathbb{Z})$ and consider the Moebius transformation induced by A, that is the fractional linear transformation $A : \mathbb{C}^* \longrightarrow \mathbb{C}^*$,

$$A: \ z \mapsto \ A(z) = \begin{pmatrix} r & p \\ s & q \end{pmatrix}(z) = \frac{rz + p}{sz + q}.$$

Let $a_1, a_2, \dots$ be the sequence of partial quotients of x and $\varepsilon_1, \varepsilon_2, \dots$ be the sequence of signs. Define the matrices:

$$A_n = \begin{pmatrix} 0 & \varepsilon_{n-1} \\ 1 & a_n \end{pmatrix}, \quad n \geq 1, \ \varepsilon_0 := 1,$$

and

$$M_n = A_1 \cdots A_n = \begin{pmatrix} r_n & p_n \\ s_n & q_n \end{pmatrix}, \quad n \geq 1, \qquad M_0 = \mathbb{1}_2.$$

From $M_n = M_{n-1}A_n$, $n \geq 2$, it follows that

$$\begin{pmatrix} r_n & p_n \\ s_n & q_n \end{pmatrix} = \begin{pmatrix} r_{n-1} & p_{n-1} \\ s_{n-1} & q_{n-1} \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_{n-1} \\ 1 & a_n \end{pmatrix},$$

therefore $r_n = p_{n-1}$, $s_n = q_{n-1}$ and we find the recurrence relations:

$$\begin{array}{lll} p_{-1} := 1; & p_0 := 0; & p_n = a_n p_{n-1} + \varepsilon_{n-1} p_{n-2}, \quad n \geq 1, \\ q_{-1} := 0; & q_0 := 1; & q_n = a_n q_{n-1} + \varepsilon_{n-1} q_{n-2}, \quad n \geq 1. \end{array} \tag{1.8}$$

In the case $D = \emptyset$, we are back in the regular case, so $\varepsilon_n = 1$ for all $n \geq 0$ and all x, and the recurrence relations are given by (1.3).
Define finally $t_n = T_D^n(x)$ and $A_n^* = \left(\begin{smallmatrix} 0 & \varepsilon_{n-1} \\ 1 & a_n + \varepsilon_n t_n \end{smallmatrix}\right)$. Using (1.8) we find:

$$x = M_{n-1} A_n^*(0) = \frac{p_n + p_{n-1} t_n \varepsilon_n}{q_n + q_{n-1} t_n \varepsilon_n}. \tag{1.9}$$

Setting

$$w_n = M_n(0) = \frac{p_n}{q_n},$$

it follows, from $\det M_n = (-1)^n \prod_{k=1}^{n-1} \varepsilon_k$, that

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n \prod_{k=1}^{n-1} \varepsilon_k = \pm 1, \quad n \geq 1 \tag{1.10}$$

thus

$$\gcd(p_n, q_n) = 1, \quad n \geq 1$$

and in particular $p_n \neq 0 \neq q_n$ for $n \geq 1$.
The sequence $(\omega_n)_n$ is obtained from the expansion of x by finite truncation. As in the regular case, it is possible to prove that $(\omega_n)_n$ converges to x. We will prove this result in the following section.

1.3 Singularization and insertion

Singularization and insertion are two algebraic procedures that work via manipulation of the partial quotients. They are both very classical (cf. [P50]), but they have been used a lot lately (see for example [HK02], [DK00] or [S04]). We will use them to built a link between D-and regular continued fractions.

Let a, b be positive integers, $\varepsilon = \pm 1$ and $\xi \in [0,1]$. A *singularization* is based on the identity

$$a + \cfrac{\varepsilon}{1 + \cfrac{1}{b + \xi}} = a + \varepsilon + \frac{-\varepsilon}{b + 1 + \xi}. \tag{1.11}$$

To see the effect of a singularization on a continued fraction expansion, let $x \in [0,1]$ with expansion (1.7), and suppose that for some $n \geq 0$ one has

$$a_{n+1} = 1; \ \varepsilon_{n+1} = 1, \ a_n + \varepsilon_n \neq 0$$

i.e.,

$$[0; \varepsilon_0/a_1, \ldots, a_n, \ldots] = [a_0; \varepsilon_0/a_1, \ldots, \varepsilon_{n-1}/a_n, \varepsilon_n/1, 1/a_{n+2}, \ldots]. \tag{1.12}$$

Singularization then changes (1.7) into

$$[a_0; \varepsilon_0/a_1, \ldots, \varepsilon_{n-1}/(a_n + \varepsilon_n), -\varepsilon_n/(a_{n+2} + 1), \ldots]. \tag{1.13}$$

As we mentioned in Proposition (1.2), finite truncation yields the sequence of convergents $(p_k/q_k)_{k \geq -1}$. Let $(r_k/s_k)_{k \geq -1}$ be the sequence of convergents relative to (1.7). It was shown in [K91] that the sequence of vectors $(p_k/q_k)_{k \geq -1}$ relative to (1.13) is obtained from $(r_k/s_k)_{k \geq -1}$ by removing the term (r_n/s_n) from the latter, i.e.,

$$(p_k/q_k)_{k \geq -1} = (r_{-1}/s_{-1}), (r_0/s_0), \ldots, (r_{n-1}/s_{n-1}), (r_{n+1}/s_{n+1}), \ldots.$$

Example 1.5. Let e be the base of the natural logarithm. Its continued fraction expansion is

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \dots, 1, 2k, 1, \dots], \qquad k \geq 1,$$

where the first digits correspond to the integer part of e.
Singularizing the first 1 of every block $\dots, 1, 2k, 1, \dots$ we get:

$$e = [3; -1/3, 1/2, -1/5, 1/2, -1/7, 1/2, \dots, -1/(2k+1), 1/2, \dots], \qquad k \geq 1.$$

An operation that can be considered as an inverse of singularization is the so-called *insertion*. An insertion is based upon the identity

$$a + \frac{1}{b+\xi} = a + 1 + \cfrac{-1}{1 + \cfrac{1}{b-1+\xi}}. \tag{1.14}$$

where $\xi \in [0, 1]$ and a, b are positive integers with $b \geq 2$.
Let (1.7) be the expansion of $x \in [0, 1]$ with $(r_k/s_k)_{k\geq -1}$ as sequence of convergents, and suppose that for some $n \geq 0$ one has

$$a_{n+1} > 1; \ \varepsilon_n = 1.$$

An insertion between a_n and a_{n+1} will change (1.7) into

$$[a_0; \varepsilon_0/a_1, \dots, \varepsilon_{n-1}/(a_n + 1), -1/1, 1/(a_{n+1} - 1), \dots].$$

Let $(p_k/q_k)_{k\geq -1}$ be the convergents relative to this last expansion. Using some matrix-identities it was shown in [K91] that the sequence $(p_k/q_k)_{k\geq -1}$ is obtained from $(r_k/s_k)_{k\geq -1}$ by inserting the term $(r_n + r_{n-1}/s_n + s_{n-1})$ before the nth term of the latter sequence, i.e.,

$$(p_k/q_k)_{k\geq -1} = (r_{-1}/s_{-1}), (r_0/s_0), \dots, (r_{n-1}/s_{n-1}), (r_n + r_{n-1}/s_n + s_{n-1}), (r_n, s_n), \dots. \tag{1.15}$$

Remark 1.6. Let x be a real irrational number with RCF-expansion $[a_0; a_1, \dots]$ and sequence of convergents $(p_n/q_n)_{n\geq 0}$. It follows from a theorem by Dirichlet of 1842, that there exists infinitely many pairs of integer p and q, with $q \geq 1$ and $(p, q) = 1$, such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}. \tag{1.16}$$

From a theorem of Borel of 1903, it is possible to deduce that infinitely many of these solutions are regular continued fraction convergents p_n/q_n of x.
Barbolosi and Jager characterized in [BJ94] the rationals p/q that are not convergents but still satisfy (1.16). They showed that all rational pairs satisfying (1.16) belong to the set of *intermediate convergents* of x, defined by

$$\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}}, \qquad 1 \geq k \geq a_{n+1} - 1, \ n \geq 0.$$

Observe finally that if we insert m times between a_n and a_{n+1}, we get the following sequence of convergents for x (using the notation above)

$$\left(\frac{p_k}{q_k}\right)_{k\geq -1} = \frac{r_{-1}}{s_{-1}},\ \frac{r_0}{s_0},\ \dots\ ,\ \frac{r_{n-1}}{s_{n-1}},\ \frac{r_n+r_{n-1}}{s_n+s_{n-1}},\ \frac{2r_n+r_{n-1}}{2s_n+s_{n-1}},\ \dots\ ,\ \frac{mr_n+r_{n-1}}{ms_n+s_{n-1}},\ \frac{r_n}{s_n},\ \dots\ ,$$

where all the new inserted convergents are intermediate convergents.

Example 1.7. Let us consider, as in the previous example, the expansion of e, $e = [2;1,2k,1]$, $k \geq 1$. Inserting $-1/1$ before every digit different from one we get the following equivalent expansion of e

$$e = [2;1/2,-1/1,1/1,1/1,1/2,-1/1,1/3,1/1,\dots,1/2,-1/1,1/(2k-1),1/1,\dots], \quad k \geq 1.$$

As we have already mentioned, if now we singularize each term $-1/1$ in the last expansion, we retrieve the original expansion of e. This is what we meant saying that singularization and insertion can be considered one the inverse of the other.

Every time we insert between a_n and a_{n+1} we decrease a_{n+1} by 1, i.e. the new $(n+2)$th digit equals $a_{n+1}-1$. This implies that for every n we can insert between a_n and a_{n+1} at most $(a_{n+1}-1)$ times.
On the other side, suppose that $a_{n+1}=1$ and that we singularize it. Then both a_n and a_{n+2} will be increased by 1, so we can singularize at most one out of two consecutive digits. As an example we look at the odd and even continued fraction expansion.

Examples 1.8.

Odd continued fractions. Let $x \in [0,1]$. Then x can be written as a finite (if x is rational) or infinite (if x is irrational) continued fraction of the form:

$$x = \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{a_2 + \dots}} = [0;1/a_1,\varepsilon_1/a_2,\varepsilon_2/a_3,\dots] \tag{1.17}$$

where $\varepsilon_n = \pm 1$ and a_n is a positive odd integer for every $n \geq 1$ and $a_n + \varepsilon_n > 1$ for every $n \geq 1$. We call such an expansion *odd continued fraction* (OddCF-) expansion of x.
In [HK02] an algorithm is given which turns the RCF-expansion of any $x \in [0,1]$ into the OddCF-expansion of x. The algorithm follows directly from the definitions of singularization and insertion and works as follows. Let $x \in [0,1]$ with RCF-expansion $x = [0;a_1,a_2,\dots]$. Then:

1. Let $m := \inf\{n \in \mathbb{N} \mid a_n \text{ is even}\}$.

   (i) If $a_{m+1} > 1$, insert via (1.14) after a_m to obtain:

$$[0;\dots,1/a_{m-1},1/(a_m+1),-1/1,1/(a_{m+1}-1),1/a_{m+2},\dots].$$

(ii) If $a_{m+1} = 1$, let $k := \inf\{n > m \;:\; a_n > 1\}$ ($k = \infty$ is allowed) and singularize the first, the third, the fifth, etc. partial quotients equal to 1 in the block of partial quotients

$$a_{m+1} = 1, a_{m+2} = 1, \dots, a_{k-1} = 1.$$

Now, in case $k - m - 1$ is odd or $k = \infty$ we arrive respectively at:

$$[0; \dots, 1/a_{m-1}, 1/(a_m + 1), \underbrace{-1/3, \dots, -1/3}_{\frac{k-m-2}{2}-\text{times}}, -1/(a_k + 1), 1/a_{k+1}, \dots]$$

or

$$[0; \dots, 1/a_{m-1}, 1/(a_m + 1), -1/3, \dots, -1/3, \dots],$$

while in case $k - m - 1$ is even we get:

$$[0; \dots, 1/a_{m-1}, 1/(a_m + 1), \underbrace{-1/3, \dots, -1/3}_{\frac{k-m-3}{2}-\text{times}}, -1/2, -1/a_k, 1/a_{k+1}, \dots],$$

and we have to insert again using (1.14) to arrive at:

$$[0; \dots, 1/a_{m-1}, 1/(a_m + 1), \underbrace{-1/3, \dots, -1/3}_{\frac{k-m-1}{2}-\text{times}}, -1/1, -1/(a_k - 1), 1/a_{k+1}, \dots].$$

2. Let $m \geq 1$ be the first index in the new expansion x obtained in 1. for which the correspondent digit is even. Repeat then the above procedure with the new expansion of x and with this value of m.

Repeating the process in 2. for every m, we obtain the OddCF-expansion of x.

Consider the following example with $x \in [0, 1]$ with RCF-expansion

$$x = [0; 1/1, 1/6, 1/3, (1/1)^4, 1/5, \dots],$$

where $(1/1)^n$ is the abbreviation of $\underbrace{1/1, \dots, 1/1}_{n-times}$ and where we omit the term in case $t = 0$.

(i) Apply the algorithm with $m = 2$. Since $a_3 > 1$, we apply (1.14) after 1/6 to obtain

$$[0; 1/1, 1/7, -1/1, 1/2, (1/1)^4, 1/5, \dots].$$

(ii) Apply the algorithm with $m = 4$ in the new expansion. Since $a_5 = \cdots = a_9 = 1$, we first singularize a_5, a_7, a_9 to arrive at

$$[0; 1/1, 1/7, -1/1, 1/3, -1/3, -1/2, 1/5, \dots]$$

and then insert to eventually get

$$[0; 1/1, 1/7, -1/1, 1/3, -1/3, -1/3, -1/1, 1/4, \dots].$$

(iii) Apply now the algorithm with $m = 8$ and continue until the algorithm terminates.

Even continued fractions. As in the previous case, we give an algorithm that maps a RCF-expansion of a real number x into an expansion where only even digits are allowed. Let $x \in [0,1]$ with RCF-expansion $[0; a_1, \dots, a_n, \dots]$.

1. Let $m := \inf\{n \in \mathbb{N} \mid a_n \text{ is odd}\}$.
   (i) If $a_{m+1} > 1$ then use the first insertion-identity to get
   $$[0; \dots, 1/(a_m+1), (-1/2)^{a_{m+1}-2}, -1/1, 1/1, 1/a_{m+2}, \dots],$$
   where $(-1/2)^t$ is an abbreviation of $\underbrace{-1/2, \dots, -1/2}_{t-times}$ for $t \geq 1$ and where we omit the term in case $t = 0$.
   Singularize now $1/1$ to obtain
   $$[0; \dots, 1/(a_m+1), (-1/2)^{a_{m+1}-1}, -1/(a_{m+2}+1), \dots].$$
   (ii) If $a_{m+1} = 1$, then singularize a_{m+1} to arrive at
   $$[0; \dots, 1/(a_m+1), -1/(a_{m+2}+1), \dots].$$
2. Let $m \geq 1$ be the first index in the new expansion $[b_0; \varepsilon_0/b_1, \dots]$ of x obtained as in 1. for which b_m is odd. Repeat the procedure described in 1. to this new expansion of x with this value of m.

Repeating the process in 2. for every m, we obtain the EvenCF-expansion of x.

As an example, let us pick the same number as before, that is, take $x = [0; 1/1, 1/6, 1/3, (1/1)^4, 1/5,$ Applying the algorithm with first $m = 1$ and then $m = 8$ we arrive at
$$[0; 1/2, (-1/2)^5, -1/4, 1/2, -1/2, 1/1, 1/5, \dots].$$

We can now use $m = 10$ and go on until the algorithm terminates.

As we remarked in Example 1.4.3, OddCF's and EvenCF's can be obtained via singularization and insertion from D-expansions, choosing a suitable set D. Also the D-expansion in Example 1.4.2 can be obtained via insertion and singularization, using the algorithm that we present in the next subsection. Obviously, there must be a strong connection between D-continued fractions on one hand, and singularization and insertion on the other.

1.3.1 The D-algorithm

For $n \in \mathbb{N}$, let $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$, so that the expansion of x looks like
$$x = \cfrac{1}{n + \cfrac{1}{\ddots}},$$

and suppose $\left(\frac{1}{n+1}, \frac{1}{n}\right] \subseteq D$.
Consider the two cases:

(i) Let $x \in \left(\frac{1}{n+1}, \frac{2}{2n+1}\right] \cap D$.
In this case, because of (1.5), the RCF-expansion of x is given by

$$x = \cfrac{1}{n + \cfrac{1}{1 + \cfrac{1}{a + \xi}}},$$

where $\xi \in [0, 1]$ and where the first partial quotient is due to the fact that $x \in (1/(n+1), 1/n]$, and the second one to the fact that $T(x) \in [1/2, 1]$ (see figure 1.5).
Singularizing the second digit, equal to 1, in the previous expansion we find

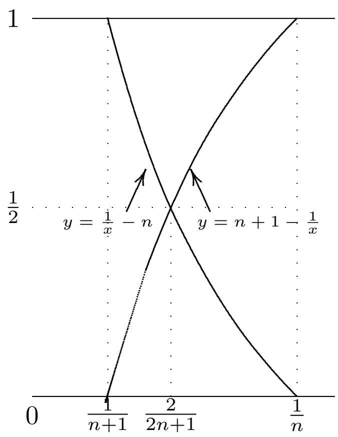

Figure 1.5: D-and regular map in the interval $(\frac{1}{n+1}, \frac{1}{n})]$.

$$x = \cfrac{1}{n + 1 + \cfrac{-1}{a + 1 + \xi}}. \tag{1.18}$$

Let us move now to the D-expansion. Since $x \in (1/(n+1), 1/n] \cap D$, the D-expansion of x will look like

$$x = \cfrac{1}{n + 1 + \cfrac{-1}{\ddots}}.$$

Let us compute $T_D(x)$ to see how to continue. We have

$$T_D(x) = n + 1 - \frac{1}{x} = 1 - T(x) = 1 - \cfrac{1}{1 + \cfrac{1}{a + \xi}} = \frac{1}{a + 1 + \xi},$$

so that the D-expansion of x is

$$x = \cfrac{1}{n+1+\cfrac{-1}{a+1+\xi}}$$

which is equal to (1.18), showing that T_D works as a singularization on $\left(\frac{1}{n+1}, \frac{2}{2n+1}\right] \cap D$.

(ii) Now let $x \in \left(\frac{2}{2n+1}, \frac{1}{n}\right] \cap D$.
In this case the RCF-expansion of x is

$$x = \cfrac{1}{n+\cfrac{1}{c+\xi}},$$

where $\xi \in [0,1]$ and with $c \geq 2$ because $T(x) \leq 1/2$ (see figure 1.5).
An insertion after the first partial quotient yields

$$x = \cfrac{1}{n+1+\cfrac{-1}{1+\cfrac{1}{c-1+\xi}}}. \tag{1.19}$$

Going to the D expansion, since $x \in (1/(n+1), 1/n] \cap D$, the D-expansion of x will look like

$$x = \cfrac{1}{n+1+\cfrac{-1}{\ddots}}.$$

Computing $T_D(x)$ we find

$$T_D(x) = 1 - T(x) = 1 - \frac{1}{c+\xi} = \cfrac{1}{1+\cfrac{1}{c-1+\xi}},$$

so that the D-expansion of x is

$$x = \cfrac{1}{n+1+\cfrac{-1}{1+\cfrac{1}{c-1+\xi}}}$$

that is equal to (1.19), showing that T_D works as a insertion on $\left(\frac{2}{2n+1}, \frac{1}{n}\right] \cap D$.

We have showed that the D-expansion of a real number x can be obtain from its regular expansions via singularization and insertion. Under specific conditions on the set D, we are able to give the algorithm, through which we can get the explicit D-expansion. We present it in the following theorem.

Theorem 1.9. *Let x be a real irrational number with RCF-expansion $x = [a_0; 1/a_1, 1/a_2, \dots]$ and with tails $t_n = T^n(x) = [0; 1/a_{n+1}, 1/a_{n+2}, \dots]$. Let D be a subset of $[0,1]$ of the form $D = \bigcup_{i=0}^{\infty} \left(\frac{1}{n_i+1}, \frac{1}{n_i}\right]$, where n_i is a positive integer for all $i = 0, 1, \dots$. Then the following algorithm yields the D-continued fraction expansion of x.*

1. *Let $m := \inf\{m \in \mathbb{N} \cup \{\infty\} \ : \ t_m \in D \text{ and } \varepsilon_k = 1\}$.*

   (i) If $a_{m+2} = 1$, singularize the digit a_{m+2} in order to get

$$x = [a_0; \dots, 1/(a_{m+1} + 1), -1/a_{m+3}, \dots].$$

   (ii) If $a_{m+2} \neq 1$, insert -1/1 after a_{m+1} to get

$$x = [a_0; \dots, 1/a_{m+1} + 1, -1/1, 1/(a_{m+2} - 1), \dots].$$

2. *Replace the RCF-expansion of x with the continued fraction obtained in 1. and repeat the above procedure.*

Repeating this procedure for every m, we obtain the D-continued fraction expansion of x via insertion and singularizations.

Since the D-expansion of x can be obtained from the RCF-expansion of x via suitable singularizations and insertions, we immediately have the following corollary.

Corollary 1.10. *Let $D \subseteq [0,1]$ be a measurable set and let $x \in [0,1]$. Then the D-continued fraction expansion of x is infinite if and only if x is irrational.*

1.3.2 More about D-expansions

Remark 1.11. Let x be a real number, $(r_n/s_n)_n$ be the convergents relative to its regular expansion. From (1.3) we can easily see that in the regular case the s_i are an increasing sequence. This property, which plays a fundamental role in the proof of Proposition 1.2, no longer holds when we move from the regular case, since we have seen in (1.15) that inserting in the nth position, we might get convergents $(p_k/q_k)_k$ with $q_{n+1} < q_n$. However, it is still possible to prove the convergence of the w_n's, $\omega_n = p_n/q_n$, to x.
Moreover, as we will see in (1.20), the q_i's decrease only after an insertion, or, more precisely, at the end of a chain of insertion, so we can't decrease more than one time in a row. If then we singularize at the end of a chain of insertion, we preserve the property of the q_i's being increasing, as we are going to show.
Let $x \in [0,1]$ with an expansion like (1.7) and with $a_{n+1} > 1$; $\varepsilon_n = 1$. Suppose that we insert m-times between the digit a_n and a_{n+1}, with $m \leq a_{n+1} - 1$, and we singularize at the end. Call $(p_k/q_k)_{k \geq -1}$ the sequence of convergents of this new expansion of x and

$(r_k/s_k)_{k\geq -1}$ the one corresponding to the original expansion. After the chain of insertion the sequence $(p_k/q_k)_{k\geq -1}$ will look like:

$$\left(\frac{p_k}{q_k}\right)_{k\geq -1} = \frac{r_{-1}}{s_{-1}}, \ \dots \ , \frac{r_{n-1}}{s_{n-1}}, \frac{r_n + r_{n-1}}{s_n + s_{n-1}}, \ \dots \ , \frac{mr_n + r_{n-1}}{ms_n + s_{n-1}}, \frac{r_n}{s_n}, \frac{r_{n+1}}{s_{n+1}}, \ \dots . \tag{1.20}$$

With the final singularization we cut off the convergent (r_n/s_n) so that the sequence becomes:

$$\left(\frac{p_k}{q_k}\right)_{k\geq -1} = \frac{r_{-1}}{s_{-1}}, \ \dots \ , \frac{r_{n-1}}{s_{n-1}}, \frac{r_n + r_{n-1}}{s_n + s_{n-1}}, \ \dots \ , \frac{mr_n + r_{n-1}}{ms_n + s_{n-1}}, \frac{r_{n+1}}{s_{n+1}}, \ \dots . \tag{1.21}$$

Now, $s_{n+1} = a_{n+1}s_n + \varepsilon_n s_{n-1} = a_{n+1}s_n + s_{n-1} > ms_n + s_{n+1}$, where in the last inequality we have used that $a_{n+1} - 1 > m$.

Now we give the theorem that proves that we still have convergence.

Theorem 1.12. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$ with a D-expansion as in (1.7), and let $(\omega_n)_n$ be obtained via finite truncation, cf. Section 1.2. Then $\lim_{n\to\infty} \omega_n = x$.*
Moreover, set $\omega_n = p_n/q_n$ with $\gcd(p_n, q_n) = 1$ and $q_n > 0$. Then the coefficient p_n and q_n satisfy the following recurrence relations

$$\begin{array}{lll} p_{-1} = 1; & p_0 = 0; & p_n = a_n p_{n-1} + \varepsilon_{n-1} p_{n-2}, \quad n \geq 1, \\ q_{-1} = 0; & q_0 = 1; & q_n = a_n q_{n-1} + \varepsilon_{n-1} q_{n-2}, \quad n \geq 1. \end{array}$$

Note that it is because of this theorem that the writing $x = [a_0; \varepsilon_0/a_1, \dots, \varepsilon_{n-1}/a_n, \dots]$ makes sense.

Proof. For the recurrence relations, see (1.8). We only need to prove the convergence. From (1.9) and (1.10) we find

$$x - \frac{p_n}{q_n} = \frac{(-1)^n (\prod_{k=1}^n \varepsilon_k) t_n}{q_n(q_n + q_{n-1}\varepsilon_n t_n)},$$

hence

$$\left| x - \frac{p_n}{q_n} \right| = \left| \frac{t_n}{q_n(q_n + q_{n-1}\varepsilon_n t_n)} \right| < \frac{1}{|q_n(q_n + q_{n-1}\varepsilon_n t_n)|}. \tag{1.22}$$

If $\varepsilon_n = 1$ for all n we are back in the regular case. Let us suppose then $\varepsilon_n = -1$ for some $n \geq 1$.
The case $\varepsilon_n = -1$ occurs as a result of either a singularization or an insertion, cf. Section 1.3.1. Let us consider the two cases separately.
If there has been a singularization, we know that the sequence of convergents has the form

$$(p_k/q_k)_{k\geq -1} = (r_{-1}/s_{-1}), \dots, (r_{n-1}/s_{n-1}), (r_{n+1}/s_{n+1}), \dots,$$

where $(r_k/s_k)_{k\geq -1}$ are the convergents relative to the regular expansion of x. We can rewrite the denominator of (1.22) as

$$q_n(q_n+q_{n-1}\varepsilon_n t_n)=s_{n+1}(s_{n+1}-t_n s_{n-1})>s_{n+1}(s_{n+1}-s_{n-1})\geq s_{n+1}s_{n-2}>s_{n-2}^2,$$

where we have used that $\varepsilon_n=-1$, $t_n<1$ and (1.12).
If there has been a (chain of) insertions, then the sequence of convergents is

$$\left(\frac{p_k}{q_k}\right)_{k\geq -1}=\frac{r_{-1}}{s_{-1}},\ \ldots\ ,\ \frac{r_{n-1}}{s_{n-1}},\ \frac{r_n+r_{n-1}}{s_n+s_{n-1}},\ \ldots\ ,\ \frac{mr_n+r_{n-1}}{ms_n+s_{n-1}},\ \frac{r_n}{s_n},\ \frac{r_{n+1}}{s_{n+1}},\ \ldots,$$

where $m\geq 1$ is the length of the chain. Let us set

$$\begin{aligned} q_{n-1}&=(l-1)s_n+s_{n-1}\\ q_n&=ls_n+s_{n-1}, \end{aligned}$$

with $1\leq l\leq m$. Then again we have

$$q_n(q_n+q_{n-1}\varepsilon_n t_n)>q_n(q_n-q_{n-1})=q_n(ls_n-(l-1)s_n)=q_n s_n>s_{n-2}^2.$$

Thus, in both cases we obtain the inequality

$$\left|x-\frac{p_n}{q_n}\right|<\frac{1}{|q_n(q_n+q_{n-1}\varepsilon_n t_n)|}<\frac{1}{s_{n-2}^2}$$

where the last expression converges to 0, since s_n tends to ∞ as $n\to\infty$. □

1.4 Japanese continued fractions

In this section we will recall the definition of others non-regular continued fraction expansions, the Nakada's α-expansions. Later we will investigate the link between the D-continued fraction expansion and the α-expansions of Nakada.

1.4.1 Nakada's α-continued fraction expansions

RCF-expansions and NICF-expansions are a particular case of the *Nadaka's α-expansions*, introduced in 1981 by H. Nakada in [N81]. For $\alpha\in[0,1]$, the α-expansion of a real number is defined as follows.
Define the *modified integer part* of x as

$$[x]_\alpha=\min\{p\in\mathbb{Z}\ :\ x<\alpha+p\}$$

that is

$$[x]_\alpha=p\quad\text{if and only if}\quad\alpha-1+p\leq x<\alpha+p.$$

Note that

$$[x]_\alpha = \lfloor x - \alpha + 1 \rfloor,$$

where $\lfloor \cdot \rfloor = [\cdot]_1$ is the usual integer part of a real number. Define the *modified fractional part* of x as

$$\{x\}_\alpha = \{x - \alpha + 1\}_1 + \alpha - 1,$$

and consider the map $T_\alpha : [\alpha - 1, \alpha) \to [\alpha - 1, \alpha)$ given by:

$$T_\alpha(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor = \left|\frac{1}{x}\right| - \left[\left|\frac{1}{x}\right|\right]_\alpha, \text{ if } x \neq 0; \qquad T_\alpha(0) := 0.$$

Iterations of this map give rise to a semi-regular continued fraction expansion of $x \in [\alpha - 1, \alpha)$ with partial quotients:

$$a_1 = \left[\left|\frac{1}{x}\right|\right]_\alpha; \quad a_n = a_n(x) = a_1(T_\alpha^{n-1}(x)),$$

and signs

$$\varepsilon_n = \varepsilon_n(x) = \mathrm{sgn}(T_\alpha^{n-1}(x)), \quad \text{whenever } n \text{ is such that } T_\alpha^{n-1}(x) \neq 0.$$

This yields

$$x = \frac{\varepsilon_1}{a_1 + T_\alpha^{n-1}} = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cdots + \cfrac{\varepsilon_n}{a_n + T_\alpha^n}}}.$$

Observe that for $\alpha = 1$ we find the regular continued fraction expansion, while for $\alpha = 1/2$ we have the expansion to the nearest integer described above.

1.4.2 D- and folded α-continued fractions

The Nakada's α-expansion can be slightly modified to get expansions which have been sometimes called *japanese continued fractions* and that we will also call *folded* α-expansion. They are so called because originally they were obtained by 'folding' the interval $[\alpha - 1, \alpha]$ to obtain $[0, \alpha^+]$, where $\alpha^+ = \max\{\alpha, 1 - \alpha\}$, so that α^+ always belongs to the interval $[1/2, 1]$. However, we now define them in a slightly more general way, allowing α to vary in all $[0, 1]$.

Let α be a fixed real number such that $0 \leq \alpha \leq 1$. We will consider the iteration of:

$$\widetilde{T_\alpha} : (0, \alpha) \to [0, \alpha] \tag{1.23}$$

$$x \mapsto \left|\frac{1}{x} - \left[\frac{1}{x}\right]_\alpha\right|, \tag{1.24}$$

so that $\widetilde{T_\alpha} = |T_\alpha|$, for the values of α for which T_α is defined.

A more detailed description states that the map $\widetilde{T_\alpha}$ is made of the following branches:

branch k^+: $\widetilde{T}_\alpha(x) = \frac{1}{x} - k$ for $\frac{1}{k+\alpha} < x \le \frac{1}{k}$,

branch k^-: $\widetilde{T}_\alpha(x) = k - \frac{1}{x}$ for $\frac{1}{k} < x \le \frac{1}{k+\alpha-1}$,

as we can see in picture 1.6. Note that, although the branches are defined for number in the all interval $[0,1]$, after at most one iteration we are in the square $[0,\alpha]$.

When $\frac{1}{2} < \alpha \le 1$, the function $\widetilde{T}_\alpha$ maps the interval $[0,\alpha)$ to itself, whereas when

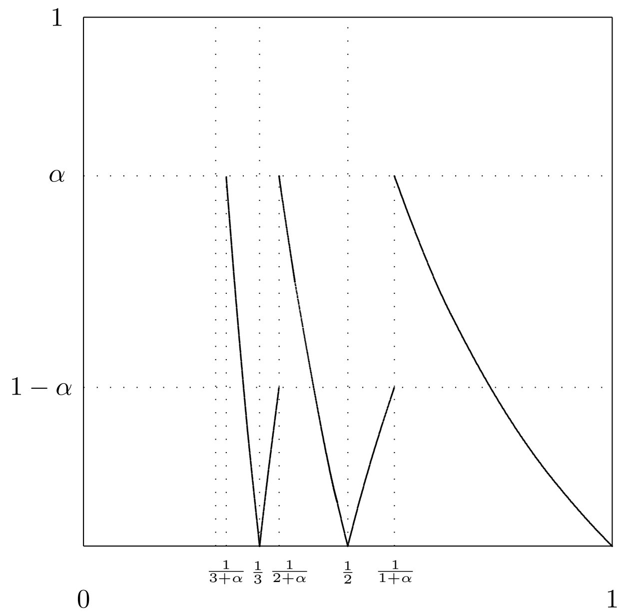

Figure 1.6: "Folded" α-continued fraction map.

$0 < \alpha \le \frac{1}{2}$, it maps the interval $[0, 1-\alpha]$ to itself. In both cases it is convenient to set

$$\widetilde{T}_\alpha(0) := 0,$$

and we get a map which is infinitely differentiable by pieces, with discontinuities accumulating to 0. Moreover, if $\alpha \neq 0$, the map is expanding, which means that its derivative is strictly greater than one everywhere it is defined. This is no longer true for $\alpha = 0$, since in this case the derivative at 1 equals 1.

As we said before, given a real irrational number x, we associate to x a continued fraction expansion by iterating $\widetilde{T}_\alpha$ as follows. Let

$$x_0 = |x - [x]_\alpha|, \qquad a_0 = [x]_\alpha,$$

then one has

$$x = a_0 + \varepsilon_0 x_0, \text{ where } \varepsilon_0 = \begin{cases} +1 & \text{if } x \ge [x]_\alpha, \\ -1 & \text{otherwise.} \end{cases}$$

We now define inductively for all $n \geq 0$,

$$x_{n+1} = \widetilde{T}_\alpha(x_n), \qquad a_{n+1} = \left[\frac{1}{x_{n+1}}\right]_\alpha \geq 1,$$

thus

$$x_n^{-1} = a_{n+1} + \varepsilon_{n+1} x_{n+1},$$

where the sign ε_{n+1} is determined by

$$\varepsilon_{n+1} = \begin{cases} +1 & \text{if } x_n^{-1} \geq a_{n+1}, \\ -1 & \text{otherwise.} \end{cases}$$

Therefore we have

$$x = a_0 + \varepsilon_0 x_0 = a_0 + \frac{\varepsilon_0}{a_1 + \varepsilon_1 x_1} = a_0 + \cfrac{\varepsilon_0}{a_1 + \cfrac{\varepsilon_1}{a_1 + \cdots + \cfrac{\varepsilon_{n-1}}{a_n + \varepsilon_n x_n}}}.$$

Note that for $\alpha = 1$ we find again the standard continued fraction expansion defined through the iteration of the map $T(x) = x^{-1} - \lfloor x^{-1} \rfloor$, and all signs $\varepsilon_n = 1$, while when $\alpha = 1/2$ we have the nearest integer continued fraction.
Comparing the domain of definition of $\widetilde{T}_\alpha$ with the possible domains of the various branches given above, we find that the branch 1^- never occurs for $\alpha \in [0, 1]$, and that the branch 1^+ occurs only of $\alpha > G = (\sqrt{5} - 1)/2$ (the golden mean). In particular, we always have $a_n \geq 2$ whenever $\varepsilon_n = -1$. Moreover, if $\alpha > G$ and $\varepsilon_n = -1$, then $a_{n+1} \geq 3$.
Observe then that if we admit x to be rational, then for $\alpha \neq 0$, the continued fractions that represent any rational number stop at finite order. For $\alpha = 0$ instead, all signs ε_n are equal to -1 and we never have $x_n = 0$, so that the expansion never stops. In this case, the rational numbers are represented by infinite continued fraction expansion with constant tail.

As a final remark we give the set D that yield the link between the japanese continued fractions and the D-continued fractions.
Let $0 < \alpha \leq 1$ and consider the set $D = \bigcup_n \left(\frac{1}{n}, \frac{1}{n-1+\alpha}\right]$. Then the continued fraction expansion of a fixed real number x given by the iteration of the D-map T_D, coincides with the expansion of x via the folded α-map $\widetilde{T}_\alpha$ just described. Note that the first iteration of the map may yield values greater that α, but at most after one iteration, we fall in the interval $[0, \alpha)$.

1.5 Periodicity

We conclude the chapter extending some remarks about periodic continued fractions in the case of D-expansions.

We say that a regular continued fraction is *periodic* or *eventually periodic* if it consists of an initial block of length n followed by a repeating block of length m, i.e., if it is of the form

$$[a_0; a_1, \dots, a_n, \overline{a_{n+1}, \dots, a_{n+m}},]$$

where $[\overline{a_{n+1}, \dots, a_{n+m}}]$ means that $a_{n+1+km} = a_{n+1}, \dots, a_{n+(k+1)m} = a_{n+m}$, for every $k \geq 1$. Moreover, no block of length shorter than m has this property and the initial block does not end with a copy of the repeating block.
If the initial block has length 0, we say that the continued fraction is *purely periodic.*

We recall also the definition of quadratic irrational number:
A number x is called *quadratic irrational* if it is a root of a polynomial $ax^2 + bx + c$ with $a, b, c \in \mathbb{Z}$, $a \neq 0$, and $b^2 - 4ac$ is not a perfect square.

There is the following classical result; see [HW] for a proof.

Theorem 1.13. *A number is a quadratic irrational number if and only if it has an eventually periodic continued fraction expansion.*

Example 1.14 (Golden mean)**.** Consider a rectangle having sides in the ratio, say, $1 : x$ and divide into two parts, a square of side length 1 and another rectangle. The *golden ratio* is the defined as the number x such that, in such a construction, the smaller rectangle has again sides in the ratio $1 : x$. Such a rectangle is called a *golden rectangle* and successive iteration of this construction give rise to a figure known as a *whirling square* (cf. Figure 1.7). Let G be the golden ratio. By abuse of language, we will refer in the text to $g = G^{-1}$

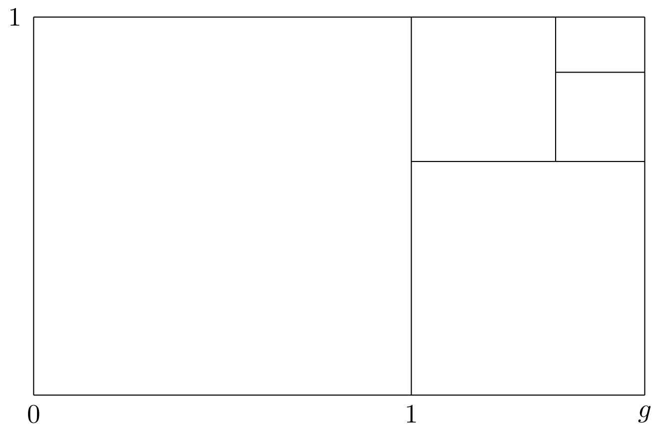

Figure 1.7: Whirling square.

as the *golden mean.* From the above definition we have:

$$G = \frac{G}{1} = \frac{1}{x} = \frac{x}{1-x}.$$

Since $G > 1$ we also have:

$$\frac{1}{x} = 1 + \left(\frac{1}{x} - 1\right) = 1 + \frac{1-x}{x} = 1 + \cfrac{1}{\cfrac{x}{1-x}}$$

which yields

$$G = 1 + \frac{1}{G} \tag{1.25}$$

that implies that G is a solution of the quadratic equation $y^2 - y - 1 = 0$, i.e. $G = (1+\sqrt{5})/2$. Iterations of (1.25) lead to the periodic continued fraction

$$G = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}} = [1; 1, 1, \dots].$$

A D-continued fraction is *purely periodic* of period m if the initial block of m partial quotients is repeated throughout the expansion, that is, if $a_{km+1} = a_1, \dots, a_{(k+1)m} = a_m$ and $\varepsilon_{km+1} = \varepsilon_1, \dots, \varepsilon_{(k+1)m} = \varepsilon_m$ for every $k \geq 1$, and no shorter block has this property. The notation for such a continued fraction is

$$[a_0; \varepsilon_0/\overline{a_1, \varepsilon_1/a_2, \dots, \varepsilon_{m-1}/a_m, \varepsilon_m}]. \tag{1.26}$$

An *(eventually) periodic* continued fraction consists of an *initial block* of length $n \geq 0$ followed by a *repeating block* of length m and it is written as

$$[a_0; \varepsilon_0/a_1, \varepsilon_1/a_2, \dots, \varepsilon_{n-1}/a_n, \varepsilon_n/\overline{a_{n+1}, \dots, \varepsilon_{n+m-1}/a_{n+m}, \varepsilon_{n+m}}], \tag{1.27}$$

where there is no shorter such repeating block and the initial block does not end with a copy of the repeating block.

Note that the same definitions hold for semi-regular continued fraction in the standard notation (1.6), but in this case the notation would be

$$[a_0; \varepsilon_1/a_1, \varepsilon_2/a_2, \dots, \varepsilon_n/a_n, \overline{\varepsilon_{n+1}/a_{n+1}, \dots, \varepsilon_{n+m}/a_{n+m}}].$$

Theorem 1.15. *Let $x \in [0,1]$ have a periodic D-expansion. Then x is a quadratic irrational number.*

Proof. Suppose first that x is purely periodic and has the form given by (1.26). Recall that from (1.9) we have

$$x = \frac{p_k + p_{k-1}\varepsilon_k t_k}{q_k + q_{k-1}\varepsilon_k t_k}, \quad \text{for } k \geq 0.$$

In our case $x = t_0 = t_{km}$, so

$$x = \frac{p_m + p_{m-1}\varepsilon_m x}{q_m + q_{m-1}\varepsilon_m x},$$

thus

$$q_{m-1}\varepsilon_m x^2 + (q_m - \varepsilon_m p_{m-1})x - p_m = 0.$$

Since the q_i's are different from 0 then x is quadratic irrational.

Let now $y = [0; 1/b_1, \varepsilon_1/b_2, \dots, \varepsilon_{p-1}/b_p, x]$ be periodic, with x as in (1.26) purely periodic. Clearly for every integer b the expression $(b \pm \frac{1}{x})$ is again quadratic irrational (just write

$x = (a + \sqrt{D})/b$ and work out the sum).
The fact that y is quadratic irrational follows then by induction, writing

$$y = \cfrac{1}{b_1 + \cfrac{\varepsilon_1}{\left(b_2 + \cfrac{\varepsilon_2}{\ddots}\right)}},$$

where the sub-fraction in the brackets is quadratic irrational by induction hypothesis. Finally, y is irrational because it has an infinite continued fraction expansion, cf. Corollary 1.10. $\square$

Theorem 1.16. *Let $x \in [0,1]$ be a quadratic irrational. Then x has an eventually periodic D-expansion.*

Proof. In Section 1.3.1 we have seen how every D-continued fraction can be derived via insertion and singularization from a regular continued fraction. Moreover, given a regular periodic continued fraction $y \in [0,1]$, $y = [0; b_1, \ldots, b_{p-1}, \overline{b_p, \ldots, b_{p+l-1}}]$, recall that between b_n and b_{n+1} we can insert at most $b_{n+1} - 1$ times. In particular, the set of the partial quotients of y is a finite set.
Let x be a quadratic irrational number with D-expansion $[0; 1/a_1, \varepsilon_1/a_2, \ldots]$. In Section (1.3.1) we showed that every D-expansion of x is derived via insertion and singularization from the regular expansion of x. Moreover, the RCF-expansion of x is eventually periodic by theorem 1.13. Let $x = [0; b_1, \ldots, b_{p-1}, \overline{b_p, \ldots, b_{p+l-1}}]$ be the RCF-expansion of x and consider the orbit of the point x, $\mathcal{O} = \{t_1, t_2, \ldots, t_n, \ldots\} = \{T(x), T^2(x), \ldots, T^n(x), \ldots\}$, obtained by iterations of the Gauss map T.
Recall that with every singularization we loose a point in the orbit, while we get a new point every time we insert. However, the maximum number of insertion between two digits is bounded, since we can insert at most $b_{n+1} - 1$ times between b_n and b_{n+1}. This, together with the fact that the set $\{b_i,\ i \geq 1\}$ of the partial quotients is finite, implies that $\mathcal{O}$ is a finite set as well.
Thus, there must exist $\overline{m}, \overline{n}$ with $\overline{m} > \overline{n}$ such that $t_{\overline{m}} = t_{\overline{n}}$. In particular, the tails will coincide, i.e., $t^{-1}_{\overline{m}+1} = t^{-1}_{\overline{n}+1}$.
Pick now the smallest m such that there exists such an n (note that m doesn't have to be necessarily equal to 1, since we require the fraction to be only eventually periodic), and choose n such that $k = m - n$ is minimal with respect to this property. Then x is periodic with period k.

$\square$

1.5.1 Hurwitzian numbers

We can generalize the concept of eventually periodic continued fraction introducing the notion of Hurwitzian numbers.

Let x be a real irrational number with RCF expansion $[a_0; a_1, \dots]$. The number x is called *Hurwitzian* if it can be written as

$$x = [a_0; a_1, \dots, a_p, \overline{a_{p+1}(k), \dots, a_{p+l}(k)}]_{k=0}^{\infty},$$

where $a_{p+1}(k), \dots, a_{p+l}(k)$ (the so-called *quasi period* of x) are polynomials with rational coefficients which take positive integral values for $k = 0, 1, \dots,$ and at least one of them is not constant.
A classical example of such a number is $e = [2; \overline{1, 2k+2, 1}]_{k=0}^{\infty}$.

We have already recalled in this work that every irrational real number x admits, besides the RCF expansion, other kinds of continued fraction expansions, such as the D-, the backward, the Nakada's α continued fraction expansion (see section 1.4).
In [HaK] is given the definition of Hurwitzian number relative to each one of these expansions, and the following result is obtained:

Theorem 1.17. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then x is Hurwitzian if and only if x has a Nakada's α-(resp. backward-) Hurwitzian expansion.*

The definition of Hurwitzian number can be extended to the D-continued fractions as follows.

Definition 1.18. *Let $x \in \mathbb{R} \setminus \mathbb{Q}$, D be a subset of $[0, 1]$ and T_D be the D-continued fraction map corresponding to D. Then x has a D-Hurwitzian expansion if*

$$x = [a_0; \varepsilon_0/a_1, \dots, \varepsilon_{p-1}/a_p, \varepsilon_p/\overline{a_{p+1}(k), \dots, \varepsilon_{p+l-1}/a_{p+l}(k), \varepsilon_{p+l}}]_{k=0}^{\infty},$$

where $a_0 = \left\lfloor \frac{1}{x} \right\rfloor$, $\varepsilon_n = \pm 1$ and $a_n \in \mathbb{N}$ are given by the D-continued fraction map T_D. Moreover, for $i = 1, \dots, l$ we have that $a_{p+i}(k)$ are polynomial with rational coefficients which take positive integral values for $k = 0, 1, \dots,$ and at least one of them is not constant.

It is very easy to show that the previous theorem does not apply to the D-case, just consider in fact the base of the natural logarithm $e = [2; \overline{1, 2k, 1}]_{k=1}^{\infty}$ and apply theorem (1.9) with $D = \bigcup_{i=0}^{\infty} \left(\frac{1}{2^i+1}, \frac{1}{2^i} \right]$.

Chapter 2

Metric and ergodic properties

2.1 Classical results

Let $(X, \mathcal{F})$ be a measure space and let μ be a measure on X. If $\mu(X) = 1$, we say that μ is a *probability measure* and that the triple $(X, \mathcal{F}, \mu)$ is a *probability space*.
A measurable map T from a probability space $(X, \mathcal{F}, \mu)$ to itself is said *measure preserving* with respect to μ (equivalently, the measure μ is *T-invariant*) if

$$\mu(T^{-1}(A)) = \mu(A)$$

for every μ-measurable set A in $\mathcal{F}$. If T is invertible, the last definition is equivalent to $\mu(T(A)) = \mu(A)$ for every μ-measurable set A in $\mathcal{F}$.
It is a classical result, that any map T on a probability space $(X, \mathcal{F}, \mu)$ is measurable and measure preserving if $\mu(T^{-1}(A)) = \mu(A)$ for any A in a semi-algebra $\mathcal{A}$ generating $\mathcal{F}$. Moreover, a transformation T on a probability space $(X, \mathcal{F}, \mu)$ remains measurable and measure-preserving when one passes to the completion $\mathcal{C}$ of $\mathcal{F}$ under μ (see [DK02], p.14, for a proof). This means that, if we look at the measure space $[0, 1]$ equipped with the Lebesgue σ-algebra $\mathcal{L}$ and a probability measure μ, to check that a map T is measurable and measure preserving with respect to μ, we just have to check on intervals $A \subset [0, 1]$.

A *dynamical system* is a quadruple $(X, \mathcal{F}, \mu, T)$, where $(X, \mathcal{F}, \mu)$ is a probability space and $T : X \to X$ is a surjective μ-preserving transformation. Further, if T is injective, we call $(X, \mathcal{F}, \mu, T)$ an invertible dynamical system.
If now $(X, \mathcal{F}, \mu, T)$ is a dynamical system, then T is called *ergodic* if for every μ-measurable set A satisfying $T^{-1}A = A$ one has $\mu(A) = 0$ or $\mu(A) = 1$. Such a set A is called T-invariant. As before, if T is invertible we can replace $T^{-1}A = A$ with the equivalent equality $TA = A$.

Definition 2.1. *Let $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ be two dynamical systems. Then $(Y, \mathcal{C}, \nu, S)$ is said to be a factor of $(X, \mathcal{F}, \mu, T)$ if there exists a measurable and surjective map $\psi : X \to Y$ such that*

(i) $\psi^{-1}\mathcal{C} \subset \mathcal{F}$ (so ψ preserves the measure structure);

(ii) $\psi T = S\psi$ *(so* ψ *preserves the dynamics);*

(iii) $\mu(\psi^{-1}(E)) = \nu(E)$, $\forall E \in \mathcal{C}$ *(so* ψ *preserve the measure).*

In this case, the dynamical system $(X, \mathcal{F}, \mu, T)$ is called an *extension* of $(Y, \mathcal{C}, \nu, S)$ and ψ is called a *factor map.*

2.1.1 Regular continued fractions

Let T be the Gauss map $T(x) = 1/x - \lfloor 1/x \rfloor$. Gauss found in 1800 an invariant measure for T, now known as the *Gauss measure* μ, which is equivalent to the Lebesgue measure λ and which is given by

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{x+1} dx,$$

for all the Lebesgue sets $A \subset [0,1]$.
As for the ergodic properties of T, the following theorem holds.

Theorem 2.2. *Let* $\Omega = [0,1]$, $\mathcal{L}$ *the collection of Lebesgue sets of* Ω *and* μ *the Gauss measure on* $(\Omega, \mathcal{L})$. *Then the dynamical system* $(\Omega, \mathcal{L}, \mu, T)$ *is an ergodic system.*

For a proof, see [DK02], Theorem 3.5.1.

Remark 2.3. From the ergodicity of T and Birkhoff's Ergodic Theorem, cf. [DK02] Theorem 3.1.7., many classical result follow, for example the theorem that now it is know as *Gauss-Kusmin* or *Gauss-Kusmin-Lévy theorem.*
In 1812, Gauss sent a letter to Laplace stating that he could prove that

$$\lim_{n\to\infty} \lambda(T^{-n}[0,z]) = \mu([0,z]), \qquad 0 \le z \le 1,$$

and asked Laplace to estimate the error term

$$r_n(z) := \lambda(T^{-n}[0,z]) = \mu([0,z]), \qquad 0 \le z \le 1, \quad n \ge 1.$$

In 1928 R. Kusmin [Kus28] showed that

$$r_n(z) = \mathcal{O}(q^{\sqrt{n}}),\ n \to \infty,$$

for some constant $0 < q < 1$, while, independently, Lévy [L29] proved in 1929 that:

$$r_n(z) = \mathcal{O}(q^n),\ n \to \infty,$$

with $q \approx 0.7$.
Kusmin and Lé obtained these results using ideas from probability theory. The first link with ergodic theory was first made by W. Doeblin in 1940 and Ryll-Nardzewski in 1951, who discovered that the ergodic system underlying the regular continued fraction expansions is ergodic. In particular, Ryll-Nardzewski obtained several results of Lévy and Kusmin using ergodic theory.

2.1.2 Nakada's α-continued fraction expansions

Consider the dynamical systems relative to the intervals $I_\alpha = [\alpha - 1, \alpha]$ and Nakada's α-continued fraction maps T_α. For $\alpha \in (0, 1]$ these maps are expanding and admit a unique absolutely continuous invariant probability measure $d\mu_\alpha(x) = \rho_\alpha(x)dx$. Nakada in [N81] computed the invariant densities ρ_α for $1/2 \leq \alpha \leq 1$ by finding an explicit representation of their natural extensions (see Section 2.2.1 for the definition of natural extension).
The case $\sqrt{2} - 1 \leq \alpha \leq 1/2$ was later studied by Moussa, Cassa and Marmi [MCM] for the folded α-maps $\widetilde{T}_\alpha$ defined on the intervals $[0, \max\{\alpha, 1 - \alpha\}]$.
We will give explicitely the density and the normalizing constant relative to the probability measure μ_α. The proof of the fourth case has been obtained by Cassa ([C95]) for the maps $\widetilde{T}_\alpha$ and has been extended to the maps T_α by Luzzi and Marmi in [LM08].
Note that for given α, $\widetilde{T}_\alpha$ is a factor of T_α, with factor map given by the absolute value $\psi : x \mapsto |x|$. So, all the corresponding result for the maps $\widetilde{T}_\alpha$ can be derived from the result about T_α through this semiconjugacy.

Proposition 2.4. *Let T_α be the family of the α-maps defined on the intervals $[\alpha-1, \alpha]$, and set $\gamma = \sqrt{2} - 1$ and $G = g^{-1}$. Then T_α preserve a unique absolutely continuous probability measure $\mu_\alpha = c_\alpha \rho_\alpha$ whose density is given by:*

(i) If $\gamma \leq \alpha < \frac{1}{2}$, then

$$c_\alpha = \frac{1}{\log G},$$

and

$$\rho_\alpha(x) = \begin{cases} \frac{1}{x+G+1} & \text{if } x \in [\alpha - 1, \frac{2\alpha-1}{1-\alpha}], \\ \frac{1}{x+2} + \frac{1}{x+G+1} + \frac{1}{x+G} & \text{if } x \in (\frac{2\alpha-1}{1-\alpha}, \frac{1-2\alpha}{\alpha}), \\ \frac{1}{x+G} & \text{if } x \in [\frac{1-2\alpha}{\alpha}, \alpha). \end{cases}$$

(ii) If $\frac{1}{2} \leq \alpha < g$, then

$$c_\alpha = \frac{1}{\log G},$$

and

$$\rho_\alpha(x) = \begin{cases} \frac{1}{x+G+1} & \text{if } x \in [\alpha - 1, \frac{1-2\alpha}{\alpha}], \\ \frac{1}{x+2} & \text{if } x \in (\frac{1-2\alpha}{\alpha}, \frac{2\alpha-1}{1-\alpha}), \\ \frac{1}{x+G} & \text{if } x \in [\frac{2\alpha-1}{1-\alpha}, \alpha). \end{cases}$$

(iii) If $g \leq \alpha \leq 1$, then

$$c_\alpha = \frac{1}{\log(1 + \alpha)},$$

and

$$\rho_\alpha(x) = \begin{cases} \frac{1}{x+2} & \text{if } x \in [\alpha - 1, \frac{1-\alpha}{\alpha}], \\ \frac{1}{x+1} & \text{if } x \in (\frac{1-\alpha}{\alpha}, \alpha). \end{cases}$$

Notice that if $\alpha = 1$, one recovers the Gauss' measure, of density

$$\rho(x) = \frac{1}{\log 2} \frac{1}{1+x},$$

while, if $\alpha = \frac{1}{2}$, one finds the density

$$c_{1/2}\rho_{1/2} = \frac{1}{\log G} \left(\frac{1}{G+x} + \frac{1}{G+1-x} \right)$$

corresponding to the *nearest integer expansion*, a particular case of the Nakada's α-expansion. Moreover, it follows from the result above that the measures μ_α have everywhere a nonzero density, and therefore are equivalent to Lebesgue measure. The same holds for the invariant measure $\widetilde{\mu}_\alpha$ relative to $\widetilde{T}_\alpha$.

In the left range $[0, \gamma]$, the existence of the invariant density is not known. Moussa *et al* in [MCM] proved the existence of the invariant measure for some particular value of α in this interval (e.g. $\alpha = 2 - g$ or $\alpha = 2/5$). They noted that the explicit form in this case involves an infinite number of terms of the form $1/(x + a_k)$. However, the change of behaviour around $\alpha = \gamma$ has not been explained nor understood yet.

2.2 S-expansions and natural extensions

In [N81], Nakada obtained the natural extension for each α-expansion with $\alpha \in [1/2, 1]$. This result has been re-obtained by C. Kraaikamp in [K91] in a completely different way, that is, via singularizations. In particular, α-expansions are a special case of the so-called S-expansions, introduced by Kraaikamp in the aforementioned paper.

We will now show how singularizations and insertions can be used to deal with the case $\alpha \in [\sqrt{2} - 1, 1/2]$.

First, we recall the definition of natural extension, and then give a brief introduction to S-expansion. Next, we use the theory of S-expansion to re-obtain the natural extension of the α-expansions using only insertion and singularization.

2.2.1 Natural extensions

Definition 2.5. *Let $(Y, \mathcal{C}, \nu, S)$ be a non-invertible measure-preserving dynamical system. An invertible measure preserving dynamical system $(X, \mathcal{F}, \mu, T)$ is called a natural extension of $(Y, \mathcal{C}, \nu, S)$ if Y is a factor of X and the factor map ψ satisfies*

$$\bigvee_{m=0}^{\infty} T^m \psi^{-1} \mathcal{C} = \mathcal{F},$$

where $\bigvee_m T^m \psi^{-1} \mathcal{C}$ is the smallest σ-algebra containing the σ-algebras $T^k \psi^{-1} \mathcal{C}$.

Natural extensions are extremely helpful in understanding the dynamics of expansions, and - in the case of continued fractions - help us to find in an easy way various Diophantine results; for more details, see [DKS96] where the invariant Parry-measure for β-transformations is obtained, and Chapter 4 in [DK02], where classical and more recent results in Diophantine approximation are derived using the natural extension of the regular continued fraction expansion.

Example 2.6. We give as an example the natural extension of the RCF-expansions.
Let x be a real irrational number in $[0,1]$ with RCF-expansion $[0; a_1, a_2, \dots]$ and convergents $(p_n/q_n)_{n\geq 0}$. We write t_n and v_n for respectively the "future" and the "past" of $\frac{p_n}{q_n}$,

$$t_n := [0; a_{n+1}, a_{n+2}, \dots] \textit{ and } v_n := [0; a_n, \dots, a_1].$$

It is easy to check that

$$v_n = \frac{q_{n-1}}{q_n}.$$

Note that $t_n \in [0,1]$ for $x \notin \mathbb{Q}$ and $v_n \in [0,1]$, so we can introduce the space
$\Omega := ([0,1) \setminus \mathbb{Q}) \times [0,1]$.
Define now the map $\mathcal{T} : \Omega \to \Omega$ by

$$\mathcal{T}(x,y) := \left(T(x), \frac{1}{\lfloor \frac{1}{x} \rfloor + y} \right).$$

One has

$$\begin{aligned}
\mathcal{T}(x,0) &= (T(x), \frac{1}{a_1+0}) = (t_1, v_1)\,, \\
\mathcal{T}^2(x,0) &= \mathcal{T}(\mathcal{T}(x,0)) \;=\; (t_2, \frac{1}{a_2+v_1}) \;=\; (t_2, v_2)\,, \\
&\;\;\vdots \\
\mathcal{T}^n(x,0) &= (t_n, v_n), \quad n \geq 1.
\end{aligned}$$

We have the following theorem:

Theorem 2.7 (Nakada, Ito, Tanaka, [NIT77], [N81])**.** *Let $\overline{\mu}$ be the probability measure on Ω with density*

$$d(x,y) := \frac{1}{\log 2}\frac{1}{(1+xy)^2}, \quad (x,y) \in \Omega.$$

Then $\overline{\mu}$ is the invariant measure for $\mathcal{T}$. Furthermore, the dynamical system $(\Omega, \overline{\mu}, \mathcal{T})$ is an ergodic system.

2.2.2 S-expansions

A simple way to derive a strategy for singularization is given by a *singularization area.*

Definition 2.8. *Let $(\Omega, \overline{\mathcal{B}}, \overline{\mu}, \mathcal{T})$ be the natural extension of the RCF. A subset S of Ω is called a singularization area if it satisfies*

(i) $S \subset [\frac{1}{2}, 1) \times [0, 1]$;

(ii) $\mathcal{T}(S)$ *and* S *are either disjoint or intersect in the single point* (g, g);

(iii) $S \in \mathcal{B}$ *and* $\mu(\partial S) = 0$.

Note that (g, g) is a fixed point for $\mathcal{T}$.

Let now $x \in [0, 1]$ be an irrational number with RCF-expansion $x = [0; a_1, a_2, \dots]$, and let S be a singularization area. Consider the following algorithm:

$$\text{singularize } a_{n+1} \text{ if and only if } (\mathcal{T}^n(x), 0) \in S,\ n \geq 0.$$

The iteration of the algorithm algorithm yields a semi-regular continued fraction expansion converging to x known as the S*-expansion* of x. Definition 2.8 reflects in *(i)* that we only singularize partial quotients equal to 1, and in *(ii)* that we never singularize two consecutive partial quotients equal to 1.

Write now the S-expansion of x as

$$[b_0; \varepsilon_1/b_1, \varepsilon_2/b_2, \dots] = b_0 + \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \ddots}}.$$

For $k \geq 1$ define

$$\begin{aligned} t_k &:= t^k(x - b_0) = [0; \varepsilon_{k+1}/b_{k+1}, \varepsilon k + 2/b_{k+2}, \dots] \\ v_k &:= s_{k-1}/s_k = [0; 1/b_k, \varepsilon_k/b_{k-1}, \dots, \varepsilon_2/b_1], \end{aligned}$$

where

$$\begin{aligned} t^0(x - b_0) = t(x - b_0) &:= [0 : \varepsilon_2/b_2, \varepsilon_3/b_3, \dots] \\ v_0 &= 0, \end{aligned}$$

and where $(r_k/s_k)_k$ are the convergents relative to the S-expansion.

We have the following theorems, which are given without a proof; see also [K91].

Theorem 2.9. *Let x be a real number with RCF-expansion $[a_0; a_1, \dots]$ and RCF-convergents $(p_k/q_k)_k$, and S be a singularization area in $(\Omega, \overline{\mathcal{B}}, \overline{\mu}, \mathcal{T})$. Set $\Delta_S = \Omega \setminus S$, $\Delta_S^- = \mathcal{T}S$, $\Delta_S^+ = \Delta_s \setminus \Delta_s^-$, then:*

1. *The system* $(\Delta_S, \mathcal{L}, \rho_S, \mathcal{O}_S)$ *forms an ergodic system, where* ρ_S *is the probability measure on* $(\Delta_S, \mathcal{L})$ *with density*

$$\frac{1}{\log 2(1-\mu(S))}\frac{1}{(1+xy)^2}$$

and where

$$\mathcal{O}_S(x,y) = \begin{cases} \mathcal{T}(x,y), & \text{if } \mathcal{T}(x,y) \in \Delta_S, \\ \mathcal{T}^2(x,y), & \text{if } \mathcal{T}(x,y) \in S, \end{cases}$$

is the map induced by $\mathcal{T}$ *on* Δ_S*.*

2. $\mathcal{T}^n(x,0) \in S$ *if and only if* p_n/q_n *is not an* S*-convergent.*

3. *If* p_n/q_n *is an* S *convergent, then both* p_{n-1}/q_{n-1} *and* p_{n+1}/q_{n+1} *are* S*-convergents.*

4. $\mathcal{T}^n(x,0) \in \Delta_S^+$ *if and only if* $\exists k$ *such that* $\mathcal{T}^n(x,0) = (t_k, v_k)$ *and*

$$\begin{cases} r_{k-1} = p_{n-1}, & r_k = p_n \\ s_{k-1} = q_{n-1} & s_k = q_n, \end{cases}$$

where $(r_k/s_k)_k$ *are the* S*-convergents of* x*.*

5. $\mathcal{T}^n(x,0) \in \Delta_S^-$ *if and inlay if* $\exists k$ *such that* $\mathcal{T}^n(x,0) = (-\frac{t_k}{1+t_k}, 1-v_k)$ *and*

$$\begin{cases} r_{k-1} = p_{n-2}, & r_k = p_n \\ s_{k-1} = q_{n-2} & s_k = q_n, \end{cases}$$

where $(r_k/s_k)_k$ *are the* S*-convergents of* x*.*

In view of the theorem, define the following map $\mathcal{M} : \Delta_S \to \mathbb{R}^2$,

$$\mathcal{M}(T,V) := \begin{cases} (T,V) & \text{if } (T,V) \in \Delta_S^+, \\ \left(-\frac{T}{1+T}, 1-V\right) & \text{if } (T,V) \in \Delta_S^-. \end{cases}$$

Keeping all the notations above, we give the second theorem.

Theorem 2.10. *Let* S *be a singularization area and consider the space* $(\Omega_S, \mathcal{L})$*, where* $\Omega_S := \mathcal{M}(\Delta_S)$ *and* $\mathcal{L}$ *is the collection of Lebesgue subsets of* Ω_S*. Define then the probability measure* μ_S *on* $(\Omega_S, \mathcal{L})$ *as*

$$\mu_S(E) := \rho_S(\mathcal{M}^{-1}(E)), \ E \in \mathcal{L},$$

and the $\mathcal{T}_S : \Omega_S \to \Omega_S$ *by*

$$\mathcal{T}_S(t,v) := \mathcal{M}(\mathcal{O}_S(\mathcal{M}^{-1}(t,v))), \ (t,v) \in \Omega_S.$$

Then $\mathcal{T}_S$ is isomorphic to $\mathcal{O}_S$ by $\mathcal{M}$, and $(\Omega_S, \mathcal{L}, \mu_S, \mathcal{T}_S)$ forms an ergodic system with density

$$\frac{1}{\log 2(1-\mu(S))} \frac{1}{(1+tv)^2}.$$

Finally, for almost all $x \in [0,1)$ the sequence $(t_k, v_k)_{k\geq 0}$ is distributed over Ω_S according to this density.

One can show that $\mathcal{T}_S$ can be written in the following way:

$$\mathcal{T}_S(t,v) = \left(\left|\frac{1}{t}\right| - f_S(t,v), \frac{1}{sgn(t)\cdot v + f_S(t,v)} \right), \quad \text{for } (t,v) \in \Omega_S.$$

Furthermore for the coefficient holds

$$b_{k+1} = f_S(t_k, v_k), \ K \geq 0, \ \text{where } (t_0, v_0) = (x - b_0, 0).$$

Thus we see that the S-expansions are the process associated with $\mathcal{T}_S$ and f_S.
For the α-expansion the formula becomes

$$f_\alpha(t,v) = \left\lfloor \left|\frac{1}{t}\right| + 1 - \alpha \right\rfloor, \ \text{where } \alpha \in \left[\frac{1}{2}, 1\right].$$

2.2.3 Entropy

Before analysing the special case of the α-expansion with $\alpha \in \left[\sqrt{2}-1, \frac{1}{2}\right]$, we give a short introduction about entropy. Heuristically, entropy measures the amount of randomness generated by T.
Let $(X, \mathcal{F}, \mu)$ be a probability space and let $\alpha = \{A_0, \dots, A_{n-1}\}$ be a partition of X, so that, X can be written as disjoint union (up to set of measure zero) of the element of α. We define the *entropy of the partition* α as:

$$H(\alpha) = H_\mu(\alpha) := -\sum_{i=0}^{n-1} \mu(A_i) \log \mu(A_i).$$

Consider now the partition

$$\bigvee_{i=0}^{n-1} T^{-i}\alpha, \tag{2.1}$$

whose atoms are sets of the form $A_{i_o} \cap T^{-1}A_{i_1} \cap \dots \cap T^{-(n-1)}A_{i_{n-1}}$, consisting of all the points $x \in X$ such that $x \in A_{i_0}, Tx \in A_{i_1}, \dots, T^{-(n-1)}x \in A_{i_{n-1}}$, $A_{i_k} \in \alpha$ for all k. We can give the following definition.

Definition 2.11. *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. The entropy of the measure preserving transformation T with respect to the partition α is given by*

$$h(\alpha, T) = h_\mu(\alpha, T) := \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-1}\alpha \right),$$

where

$$H\left(\bigvee_{i=0}^{n-1} T^{-1}\alpha\right) = -\sum_{D\in\bigvee_{i=0}^{n-1}T^{-i}\alpha} \mu(D)\log(\mu(D)).$$

Finally, the entropy of the transformation T is given by

$$h(T) = h_\mu(T) := \sup_\alpha h(\alpha, T),$$

where the supremum is taken over all finite partitions α of finite entropy.

Note that the last definition of entropy is independent on the choice of the partition. Moreover, the limit $\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1} T^{-1}\alpha\right)$ exists since H is a sub-additive sequence. We have the following theorem (cf. [DK02], Theorem 6.1.7).

Theorem 2.12. *Entropy is invariant under isomorphism.*

Examples 2.13.

1. Let T be the continued fraction map. Then the entropy of T is given by

$$h(T) = \frac{\pi^2}{6\log 2}.$$

2. In [N81] Nakada constructed the two-dimensional representation of the natural extension of the α-continued fraction transformation T_α for $\alpha \in [1/2, 1]$ and showed that the entropy $h(\alpha) = h(T_\alpha)$ of T_α with respect to the absolutely continuous invariant measure is given by

$$h(\alpha) = \begin{cases} \frac{\pi^2}{6}\frac{1}{\log(1+g)} & \text{for } \alpha \in [\frac{1}{2}, g], \\ \frac{\pi^2}{6}\frac{1}{\log(1+\alpha)} & \text{for } \alpha \in (g, 1], \end{cases}$$

where $g = (\sqrt{(5)} - 1)/2$ is the golden mean. This means that $h(\alpha)$ is constant on $[1/2, g]$ and decreasing on $(g, 1]$. Later, Moussa *et al* in [MCM] extended this result to $\alpha \in [\gamma, 1/2)$, with $\gamma = \sqrt{2} - 1$, proving that here $h(\alpha)$ is constant, with the same value as in the interval $[1/2, g]$.

2.2.4 The case $\alpha \in \left[\sqrt{2} - 1, \frac{1}{2}\right)$

Remark 2.14. In this section, as anticipated, we obtain the natural extension of the Nakada's α expansions through singularizions and insertions. Recall that the case with $\alpha \in [1/2, 1]$, as treated in [K91], used singularization only.

Let $\alpha = 1/2$ and consider the corresponding Nakada's α-map, $T_{\frac{1}{2}} : \left[-\frac{1}{2}, \frac{1}{2}\right) \to \left[-\frac{1}{2}, \frac{1}{2}\right)$, defined by

$$T_{\frac{1}{2}}(x) = \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + \frac{1}{2} \right\rfloor , \quad x \neq 0,$$

and $T_{\frac{1}{2}}(0) := 0$. Then for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ irrational and $n \geq 1$ we have that $T_{\frac{1}{2}}^{n-1}(x) \neq 0$. Iterations of this map generate a continued fraction expansion introduced in 1873 by Minnigerode, and studied later on by Hurwitz, called the *nearest integer continued fraction (NICF) expansion of* x.
The partial quotients are given by $a_1(x) = \left\lfloor \left|\frac{1}{x}\right| + \frac{1}{2} \right\rfloor$ and

$$a_n = a_n(x) = a_1(T_{\frac{1}{2}}^{n-1}(x)),$$

and the signs ε_n by

$$\varepsilon_n = \operatorname{sgn}(T_{\frac{1}{2}}^{n-1}(x)).$$

With this notation we can then rewrite the expression of $T_{\frac{1}{2}}$ as follows:

$$T_{\frac{1}{2}}(x) = \frac{\varepsilon_1}{x} - a_1,$$

so that

$$x = \frac{\varepsilon_1}{a_1 + T_{\frac{1}{2}}(x)} = \cfrac{\varepsilon_1}{a_1 + \cfrac{\varepsilon_2}{a_2 + \cdots + \cfrac{\varepsilon_n}{a_n + T_{\frac{1}{2}}^n(x)}}} = [0; \varepsilon_1/a_1, \varepsilon_2/a_2, \dots].$$

Since $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ we have that $|1/x| \geq 2$. Therefore $a_i \geq 2$ for every $i \geq 1$. Moreover $a_n + \varepsilon_{n+1} \geq 2$.
These properties uniquely determine the NICF-expansion of x.

It is also possible to show via singularizations that the NICF-convergents of x are a subsequence of the RCF-convergents, so that the NICF converges faster that the RCF. In particular, given a real number x, we can get the NICF-expansion of x applying the following algorithm to the RCF-expansion of x:

In every block of 1's, singularize the first one, the third one, the fifth one, etc..

The corresponding singularization area is given by

$$S_{\frac{1}{2}} = \left[\tfrac{1}{2}, g\right) \times [0, g] \ \cup \ [g, 1] \times [0, g).$$

Nakada showed in [N81] that the natural extension of the nearest integer expansion is the dynamical system $(\Omega_{\frac{1}{2}}, \bar{\mu}_{\frac{1}{2}}, \mathcal{T}_{\frac{1}{2}})$, where

$$\Omega_{\frac{1}{2}} = \left([-\tfrac{1}{2}, 0) \times [0, g^2]\right) \cup \left([0, \tfrac{1}{2}) \times [0, g]\right),$$

(see also Figure 2.1 on page 38), $\mathcal{T}_{\frac{1}{2}} : \Omega_{\frac{1}{2}} \to \Omega_{\frac{1}{2}}$ is the map given by given by

$$\mathcal{T}_{\frac{1}{2}}(x,y) = \left(T_\alpha(x), \frac{1}{a(x)+\varepsilon(x)y} \right),$$

and finally $\bar{\mu}_\alpha$ is a $\mathcal{T}_{\frac{1}{2}}$-invariant probability measure on $\Omega_{\frac{1}{2}}$ with density

$$\frac{1}{\log G}\frac{1}{(1+xy)^2}, \quad \text{for } (x,y) \in \Omega_{\frac{1}{2}}.$$

Define now for $n \geq 0$ the sequence of 'futures' $(t_n)_{n\geq 0}$ and the sequence of 'pasts' $(v_n)_{n\geq 0}$ of x by: $v_0 = 0$, and

$$(t_n, v_n) = \mathcal{T}_{\frac{1}{2}}^n(x, 0), \quad n \in \mathbb{N} \cup \{0\}.$$

Note that for $n \geq 0$,

$$\begin{aligned} t_n &= [0; \varepsilon_{n+1}/a_{n+1}, \varepsilon_{n+2}/a_{n+2}, \dots] \\ v_n &= q_{n-1}/q_n = [0; 1/a_n, \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1], \end{aligned}$$

where $(p_n/q_n)_{n\geq 0}$ are the NICF-convergents p_n/q_n of x. See [DK02, K91] for proofs of these statements.
Now let $\alpha \in [\sqrt{2}-1, 1/2])$; we will determine the natural extension obtained by [MCM], and doing so, we will see that this natural extension is metrically isomorphic to the case $\alpha = 1/2$.
Let $n \geq 0$ be such, that $t_n \in [\alpha, \frac{1}{2})$, i.e., that we have that $(t_n, v_n) \in [\alpha, \frac{1}{2}) \times [0, g]$; see Figure 2.1.

Clearly, the NICF-expansion of x then satisfies

$$\begin{aligned} x &= \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + \cfrac{1}{2 + \cfrac{1}{a_{n+2} + \cfrac{\varepsilon_{n+3}}{a_{n+3} + \ddots}}}}} \\ &= [0; \varepsilon_1/a_1, \dots, \varepsilon_n/a_n, 1/2, 1/a_{n+2}, \varepsilon_{n+3}/a_{n+3}, \dots], \end{aligned}$$

since $t_n \in [\alpha, 1/2)$ implies that $a_{n+1} = 2$ and $t_{n+1} > 0$.

Inserting $-1/1$ between a_n and $a_{n+1} = 2$, yields a new continued fraction expansion of x,

$$\begin{aligned} x &= \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + 1 + \cfrac{-1}{1 + \cfrac{1}{1 + \cfrac{1}{a_{n+2} + \cfrac{\varepsilon_{n+3}}{a_{n+3} + \ddots}}}}}} \\ &= [0; \varepsilon_1/a_1, \dots, \varepsilon_n/(a_n+1), -1/1, 1/1, 1/a_{n+2}, \varepsilon_{n+3}/a_{n+3}, \dots]. \end{aligned}$$

Singularize next in this new continued fraction expansion of x the $(n+1)$st partial quotient, and arrive at

$$x = \cfrac{\varepsilon_1}{a_1 + \ddots + \cfrac{\varepsilon_n}{a_n + 1 + \cfrac{-1}{2 + \cfrac{-1}{a_{n+2} + 1 + \cfrac{\varepsilon_{n+3}}{a_{n+3} + \ddots}}}}} \tag{2.2}$$

$$= [0; \varepsilon_1^*/d_1, \varepsilon_n^*/d_2, \dots], \tag{2.3}$$

where

$$\varepsilon_1^* = \varepsilon_1, \ \dots \ , \varepsilon_n^* = \varepsilon_n, \varepsilon_{n+1}^* = -1 = \varepsilon_{n+2}^*, \quad \text{and} \quad \varepsilon_{n+k}^* = \varepsilon_{n+k} \ \text{ for } k \geq 3,$$

and similarly, for the partial quotients,

$$\begin{aligned} & d_1 = a_1, \ \dots \ , d_{n-1} = a_{n-1}, \ d_n = a_n + 1, \ d_{n+1} = 2, \ d_{n+2} = a_{n+2} + 1, \\ \text{and} \ \ & d_{n+k} = a_{n+k} \ \text{ for } k \geq 3. \end{aligned}$$

Setting

$$t_i^* = \cfrac{\varepsilon_{i+1}^*}{d_{i+1} + \ddots} = [0; \varepsilon_{i+1}^*/d_{i+1}, \dots],$$

and

$$v_i^* = \cfrac{1}{d_i + \cfrac{\varepsilon_i^*}{d_{i-1} + \ddots + \cfrac{\varepsilon_2^*}{d_1}}} = [0; 1/d_i, \varepsilon_i^*/d_{i-1}, \dots, \varepsilon_2^*/d_1],$$

we find that

$$(t_i^*, v_i^*) = (t_i, v_i), \quad \text{for } i = 1, \dots, n-1, \text{ and } i \geq n+2,$$

and that (t_n, v_n) and (t_{n+1}, v_{n+1}) got replaced by (t_n^*, v_n^*) respectively (t_{n+1}^*, v_{n+1}^*).

Since $(t_n, v_n) \in R = [\alpha, \frac{1}{2}) \times [0, g]$, we immediately have that $(t_{n+1}, v_{n+1}) \in \mathcal{T}_{\frac{1}{2}}(R) = [0, \frac{1-2\alpha}{\alpha}) \times [g^2, \frac{1}{2}]$; for the new continued fraction expansion (2.2) of x, these rectangles R and $\mathcal{T}_{\frac{1}{2}}(R)$ have been 'vacated;' see Figure 2.1.
Let us now see where (t_n^*, v_n^*) and (t_{n+1}^*, v_{n+1}^*) are defined.

From (2.2) we see that

$$t_n^* = \cfrac{-1}{2 + \cfrac{-1}{a_{n+2} + 1 + \ddots}}. \tag{2.4}$$

Since

$$\begin{aligned} t_n &= \cfrac{1}{2+\cfrac{1}{a_{n+2}+\ddots}} = 1+\cfrac{-1}{1+\cfrac{1}{1+\cfrac{1}{a_n+2}+\ddots}} \\ &= 1+\cfrac{-1}{2+\cfrac{-1}{a_{n+2}+1+\ddots}} = 1+t_n^*, \end{aligned}$$

we immediately see that $t_n^* = t_n - 1$. Furthermore,

$$v_n^* = \cfrac{1}{a_n+1\cfrac{\varepsilon_n}{a_{n-1}+\ddots}} = \frac{q_{n-1}}{(a_n+1)q_{n-1}+\varepsilon_n q_{n-2}} = \frac{q_{n-1}}{q_n+q_{n-1}} = \frac{v_n}{1+v_n}.$$

Thus,

$$(t_n, v_n) \in R = [\alpha, 1/2) \times [0, g] \quad \text{if and only if} \quad (t_n^*, v_n^*) \in [\alpha-1, -\tfrac{1}{2}) \times [0, g^2].$$

From (2.2) we also see that $t_{n+1}^* = [0; -1/(a_{n=2}+1), \varepsilon_{n=3}/a_{n+3}, \dots]$, so (2.4) yields that

$$t_n^* = \frac{-1}{2+t_{n+1}^*},$$

implying that

$$t_{n=+}^* = \frac{-1}{t_n^*} - 2. \tag{2.5}$$

On the other hand, again from (2.2) we read that

$$v_{n+1}^* = \cfrac{1}{2+\cfrac{-1}{a_n+1+\cfrac{\varepsilon_n}{a_{n-1}+\ddots}}} = [0; 1/2, -1/(a_n+1), \varepsilon_n/a_{n-1}, \dots, \varepsilon_2/a_1].$$

Since

$$v_n^* = \cfrac{1}{a_n+1\cfrac{\varepsilon_n}{a_{n-1}+\ddots}},$$

we see that

$$v_{n+1}^* = \frac{1}{2-v_n^*}. \tag{2.6}$$

from (2.5) and (2.6) we now have that

$$(t_n^*, v_n^*) \in \mathcal{S}(R) = [\alpha-1, -\tfrac{1}{2}) \times [0, g^2] \quad \Leftrightarrow \quad (t_{n+1}^*, v_{n+1}^*) \in \mathcal{S}^2(R) = [\tfrac{2\alpha-1}{\alpha}, 0) \times [\tfrac{1}{2}, g];$$

see Figure 2.1.
An easy calculation shows that

$$\mathcal{T}_{\frac{1}{2}}\left([0, \tfrac{1-2\alpha}{\alpha}) \times [g^2, \tfrac{1}{2}]\right) = \mathcal{T}_\alpha\left([\tfrac{2\alpha-1}{1-\alpha}, 0) \times [\tfrac{1}{2}, g]\right),$$

which is equivalent to the earlier observation, that

$$(t_{n+2}, v_{n+2}) = (t^*_{n+2}, v^*_{n+2}).$$

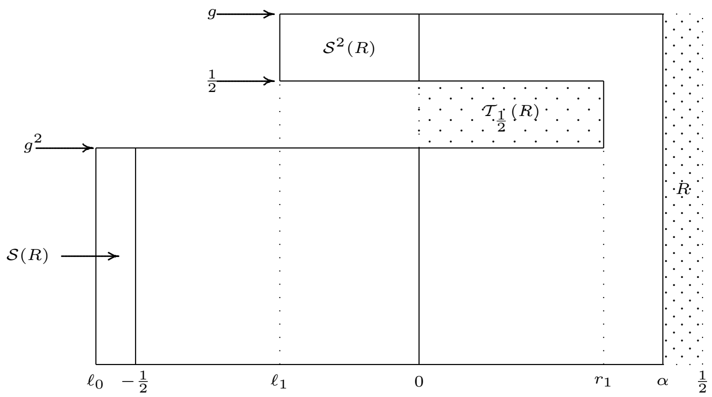

Figure 2.1: The regions $\Omega_{\frac{1}{2}}$ and Ω_α for $\alpha = 0.43$. Here $\ell_0 = \alpha - 1$, $\ell_1 = \frac{2\alpha-1}{1-\alpha}$, $r_o = \alpha$, and $r_1 = \frac{1-\alpha}{\alpha}$.

Thus we have obtained the following result, of which the first part was earlier found by Marmi, Moussa, and Yoccoz in [MMY], and of which the second part was was obtained by Nakada and Natsui in [NaNa].

Theorem 2.15. *Let $\alpha \in [\sqrt{2}-1, 1/2)$, and let Ω_α be given as in Figure 2.1. Then*

(i) The dynamical system $\mathcal{S}_\alpha = (\Omega_\alpha, \bar{\mu}_\alpha, \mathcal{T}_\alpha)$ is the natural extension of the dynamical system $([\alpha-1, \alpha), \mu_\alpha, T_\alpha)$. Here $\bar{\mu}_\alpha$ is a $\mathcal{T}_\alpha$-invariant probability measure on Ω_α with density

$$\frac{1}{\log G}\frac{1}{(1+xy)^2}, \quad \text{for } (x,y) \in \Omega_\alpha,$$

and μ_α is the projection of $\bar{\mu}_\alpha$ on the first coordinate.

(ii) For each $\alpha \in [\sqrt{2}-1, \frac{1}{2}]$ we have that the dynamical system $\mathcal{S}_\alpha$ is metrically isomorphic to the natural extension of the nearest integer continued fraction expansion $(\Omega_{\frac{1}{2}}, \bar{\mu}_{\frac{1}{2}}, \mathcal{T}_{\frac{1}{2}})$. Since this last system is weak-Bernoulli, all these natural extensions are weak-Bernoulli, and have the same entropy $h(\alpha) = \frac{1}{\log G}\frac{\pi^2}{6}$.

In the theorem we have use the definition of weak-Bernoulli. Loosely speaking, weak-Bernoulli means that the future and the distant past are approximately independent.

Remark 2.16.

1. In this section, we basically re-obtained Theorem 2.10, with $\Omega_S = \Omega_\alpha$ and $\mathcal{T}_S = \mathcal{T}_\alpha$.

2. The representation of the folded dynamical system $\mathcal{S}_\alpha$ is obtain by folding the representation of the unfolded dynamical system $\mathcal{S}_\alpha$. One first uses a folding along the y-axis, then along the x-axis.

2.3 Metric properties of D-continued fraction expansions

We know investigate the metrical properties of D-continued fraction expansions. In particular, it follows from a result of Thaler (cf. [Th80] and [Th83]), that the map T_D preserves a measure which is σ-finite, infinite if D contains a neighborhood of 1, and a probability measure on $([0,1], \mathcal{L})$ if D doesn't contain a neighborhood of 1.
As for giving the explicit formula for the invariant density, in view of the close connection between D-expansions and α-expansions, we expect it to be an extremely hard problem (recall that the explicit density of the α-expansion when $\alpha \in (0, \sqrt{2}-1]$ is still unknown).

Definition 2.17. *Let $f : A \subseteq \mathbb{R} \to \mathbb{R}$ and let $z \in \overline{A}$. We say that z is an indifferent fixed point of f if $\lim_{x\to z} f(x) = z$ and $\lim_{x\to z} |f'(x)| = 1$.*

Definition 2.18. *An indifferent fixed point is a regular source, if there exists a positive real ε such that $|f'(x)|$ is decreasing on $(z-\varepsilon, z) \cap A$ and increasing on $(z, z+\varepsilon) \cap A$.*

Let $\zeta_1 = \{B(i) \mid i \in I\}$ be a collection of disjoint sub-intervals of $[0,1]$, $|I| \geq 2$, with $\lambda(\cup_{i\in I} B(i)) = 1$, where λ denotes the Lebesgue measure. Consider then transformations $T : [0,1] \to [0,1]$ satisfying the following conditions given by M. Thaler in [Th80]:

(i) $T_{|_{B(i)}}$ is twice differentiable and $\overline{TB(i)} = [0,1]$ for all $i \in I$.

(ii) Every $B(i)$ contains exactly one fixed point x_i, and the set

$$J = \{i \in I \ : \ x_i \text{ is an indifferent fixed point}\}$$

is non-empty and finite.

(iii) $|T'(x)| \geq \rho(\varepsilon) > 1$ for all $x \in \cup_{i\in I} B(i) \setminus \cup_{j\in J}(x_j - \varepsilon, x_j + \varepsilon)$, for each $\varepsilon > 0$.

(iv) Every indifferent fixed point is a regular source.

(v) (Adler's Condition) $\frac{|T''|}{(T')^2}$ is bounded on $\cup_{i\in I} B(i)$.

Let $\mathcal{J}$ denote the class of all such transformations and $\mathcal{J}_R$ the subclass of those among them with $J = \emptyset$. It is classical (cf. [R57]) that $T \in \mathcal{J}$ belongs to $\mathcal{J}_\mathcal{R}$, if and only if T satisfies Rényi's condition (cf. Remark 2.19).
In particular, every $T \in \mathcal{J}_R$ has a finite invariant measure equivalent to the Lebesgue measure λ.
We want now to state that every $T \in \mathcal{J}$ still admits an ergodic invariant measure, which is σ-finite, infinite. To do that, we first have to introduce other notation, and in particular the definition of jump transformation, first introduced by F. Schweiger in [S75].

Let $(X, \mathcal{F})$ be a measurable space and $T : X \to X$ be a measurable transformation. Consider $A \in \mathcal{F}$, and let $n : A \to \mathbb{N}$ be a measurable map such that $T^{n(x)}(x) \in A$ for every $x \in A$. Define then a transformation $T_{A,n} : A \in A$ by

$$T_{A,n}(x) = T^{n(x)}(x).$$

Putting for $k \geq 1$

$$B_k = \{x \in A \ : \ n(x) = k\}$$

we have $T_{A,n}^{-1}(E) = \cup_{k=1}^{\infty}(B_k \cap T^{-k}E)$ for each $E \subseteq A$. This shows that $T_{A,n}$ is measurable with respect to $A \cap \mathcal{F}$.
Consider now α_1 to be an at most countable measurable partition of X and let α_n be the set of atoms of $\vee_{i=0}^{n-1}T^{-i}\alpha_1$ (see (2.1)). Take then β as an arbitrary subset of $\cup_n \alpha_n$ with $\cup_{Z\in\beta} Z = X$.
Then we call *jump transformation* over β the transformation for which:

$$n(x) = \min\{n \geq 1 \ : \ x \in Z \in (\alpha_n \cap \beta\}, \qquad x \in X.$$

It is shown in [Th83] that, if σ is a measure on $\mathcal{F}$ such that T is non singular with respect to σ, we have:

$$\text{If } T_{A,n} \text{ is ergodic with respect to } \sigma_{|A\cap\mathcal{F}}, \text{ then } T \text{ is ergodic with respect to } \sigma, \tag{2.7}$$

and that

$$\text{If } T_{A,n} \text{ is conservative with respect to } \sigma_{|A\cap\mathcal{F}}, \text{ then } T \text{ is conservative with respect to } \sigma. \tag{2.8}$$

Moreover, every measure on $A \cap \mathcal{F}$ which is invariant for $T_{A.n}$, yields an invariant measure for T. Define for $n \geq 1$

$$\begin{aligned} B(k_1, \dots, k_n) &= \bigcap_{i=1}^{n} T^{-i+1}B(k_i), \qquad (k_1, \dots, k_n) \in I^n; \\ \mathcal{Z} &= \text{ class of all cylinders } B(k_1, \dots, k_n), \ \forall n \geq 1; \\ B_n(k) &= \underbrace{B(k, \dots, k)}_{n \text{ times}}, \quad k \in I; \\ \mathcal{D}_n &= \{B_n(k) \ : \ k \in J\}; \end{aligned}$$

Let now T^* denote the jump transformation $T^*(x) = T^n(x)$ over the cylinder class $\beta = \mathcal{Z} \setminus \cup_{n=1}^{\infty} \mathcal{D}_n$. We have that $T^* \in \mathcal{J}_R$ (cf. [Th80], Theorem 2 and Corollary 2). In particular, for what we said before, T^* admits a finite ergodic invariant measure equivalent to λ, which gives rise to an infinite measure $\mu \approx \lambda$ which is invariant under $T \in \mathcal{J} \setminus \mathcal{J}_{\mathcal{R}}$. Moreover, from (2.7) and (2.8), it follows that T is conservative and ergodic with respect to $\mu(\lambda)$ (for the proof that μ is infinite, we refer to [Th80], Theorem 1 and Corollary 1, and [Th83], pg. 71).

Remark 2.19. Let x, y belong to the same cylinder $B(k_1, \ldots, k_n)$, and consider, for a given map T the expression

$$\left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| . \tag{2.9}$$

We say that T satisfies Rényi condition if (2.9) is uniformly bounded for every n and for every cylinder $B(k_1, \ldots, k_n)$.
In [R57], Rényi proved that every map T which satisfies such a condition, admits an ergodic invariant measure with density bounded away from 0 and ∞.
However, this condition is not readily checkable, since it involves higher iterates of T.
R. L. Adler, in [AF84], reformulated this condition into an equivalent one that does not involve iterates of the map. The result is now know as Adler's Theorem.
Before stating the theorem, we need the following definition.

Definition 2.20. *A map $T : X \to X$ on the measurable space $(X = [a, b], \mathcal{F})$ is Markov, if there is an at most countable collection $\{(B(k))_{k \in \mathbb{N}}\}$ of disjoint open intervals such that*

(i) T is defined on $\cup_k B(k)$ and $X \setminus \cup_k B(k)$ has measure zero.

(ii) $T_{|_{B(k)}}$ is strictly monotonic and extends to a C^2 function on $\overline{B(k)}$ for each k.

(iii) If $T(B(k)) \cap B(j) \neq \emptyset$, then $B(j) \subset T(B(k))$.

(iv) There exists R such that $B(j) \subset \cup_{n=1}^{R} T^n(B(k))$ for every k and j.

Theorem 2.21 (Adler's Theorem)**.** *Let $T : X \to X$ be Markov on the measurable space $(X = [a, b], \mathcal{F})$. Suppose then*

$$M = \sup_{B(k)} \sup_{x,y \in B(k)} \left| \frac{T''(x)}{T'(y)^2} \right| < \infty$$

and

$$\inf_x |(T^n)'(x)| > 1$$

for some n. Then T admits an invariant finite measure with density bounded away from 0 and ∞.

It is easy to see then that, taking $X = [0, 1]$, a transformation T in $\mathcal{J}_R$ satisfies Adler's theorem, and therefore Rényi condition.

Example 2.22 (D-continued fractions)**.**

1. Consider the D-continued fraction map T_D defined on $[0,1]$. We want to check when T fulfills Thaler's conditions.
   Let D be a subset of $[0,1]$ of the form $D = \bigcup_{i=0}^{\infty} \left(\frac{1}{n_i+1}, \frac{1}{n_i}\right]$, where $(n_i)_{i\geq 0}$ is a positive sequence of integers. For example, expansions without one (or more) particular digits belong to this class, as well as odd and even continued fractions.

   The map T_D might admit the unique indifferent fixed point $x=1$, which is also a regular source. We have that $x=1$ is a regular source for the map T_D if and only if $1 \in D$.
   We have:
$$|T_D'(x)| = \frac{1}{x^2} \qquad \text{and} \qquad |T_D''(x)| = \frac{2}{x^3}.$$
   Take $B(n) = [\frac{1}{n+1}, \frac{1}{n}]$, $n \geq 1$. Thaler's conditions become:

   (i) $T_{|_{B(n)}}$ is twice differentiable and $\overline{T(B(n))} = [0,1]$ for all $n \in \mathbb{N}$.

   (ii) Every $B(n)$ contains exactly one fixed point x_n. The set $J = \{i \in I : x_i \text{ is an indifferent fixed point}\}$ is finite and non-empty if and only if D contains a neighborhood of 1.

   (iii) $|T'(x)| > 1$ for all $x \in \cup_{n\in\mathbb{N}} B(n) \setminus (1-\varepsilon)$, for each $\varepsilon > 0$.

   (iv) If T_D admits indifferent fixed points, i.e., if a neighborhood of 1 is in D, then every indifferent fixed point is a regular source.

   (v) (Adler's Condition) $\frac{|T''|}{(T')^2} = 2x$, which is bounded on $[0,1]$.

   Thus we see that T_D admits an ergodic invariant measure, which is finite if and only if D doesn't contain 1, and σ-finite infinite if $1 \in D$.

2. The previous example cannot be applied directly to the folded α-continued fractions, or the Lehner expansions (cf. [DK00]) because we don't have surjectivity on the whole interval $[0,1]$. Observe however, that in the folded α-case, all iterations, except at most the first one, fall in the interval $[0,\alpha]$. We can then apply the same considerations as before in this interval, i.e. with $X = [0,\alpha]$.
   The same holds for the Lehner continued fractions; in this case $X = [1/2, 1]$.

2.4 Computer simulations

To find the density of the invariant measure is in general an extremely hard problem. Most results in this field, are merely of an existential nature, i.e. one proves, like we did, that under certain conditions there exists a unique invariant measure, generally assumed equivalent to the measure of Lebesgue.
We have seen that we are able to write the invariant density of the Nakada's α-expansions

for $\alpha \in [\sqrt{2}-1, 1]$ using two different constructions. In particular, in [K91] it is shown how the S-expansions apply to other special cases, like the optimal continued fractions (see [BK90]) and Minkowski's diagonal expansions.
Moreover, invariant densities are also known for other continued fraction expansions. In [S91], F. Schweiger proved that the odd continued fractions preserves the measure

$$\mu(A) = \frac{1}{3\log G}\left(\int_A \frac{dx}{G+x-1} + \int_A \frac{dx}{G+1-x}\right),$$

where $G = g^{-1}$, while Rényi ([Re57]) proved that the measure

$$\mu(A) = \int_A dx/x$$

is invariant and ergodic for the backward continued fractions map.
Anyway, none of the constructions used to obtain the densities above applies to the Nakada's α-expansions with $\alpha \in (0, \sqrt{2}-1)$; finding the explicit invariant density corresponding to these values of α it is still an open problem.

To get an idea of the density of the invariant measure, we could use Choe's computational approach [Ch00]. Using the help of the program R [R], we have studied and represented, for particular sets D, the distribution of the orbit $\{x, T_D x, T_D^2 x, \dots\}$ of a random point x under the map T_D.
Once the ergodicity of the maps is known, the procedure used in the codes is a straightforward application of Birkhoff's Ergodic Theorem. In particular we use that, for a measurable set A and sufficiently large n

$$\mu(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} 1_A(T^i(x)),$$

where 1_A is the characteristic function of A.
Of course, this will only give a rough idea of the invariant densities. However it still seems to be a valuable method. To check the correctness of the code and see how well the algorithm works, we first checked it on some well-known examples: the regular continued fraction expansion and the Odd-continued fraction expansions. The results are given in Figure 2.2, where the curves drawn in the graphics represent, respectively, the Gauss density (Fig. 2.2.a) and the Odd-CF density (Fig. 2.2.b).

All the other results are relative to distributions with unknown densities, and we compare them with the Gauss density. Note, in particular, that Figures 2.3.c and 2.3.d show that, if a neighborhood of 1 is in D, then 1 is an attractor for the points of the orbit, and the invariant measure seems to be infinite.
Finally, from Figures 2.3.b, it might seem that the Gauss measure is the invariant measure for the maps T_D with $D = [1/95, 1/94)$. It can be very easily checked by direct calculation that this is not correct. However, in view of the apparent similarity of the two

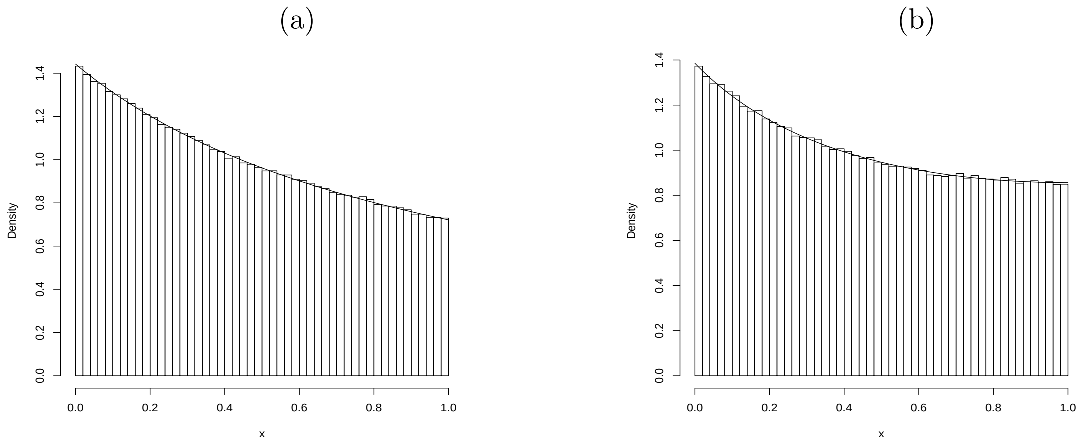

Figure 2.2: Invariant measures for (a) Regular continued fractions and (b) Odd continued fractions.

distributions, we have compared the empirical density deduced by the histogram, with the theoretical value of the Gauss density, using the Kolmogorov-Smirnov [BF51] statistical test. The result was highly significant against the hypothesis of equality, i.e., it confirmed the theoretical results that the two measures do not coincide.
For completeness, we will give the R-codes in Appendix A.

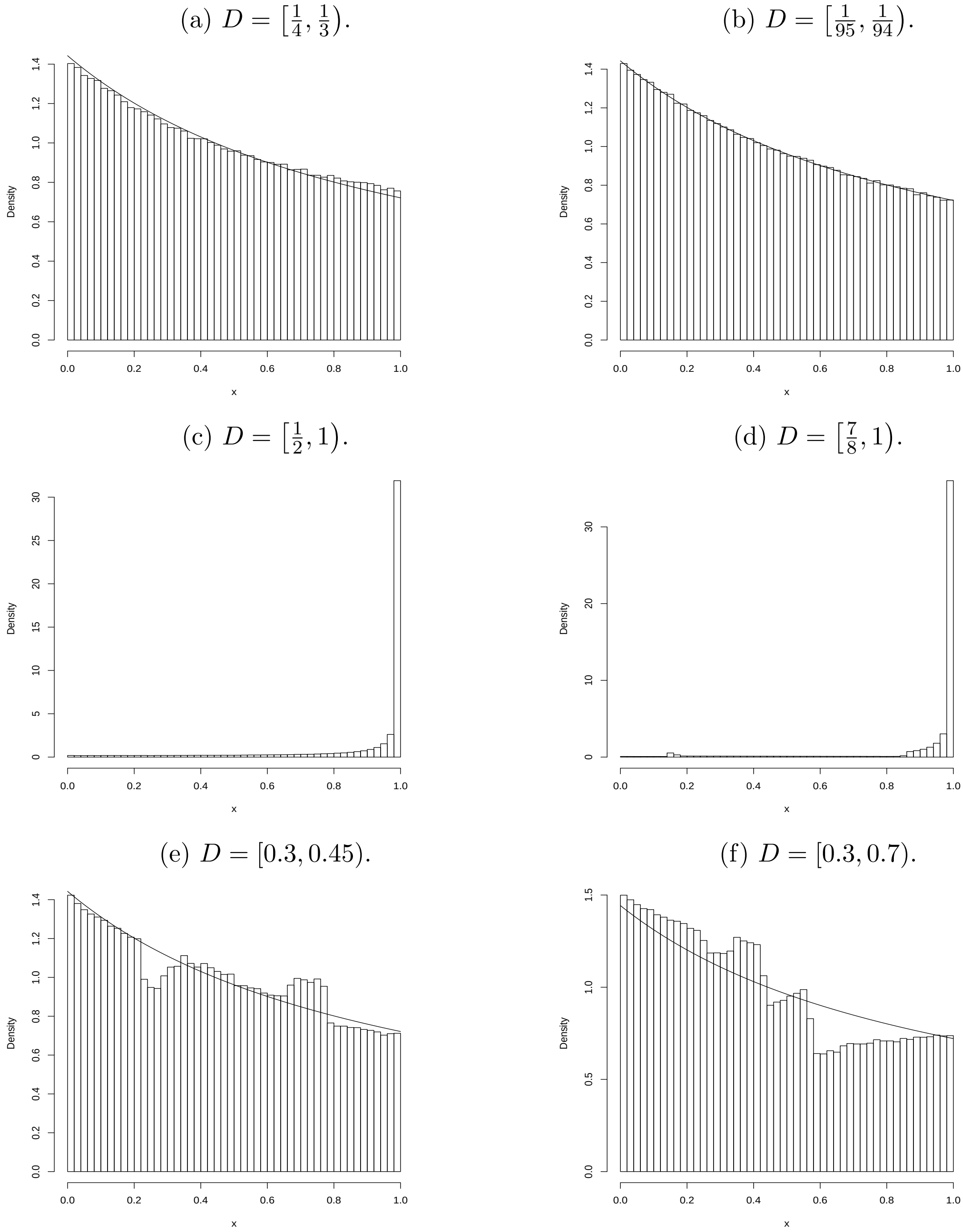

Figure 2.3: Some examples of D-distributions for particular choices of the set D.

Appendix A

R-codes

- R-code for the regular continued fraction expansion:

```
nsim <- 1000000
burnin <- 10000

# Initial conditions, where we choose a random x.

x <- rep(0,nsim+burnin)
x[1] <- runif(1)

# iteration of the regular continued fraction map.

for ( i in 1:(nsim+burnin-1)) {
        jt <- 1/x[i]
        x[i+1] <- jt - floor(jt)
}
x <- x[(burnin+1):length(x)]

# histogram

hist(x,nclass=50,prob=TRUE, main="")
curve((1/log(2))*1/(x+1),0,1,add=TRUE)
```

- R-code for the odd continued fraction expansion:

```
nsim <- 1000000
burnin <- 10000
G=(1+sqrt(5))/2
x <- rep(0,nsim+burnin)
x[1] <- runif(1)
for ( i in 1:(nsim+burnin-1)) {
```

```
        jt <- 1/x[i]
        x[i+1] <- if ( (floor(jt)%%2) < 0.5 ) floor(jt)+1-jt
                   else jt - floor(jt)
}
x <- x[(burnin+1):length(x)]
hist(x,nclass=50,prob=TRUE,main="")
curve((1/(G+1-x)+1/(G+x-1))/(3*log(G)),0,1,add=TRUE)
```

- R-code for the D-continued fractions with $D = [\frac{1}{4}, \frac{1}{3})$:

```
x <- rep(0,nsim+burnin)
x[1] <- runif(1)
D <- c(1/4,1/3)
for ( i in 1:(nsim+burnin-1)) {
        jt <- 1/x[i]
        x[i+1] <- if ( (x[i]>=D[1]) && (x[i] < D[2]) )
                      floor(jt)+1-jt
                   else jt - floor(jt)
}
x <- x[(burnin+1):length(x)]
hist(x,nclass=50,prob=TRUE,main="")
curve((1/log(2))*(1/(x+1)),0,1,add=TRUE)
```

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