THESIS FOR THE DEGREE OF MASTER

CORRESPONDENCE BETWEEN CUBIC ALGEBRAS AND TWISTED CUBIC FORMS

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Preface

The correspondence between cubic algebras and twisted binary cubic forms was discovered by Gan, Gross and Savin in [GGS]. It states that:

Proposition 0.0.1. *There is a bijection between the set of* $\mathrm{GL}_2(\mathbf{Z})$*-orbits on the space of twisted binary cubic forms with integer coefficients and the set of isomorphism classes of cubic algebras over* $\mathbf{Z}$*.*

Deligne explained in the letters [Del1] and [Del2] how to generalize this correspondence to any base schemes. In fact, he established equivalences between the following three kinds of categories, with morphisms being isomorphisms:

1. Twisted cubic forms, i.e. pairs (V, p), with V a vector bundle of rank 2 over S and $p \in \Gamma(S, \mathrm{Sym}^3(V) \otimes (\wedge^2 V)^{-1})$.

2. Geometric cubic forms, i.e. triples $(P, \mathcal{O}(1), a)$, in which $\pi : P \to S$ is a family of genus 0 curves, $\mathcal{O}(1)$ is an invertible sheaf on P of relative degree 1 over S, and $a \in \Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2 \pi_* \mathcal{O}(1))^{-1})$.

3. Cubic algebras, i.e. vector bundles over S of rank 3 endowed with a commutative algebra structure.

In this thesis, we give a detailed proof of the above equivalences following Deligne's ideas sketched in [Del1] and [Del2].

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Chapter 1

Main result

In our treatise, we use the language of algebraic stacks freely. The standard reference to algebraic stacks is [L-M], but one can also consult [Vis].

1.1 Basic definitions

Let $\mathcal{S}$ be the category of all schemes, we will endow it with Zariski topology.

Definition 1.1.1. Suppose that S is an object of $\mathcal{S}$. A *twisted cubic form* over S is a pair (V, p), where V is a locally free sheaf of $\mathcal{O}_S$-modules of rank 2 and $p \in \Gamma(S, \mathrm{Sym}^3(V) \otimes (\wedge^2 V)^{-1})$.

Definition 1.1.2. We define the *category* $\mathscr{F}$ *of twisted cubic forms* as follows. The objects of $\mathscr{F}$ are twisted cubic forms (V, p) over objects S in $\mathcal{S}$. A morphism from a twisted cubic form (V_1, p_1) over S_1 to another twisted cubic form (V_2, p_2) over S_2 consists of a morphism $f : S_1 \to S_2$ and an isomorphism $g : V_1 \to f^*V_2$ such that $g^*(f^*(p_2)) = p_1$. The functor $q : \mathscr{F} \to \mathcal{S}$ taking a twisted cubic form (V, p) over S to S makes $\mathscr{F}$ into a category over $\mathcal{S}$.

Definition 1.1.3. Suppose that S is an object of $\mathcal{S}$. By *a family of genus* 0 *curves* over S we mean a proper, smooth morphism $\pi : P \to S$ whose fibers over the geometric points are isomorphic to projective lines.

Definition 1.1.4. Suppose that S is an object of $\mathcal{S}$. A *geometric cubic form* over S is a triple $(P,\, \mathcal{O}_P(1),\, a)$, where $\pi : P \to S$ is a family of genus 0 curves, $\mathcal{O}_P(1)$ is an invertible sheaf on P of relative degree 1 over S, and $a \in \Gamma(P,\, \mathcal{O}_P(3) \otimes \pi^*(\wedge^2 \pi_*(\mathcal{O}_P(1)))^{-1})$.

Remark 1.1.5. The existence of the invertible sheaf $\mathcal{O}_P(1)$ assures that over every closed point $x \in S$, $\pi^{-1}(x)$ is isomorphic to $\mathbf{P}^1_{k(x)}$, where $k(x) = \mathcal{O}_{S,x}/\mathfrak{m}_x$.

Definition 1.1.6. We define the *category $\mathscr{G}$ of geometric cubic forms* as follows. The objects of $\mathscr{G}$ are geometric cubic forms $(P,\, \mathcal{O}_P(1),\, a)$ over objects S in $\mathcal{S}$. A morphism from a geometric cubic form $(P_1, \mathcal{O}_{P_1}(1), a_1)$ over S_1 to a geometric cubic form $(P_2, \mathcal{O}_{P_2}(1), a_2)$ over S_2 consists of two morphisms $f : S_1 \to S_2$, $g : P_1 \to P_2$ and an isomorphism $h : \mathcal{O}_{P_1}(1) \to g^*\mathcal{O}_{P_2}(1)$ such that the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{g} & P_2 \\ \downarrow & & \downarrow \\ S_1 & \xrightarrow{f} & S_2 \end{array}$$

is Cartesian, and $a_1 = h^*(g^*(a_2))$. The functor $q : \mathscr{G} \to \mathcal{S}$ taking a geometric cubic form $(P, \mathcal{O}_P(1), a)$ over S to S makes $\mathscr{G}$ into a category over $\mathcal{S}$.

Definition 1.1.7. A *cubic algebra* A over S is a sheaf over S of commutative unital $\mathcal{O}_S$-algebras, locally free of rank 3 as an $\mathcal{O}_S$-module.

Definition 1.1.8. We define the *category $\mathscr{A}$ of cubic algebras* as follows. The objects of $\mathscr{A}$ are cubic algebras A over objects S in $\mathcal{S}$. A morphism from a cubic algebra A_1 over S_1 to a cubic algebra A_2 over S_2 consists of a morphism $f : S_1 \to S_2$ and an isomorphism $g : A_1 \to f^*A_2$ as cubic algebras over S_1. The functor $q : \mathscr{A} \to \mathcal{S}$ taking a cubic algebra A over S to S makes $\mathscr{A}$ into a category over $\mathcal{S}$.

Proposition 1.1.9. *The categories $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{A}$ are all categories fibered in groupoids over $\mathcal{S}$.*

Proof. The main reason for this result is the existence of a well defined pull back. As an illustration, we verify for the category $\mathscr{F}$ the two axioms defining a category fibered in groupoids over $\mathcal{S}$. (For the two axioms, the reader is referred to chapter 2 in [L-M].)

(1) Suppose that $f : S_1 \to S_2$ is a morphism in $\mathcal{S}$ and (V_2, p_2) is a twisted cubic form over S_2. Define $V_1 = f^*V_2,\ p_1 = f^*p_2$, then (V_1, p_1) is a twisted cubic form over S_1. The pair of morphisms consisting of f and the identity map $\mathrm{id} : V_1 \to f^*V_2$ defines a morphism φ from the twisted cubic form (V_1, p_1) over S_1 to (V_2, p_2) over S_2 such that $q(\varphi) = f$.

(2) Suppose that (V_i, p_i) are twisted cubic forms over S_i, $i = 1, 2, 3$. Suppose that for $i = 1, 2$, we are given morphisms $f_i : S_i \to S_3$, isomorphisms $h_i : V_i \to f_i^*V_3$ defining a morphism from the twisted cubic form (V_i, p_i) over S_i to (V_3, p_3) over S_3. Suppose that $f : S_1 \to S_2$ is a morphism satisfying $f_1 = f_2 \circ f$, then $f_1^*V = (f_2 \circ f)^*(V) = f^*(f_2^*(V))$. Pulling back the isomorphism $h_2 : V_2 \to f_2^*V$ by f to S_1, we get an isomorphism $h_2' : f^*V_2 \to f^*(f_2^*(V)) = f_1^*V$. Define the isomorphism $h = h_2'^{-1} \circ h_1 : V_1 \to f^*V_2$, then h and f defines the desired morphism from the twisted cubic form (V_1, p_1) over S_1 to (V_2, p_2) over S_2.

□

Proposition 1.1.10. *The categories fibered in groupoids $\mathscr{F}, \mathscr{G}$ and $\mathscr{A}$ are stacks over $\mathcal{S}$.*

Proof. Since we work with Zariski topology, the two axioms of stacks holds automatically by the definition of sheaf. □

Remark 1.1.11. In fact, it will be proved in §2.2 and §3.2 that $\mathscr{A}$ and $\mathscr{F}$ are both smooth Artin stacks.

1.2 Statement of main results

Theorem 1.2.1. *The stacks $\mathscr{F}$ and $\mathscr{G}$ are equivalent.*

Proof. Suppose that (V, p) is a twisted cubic form over S, i.e. V is a locally free sheaf of $\mathcal{O}_S$-modules of rank 2 and p is a section of $\mathrm{Sym}^3(V) \otimes_{\mathcal{O}_S} (\wedge^2 V)^{-1}$. Define $P = \mathbf{P}(V) = \mathbf{Proj}(\bigoplus_{n=0}^{\infty} \mathrm{Sym}^n V)$, then $\pi : P = \mathbf{P}(V) \to S$ is a family of genus 0 curves. Let $\mathcal{O}(1)$ be Serre's twisting sheaf of P, obviously $\deg(\mathcal{O}(1)) = 1$.

By [Hart], chap. III, theorem 5.1(a) and the definition of Serre's twisting sheaf, there is a natural identification $\pi_*\mathcal{O}(1) = V$. As a consequence,

$$\Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*(\mathcal{O}(1)))^{-1}) = \Gamma(S, \mathrm{Sym}^3(V) \otimes (\wedge^2 V)^{-1}).$$

Denote by a the element in $\Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*(\mathcal{O}(1))^{-1})$ that corresponds to p. The triple $(P, \mathcal{O}(1), a)$ defines a geometric cubic form over S, which is canonically associated to the twisted cubic form (V, p).

Conversely, given a geometric cubic form $(P, \mathcal{O}(1), a)$ over S, where $\pi : P \to S$ is a family of genus 0 curves, $\mathcal{O}(1)$ is an invertible sheaf of relative degree 1, $a \in \Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*(\mathcal{O}(1))^{-1})$. Let $V = \pi_*(\mathcal{O}(1))$, then V is a locally free sheaf of rank 2 over S. As above, we have $\Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*(\mathcal{O}(1))^{-1}) = \Gamma(S, \mathrm{Sym}^3(V) \otimes (\wedge^2 V)^{-1})$. Let p be the element in $\Gamma(S, \mathrm{Sym}^3(V) \otimes (\wedge^2 V)^{-1})$ corresponding to $a \in \Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*(\mathcal{O}(1))^{-1})$. Then (V, p) defines a twisted cubic form on S.

It is obvious that the above two constructions are inverse to each other, so we get an equivalence of stacks $\mathscr{F}$ and $\mathscr{G}$. $\square$

The main result in this thesis is

Main Theorem 1. *The stacks $\mathscr{A}$ and $\mathscr{G}$ are equivalent.*

As a corollary of theorem 1.2.1 and the main theorem 1, we have

Corollary 1.2.2. *The stacks $\mathscr{A}$ and $\mathscr{F}$ are equivalent.*

1.3 Sketch of the proof

To prove the main theorem, we need to construct two 1-morphisms $F_1 : \mathscr{G} \to \mathscr{A}$ and $F_2 : \mathscr{A} \to \mathscr{G}$ inverse to each other.

Suppose that $(P, \mathcal{O}(1), a)$ is a geometric cubic form over S. Let $\pi : P \to S$ be the structural morphism. Denote $\mathcal{J} := \mathcal{O}(-3) \otimes \pi^*(\wedge^2\pi_*(\mathcal{O}(1)))$. Since $a \in \Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*(\mathcal{O}(1)))^{-1})$, we can define a complex $\mathcal{J} \xrightarrow{a} \mathcal{O}$. Let

$$A := \mathbf{R}^0\pi_*(\mathcal{J} \xrightarrow{a} \mathcal{O}),$$

where $\mathbf{R}^0\pi_*$ denotes the 0-th hypercohomology group. (For the definition of hypercohomology group, the reader is referred to [EGA 0] §12.4.1.) It can be proved that A is a locally free sheaf of $\mathcal{O}_S$-modules of rank 3. What's more, A can be endowed with a product structure. It comes from a product on the complex $\mathcal{J} \xrightarrow{a} \mathcal{O}$, deduced from the $\mathcal{O}$-module structure of $\mathcal{J}$. Calculating with Čech cocycle, we can prove that A becomes a cubic algebra over S with this product structure. The details of this construction can be found in §4.1.

This construction commutes with arbitrary base change, so we get a 1-morphism $F_1 : \mathscr{G} \to \mathscr{A}$.

The construction of the 1-morphism F_2 is more complicated. First of all, we restrict ourselves to the sub-category fibered in groupoids $\mathscr{V}$ of Gorenstein cubic algebras. Let A be a Gorenstein cubic algebra over S. Then we can associate to A a family of genus 0 curves $\pi : P \to S$ together with a closed immersion $\phi : \mathrm{Spec}(A) \to P$. Let D be the image of ϕ. It is an effective relative divisor of P over S of degree 3 and is isomorphic to $\mathrm{Spec}(A)$. Define $\mathcal{O}(1) := \mathcal{O}(D) \otimes \Omega^1_{P/S}$. The section 1 of $\mathcal{O}(D)$ defines a section a of $\mathcal{O}(1) \otimes (\Omega^1_{P/S})^\vee \cong \mathcal{O}(3) \otimes \pi^*(\wedge^2(\pi_*\mathcal{O}(1)))^{-1}$. In this way we obtain a geometric cubic form $(P, \mathcal{O}(1), a)$ over S from a Gorenstein cubic algebra A over S. This construction commutes with arbitrary base change, so we get a 1-morphism of algebraic stacks $F_2' : \mathscr{V} \to \mathscr{G}$. In the general case, we use the fact that $\mathscr{V}$ is an open sub algebraic stack of $\mathscr{A}$ and its complement has codimension 4 in $\mathscr{A}$ to extend F_2' to a 1-morphism $F_2 : \mathscr{A} \to \mathscr{G}$. The details of this construction can be found in §4.2.

In §4.3, we prove that the two 1-morphisms are inverse to each other. The idea is to restrict to the "nice" open sub algebraic stacks of primitive geometric cubic forms and Gorenstein cubic algebras. In these two cases, we have concrete and simple descriptions of the functors F_1 and F_2 respectively.

Chapter 2

The algebraic stack of cubic algebras

2.1 Classification over algebraically closed fields

In this section we suppose that k is an algebraically closed field.

Lemma 2.1.1. *Suppose that A is a cubic algebra over a commutative ring R, free as an R-module. Suppose further that $A/R \cdot 1$ is free. There exist $\alpha,\ \beta \in A$ such that $(1,\ \alpha,\ \beta)$ is a basis of A over R and $\alpha\beta \in R$.*

Proof. Take $x,\ y \in A$ such that $(1,\ x,\ y)$ forms a basis of A. Suppose that $xy = ax+by+c$, $a, b, c \in R$, then $(x-b)(y-a) = ab+c \in R$. Obviously $(1,\ x-b,\ y-a)$ is again a basis of A over R, so we can take $\alpha = x-b,\ \beta = y-a$. □

We call a basis $(1, \alpha, \beta)$ a *good basis* of A over R if $\alpha\beta \in R$.

Theorem 2.1.2. *A cubic algebra over k is isomorphic to one of the following:*

(1) $A_1 = k \times k \times k$.

(2) $A_2 = k \times (k[\varepsilon]/(\varepsilon^2))$.

(3) $A_3 = k[\varepsilon]/(\varepsilon^3)$.

(4) $A_4 = k[\alpha, \beta]/(\alpha^2, \alpha\beta, \beta^2)$.

Proof. Suppose that A is a cubic algebra over k. For any $x \in A$, let $P(T)$ be the minimal polynomial of x.

(i) Suppose that there exists $x \in A$ such that $\deg(P(T)) = 3$, then $A = k[x] \cong k[T]/(P(T))$ by dimension reason. Since k is algebraically closed, $P(T) = (T - a_1)(T - a_2)(T - a_3)$, for some $a_1, a_2, a_3 \in k$.

If all the three roots of $P(T)$ are distinct, we get $A \cong k[T]/((T - a_1)(T - a_2)(T - a_3)) \cong k \times k \times k$.

If there is one root of $P(T)$ appearing with multiplicity 2, for example $a_1 = a_2 \neq a_3$, we get $A \cong k[T]/((T - a_1)^2(T - a_3)) \cong k \times (k(\varepsilon)/(\varepsilon^2))$

If $a_1 = a_2 = a_3$, $A \cong k[T]/(T - a_1)^3 \cong k[\varepsilon]/(\varepsilon^3)$.

(ii) Suppose that for any $x \in A$, $\deg(P(T)) \leq 2$. By lemma 2.1.1, we can find a basis $(1,\ \alpha,\ \beta)$ of A over k such that $x := \alpha\beta \in k$. By assumption, there are $a,\ b,\ c,\ d \in k$ such that $\alpha^2 - a\alpha - c = 0$ and $\beta^2 - b\beta - d = 0$.

Consider the homomorphism of algebras $\rho : A \to \mathrm{End}_k(A)$ defined by $\rho(m)(n) = mn,\ \forall m, n \in A$. Under the basis $(1,\ \alpha,\ \beta)$, we find that

$$\rho(\alpha) = \begin{pmatrix} 0 & 1 & \\ c & a & \\ x & & 0 \end{pmatrix}, \quad \rho(\beta) = \begin{pmatrix} 0 & 0 & 1 \\ x & 0 & \\ d & & b \end{pmatrix},$$

and $\rho(1) = \mathrm{id}$. By calculation,

$$\rho(\alpha\beta) = \rho(\alpha)\rho(\beta) = \begin{pmatrix} x & & \\ ax & 0 & c \\ & & x \end{pmatrix}, \quad \rho(\beta\alpha) = \rho(\beta)\rho(\alpha) = \begin{pmatrix} x & & \\ & x & \\ bx & d & 0 \end{pmatrix}.$$

By assumption, $\rho(\alpha\beta) = \rho(\beta\alpha) = \rho(x) = \mathrm{diag}(x, x, x)$. Compare these identities we get $x = c = d = 0$. So $\alpha\beta = 0,\ \alpha^2 = a\alpha,\ \beta^2 = b\beta$.

We claim that $a = b = 0$. In fact, for any $r, s \in k$, by assumption the element $r\alpha + s\beta$

satisfies an equation

$$(r\alpha + s\beta)^2 + m(r\alpha + s\beta) + n = 0,$$

for some $m, n \in k$ which depend on r and s. Expanding the left hand side, we get equalities

$$0 = r^2 a + mr = s^2 b + ms = n,$$

for any $r, s \in k$, so we must have $a = b = m = 0$.

Now we have $\alpha^2 = \beta^2 = \alpha\beta = 0$, so $A = k[\alpha, \beta]/(\alpha^2, \beta^2, \alpha\beta)$. For any $r,\ s,\ t \in k$, $(r + s\alpha + t\beta)^2 = -r^2 + 2r(r + s\alpha + t\beta)$, so the minimal polynomial of any element of A will have degree at most 2.

□

Remark 2.1.3. Observe that we don't use the hypothesis that k is algebraically closed in step (ii) of the above proof. So if A is a cubic algebra over an arbitrary field k such that every element in A has minimal polynomial of degree at most 2, then $A \cong k[\alpha, \beta]/(\alpha^2, \alpha\beta, \beta^2)$.

Theorem 2.1.4. *We have the automorphism groups of the cubic algebras:*

(1) $\mathrm{Aut}_k(k \times k \times k) \cong \mathfrak{S}_3$.

(2) $\mathrm{Aut}_k(k \times (k(\varepsilon)/(\varepsilon^2))) \cong k^\times$.

(3) $\mathrm{Aut}_k(k(\varepsilon)/(\varepsilon^3)) \cong \left\{ \left(\begin{smallmatrix} b & c \\ & b^2 \end{smallmatrix} \right) \mid b \in k^\times,\ c \in k \right\}$.

(4) $\mathrm{Aut}_k(k[\alpha, \beta]/(\alpha^2, \alpha\beta, \beta^2)) \cong \mathrm{GL}_2(k)$.

Proof. (1) For $A = k \times k \times k$, take $e_1 = (1, 0, 0),\ e_2 = (0, 1, 0),\ e_3 = (0, 0, 1)$. Then

$$e_i^2 = e_i, \quad e_i e_j = 0, \quad \forall i, j = 1, 2, 3;\ i \neq j,$$

i.e. e_i are idemptents. So $f \in \mathrm{Aut}_k(A)$ if and only if $f(e_i),\ i = 1, 2, 3$ are also idemptent. It is obvious that for any $\sigma \in \mathfrak{S}_3$, the homomorphism defined by $\sigma(e_i) = e_{\sigma(i)},\ i = 1, 2, 3$, is an automorphism of A over k. Conversely, suppose that $x =$

$x_1e_1 + x_2e_2 + x_3e_3,\ x_1, x_2, x_3 \in k$ satisfies $x^2 = x$, a simple calculation tells us that $x_i^2 = x_i$, so $x_i = 0$ or 1, $i = 1, 2, 3$. Taking into account that $f(e_i) \neq 0,\ f(e_i)f(e_j) = 0,\ \forall i, j = 1, 2, 3;\ i \neq j$, we get that $f(e_i) = e_{\sigma(i)},\ i = 1, 2, 3$, for some $\sigma \in \mathfrak{S}_3$. So $\mathrm{Aut}_k(k \times k \times k) \cong \mathfrak{S}_3$.

(2) For $A = k \times (k[\varepsilon]/(\varepsilon^2))$, $1 = (1, 1)$. Take $\alpha = (1, 0),\ \beta = (0, \varepsilon)$, then $(1, \alpha, \beta)$ forms a basis of A over k and $\alpha^2 = \alpha,\ \beta^2 = 0,\ \alpha\beta = 0$. So $f \in \mathrm{Aut}_k(A)$ if and only if

$$f(1) = 1,\ f(\alpha)^2 = f(\alpha),\ f(\beta)^2 = 0,\ f(\alpha)f(\beta) = 0.$$

Suppose that $f(\alpha) = a\alpha + b\beta + c,\ f(\beta) = m\alpha + n\beta + p,\ a, b, c, m, n, p \in k$, then $f(\beta)^2 = 0$ is equivalent to $m = p = 0$. And $n \neq 0$ since f is an automorphism. By $f(\alpha)f(\beta) = 0$, we get $c = 0$. Finally by $f(\alpha)^2 = f(\alpha)$ we get $a^2 = a,\ b = 0$, so $a = 0$ or 1. But if $a = 0$, then $f(A) \subset k \cdot 1 \oplus k\beta$, contradiction to the assumption that f is an automorphism. So $a = 1$. In conclusion, $f(1) = 1,\ f(\alpha) = \alpha,\ f(\beta) = n\beta,\ n \in k^\times$. It is easy to check that any such f defines an automorphism of A over k, so $\mathrm{Aut}_k(k \times (k[\varepsilon]/(\varepsilon^2))) \cong k^\times$.

(3) For $A = k[\varepsilon]/(\varepsilon^3)$. Let $f \in \mathrm{Aut}_k(A)$, then $f(1) = 1$. Suppose that $f(\varepsilon) = a + b\varepsilon + c\varepsilon^2,\ a, b, c \in k$. By $f(\varepsilon)^3 = f(\varepsilon^3) = 0$, we get $a = 0$. By $f(\varepsilon^2) = f(\varepsilon)^2 = b^2\varepsilon^2$, we get $b \neq 0$ since f is an automorphism. In conclusion, under the base $(1, \varepsilon, \varepsilon^2)$, f will be of the form

$$\begin{pmatrix} 1 & & \\ & b & c \\ & & b^2 \end{pmatrix}, \quad b \in k^\times,\ c \in k.$$

It is easy to check that any such f defines an automorphism of A over k. So

$$\mathrm{Aut}_k(k(\varepsilon)/(\varepsilon^3)) \cong \left\{ \left(\begin{smallmatrix} b & c \\ & b^2 \end{smallmatrix}\right) \mid b \in k^\times,\ c \in k \right\}.$$

(4) For $A = k[\alpha, \beta]/(\alpha^2, \alpha\beta, \beta^2)$. Let $f \in \mathrm{Aut}_k(A)$, then $f(1) = 1$. Suppose $f(\alpha) = a + b\alpha + c\beta,\ f(\beta) = m + n\alpha + p\beta,\ a, b, c, m, n, p \in k$. By $f(\alpha)^2 = 0$, we get $a = 0$. By $f(\beta)^2 = 0$, we get $m = 0$. Because f is an automorphism, the matrix $\left(\begin{smallmatrix} b & c \\ n & p \end{smallmatrix}\right)$ is

invertible. So under the basis $(1, \alpha, \beta)$, f will be of the form

$$\begin{pmatrix} 1 & & \\ & b & c \\ & n & p \end{pmatrix} \in \mathrm{GL}_3 \, .$$

It is easy to see that any f of this form defines an automorphism of A over k. So $\mathrm{Aut}_k(A) \cong \mathrm{GL}_2(k)$.

□

2.2 Smoothness and dimension

Definition 2.2.1. Let S be an arbitrary scheme, a *based cubic algebra* over S is a pair (A, ϕ), where A is a cubic algebra over S, $\phi : \mathcal{O}_S^{\oplus 3} \to A$ is an isomorphism of $\mathcal{O}_S$-modules such that $\phi(1, 0, 0) = 1$.

Definition 2.2.2. Let S be an arbitrary scheme, a *good based cubic algebra* over S is a pair (A, ϕ), where A is a cubic algebra over S, $\phi : \mathcal{O}_S^{\oplus 3} \to A$ is an isomorphism of $\mathcal{O}_S$-modules such that $\phi(1, 0, 0) = 1$, $\phi(0, 1, 0)\phi(0, 0, 1) \in \mathcal{O}_S(S)$.

Definition 2.2.3. Let $\mathscr{B}^1$ be the *category of based cubic algebras*. Its objects are the based cubic algebras over S. A morphism from a based cubic algebra (A_1, ϕ_1) over S_1 to (A_2, ϕ_2) over S_2 consists of a morphism $f : S_1 \to S_2$, an isomorphism $h : A_1 \to f^*A_2$ such that $\phi_1 = h^{-1} \circ (f^*(\phi_2))$.

Definition 2.2.4. Let $\mathscr{B}^{1,g}$ be the *category of good based cubic algebras*. Its objects are the good based cubic algebras over S. A morphism from a good based cubic algebra (A_1, ϕ_1) over S_1 to (A_2, ϕ_2) over S_2 consists of a morphism $f : S_1 \to S_2$, and an isomorphism $h : A_1 \to f^*A_2$ such that $\phi_1 = h^{-1} \circ (f^*(\phi_2))$.

Proposition 2.2.5. *The categories $\mathscr{B}^1$ and $\mathscr{B}^{1,g}$ are both categories fibered in groupoids over $\mathcal{S}$.*

Proof. This is a simple consequence of the existence of a well defined pull-back. □

Proposition 2.2.6. *The category fibered in groupoids $\mathscr{B}^{1,g}$ is representable by $\mathbf{A}^4_{\mathbf{Z}}$ and $\mathscr{B}^1$ is representable by $\mathbf{A}^6_{\mathbf{Z}}$.*

Proof. We reproduce the proof in [Poonen], proposition 5.1. In [GGS], it was shown that if $(1, \alpha, \beta)$ is a good basis of a cubic algebra A over a ring R, then the conditions of associativity and commutativity show that

$$\begin{aligned} \alpha^2 &= -ac + b\alpha - a\beta \\ \beta^2 &= -bd + d\alpha - c\beta \\ \alpha\beta &= -ad, \end{aligned} \tag{2.1}$$

for some $a, b, c, d \in R$, and conversely by the multiplication table any $a, b, c, d \in R$ defines a good based cubic algebra. This correspondence is obviously bijective, from which we deduce $\mathscr{B}^{1,g} = \underline{\mathbf{A}^4_{\mathbf{Z}}}$.

There is a left action of $\mathbb{G}_a^2$ on $\mathscr{B}^1$ by which $(m, n) \in \mathbb{G}_a^2(S)$ maps an algebra A over S with basis $(1, \alpha, \beta)$ to A with basis $(1, \alpha + m, \beta + n)$. This action gives an isomorphism:

$$\mathbb{G}_a^2 \times \mathscr{B}^{1,g} \to \mathscr{B}^1, \tag{2.2}$$

so $\mathscr{B}^1 = \underline{\mathbf{A}^6_{\mathbf{Z}}}$. □

Denote by H the closed subgroup scheme of GL_3 stabilizing the element $(1, 0, 0)$. H acts naturally on the category fibered in groupoids $\mathscr{B}^1$. For (A, ϕ) a based cubic algebra over S, let $h\phi : \mathcal{O}_S^{\oplus 3} \to A$ be defined as $h\phi(x) = \phi(hx)$, $\forall x \in \mathcal{O}_S^{\oplus 3}$, $h \in H$. Then $h(A, \phi) := (A, h\phi)$.

Theorem 2.2.7. *The category fibered in groupoids $\mathscr{A}$ is a smooth Artin stack.*

Proof. Since every cubic algebra is locally free, it admits a basis locally. So we have $\mathscr{A} = [\mathscr{B}^1/H]$ as stack over $\mathcal{S}$. By [L-M], example 4.6.1, $[\mathscr{B}^1/H] = [\underline{\mathbf{A}^6_{\mathbf{Z}}}/H]$ is an Artin stack. Since $\mathbf{A}^6_{\mathbf{Z}}$ is smooth and H is a smooth group scheme over $\mathbf{Z}$, $[\mathscr{B}^1/H]$ is a smooth Artin stack. So $\mathscr{A}$ is a smooth Artin stack. □

Proposition 2.2.8. *The dimension of $\mathscr{A}$ is 0.*

Proof. Since $\dim(\mathbf{A}^6_{\mathbf{Z}}) = \dim(H) = 6$ and $\mathscr{A} = [\underline{\mathbf{A}^6_{\mathbf{Z}}}/H]$, we get $\dim(\mathscr{A}) = \dim(\mathbf{A}^6_{\mathbf{Z}}) - \dim(H) = 0$. □

2.3 Gorenstein cubic algebras

Let A be a cubic algebra over S, i.e. A is a sheaf of commutative unital $\mathcal{O}_S$-algebras, locally free of rank 3 as an $\mathcal{O}_S$-module. Let $X = \mathrm{Spec}(A)$, let $f : X \to S$ be the structure morphism. Then f is a finite flat morphism of degree 3 and $A = f_*\mathcal{O}_X$.

Since f is a finite flat morphism, the sheaf $\mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{O}_X,\, \mathcal{O}_S)$ is a coherent $f_*\mathcal{O}_X$-module and there is a coherent sheaf ω on X such that $f_*\omega = \mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{O}_X,\, \mathcal{O}_S)$.

Definition 2.3.1. The cubic algebra A is called *Gorenstein* over S if ω is an invertible sheaf on X.

Example 2.3.1. Suppose that k is a field, let A be a cubic algebra over k.

(1) If there exists an element $x \in A$ whose minimal polynomial is of degree 3, $A = k[x]$ by dimension reason. Define $f \in A^\vee$ by $f(1) = 0,\ f(x) = 0,\ f(x^2) = 1$. A simple calculation shows that $(f, xf, x^2 f)$ is a basis of $A^\vee$ over k. So $A^\vee$ is a free A-module with generator f and A is Gorenstein.

(2) If every element in A has minimal polynomial of degree at most 2, remark 2.1.3 shows that $A = k[\alpha, \beta]/(\alpha^2, \alpha\beta, \beta^2)$. Define $h \in A^\vee$ to be $h(a \cdot 1 + b\alpha + c\beta) = a$. It is easy to check that for any $x \in A$, $f \in A^\vee$, xf lies in the k-linear span of h in $A^\vee$. So $A^\vee$ can not be a free A-module of rank 1 and hence is not Gorenstein.

In particular, in the classification result of theorem 2.1.2, the cubic algebras $A_1,\ A_2,\ A_3$ are Gorenstein, while A_4 is not Gorenstein.

Since the pull-back of a Gorenstein cubic algebra is always Gorenstein, we can define the sub stack $\mathscr{V}$ of Gorenstein cubic algebras of $\mathscr{A}$. Let O the origin of $\mathbf{A}^4_{\mathbf{Z}}$, let V be the image of $\mathbb{G}_a^2 \times (\mathbf{A}^4_{\mathbf{Z}} \backslash O)$ in $\mathbf{A}^6_{\mathbf{Z}}$ under the isomorphism (2.2).

Theorem 2.3.2. *We have $\mathscr{V} = [\underline{V}/H]$ as stack. In particular, $\mathscr{V}$ is an open sub algebraic stack of $\mathscr{A}$ whose complement is of codimension 4.*

First of all, we prove the following lemma, which is a generalization of [GGS], proposition 5.2.

Lemma 2.3.3. *Suppose that S is a locally noetherian scheme and A is a cubic algebra over S. Then A is Gorenstein over S if and only if for every closed point x of S, the cubic algebra $A \otimes_{\mathcal{O}_S} k(x)$ is Gorenstein over* $\mathrm{Spec}(k(x))$.

Proof. The necessity is obvious. For the sufficiency, we need to prove that for any closed point x, $A_x^\vee$ is a projective A_x-module. So we can assume that $S = \mathrm{Spec}(R)$, R a noetherian local ring with residue field k. Let $M = \mathrm{Hom}_R(A, R)$, we need to prove that M is a projective A-module. Since A is a free R-module, M is a free R-module. By assumption, $A \otimes_R k$ is Gorenstein. By the result in example 2.3.1, $M \otimes_R k$ is a free $A \otimes_R k$-module of rank 1. Take $m \in M$ such that its image in $M \otimes_R k$ is a generator of $M \otimes_R k$ as an $A \otimes_R k$-module. Define $\varphi : A \to M$ by $\varphi(a) = am$, $\forall a \in A$. By Nakayama's lemma, φ is surjective. Denote its kernel by N. We have the exact sequence

$$0 \to N \to A \to M \to 0.$$

Applying the functor $\bullet \otimes_R k$ we get the exact sequence

$$\mathrm{Tor}_1^R(M, k) = 0 \to N \otimes_R k \to A \otimes_R k \xrightarrow{\varphi \otimes \mathrm{id}_k} M \otimes_R k \to 0,$$

the first equality is because M is a free R-module. Since $\varphi \otimes \mathrm{id}_k$ is an isomorphism, $N \otimes_R k = 0$. By Nakayama's lemma, $N = 0$. So M is a free A-module. ☐

Proof of theorem 2.3.2. Since $\mathscr{A} = [\mathbf{A}^6_{\mathbf{Z}}/H]$, we can suppose that the base scheme S is noetherian without any loss of generality. Suppose that A is a cubic algebra over S with good basis $(1, \alpha, \beta)$, $\alpha, \beta \in \mathcal{O}_S$. Then A together with this good basis corresponds uniquely to an S-point $Q = (a, b, c, d)$ of $\mathbf{A}^4_{\mathbf{Z}}$ as described in formula (2.1) in the proof of proposition 2.2.6. By lemma 2.3.3 and example 2.3.1, A is Gorenstein over S if and only if the image of Q doesn't intersect with the closed subscheme O of $\mathbf{A}^4_{\mathbf{Z}}$. So Q factors through the open subscheme $\mathbf{A}^4_{\mathbf{Z}} \backslash O$ and the category fibered in groupoids of Gorenstein cubic algebras with good basis is represented by the open subscheme $\mathbf{A}^4_{\mathbf{Z}} \backslash O$ of $\mathbf{A}^4_{\mathbf{Z}}$. Similar to the proof of the second conclusion in proposition 2.2.6, we have $\mathscr{V} = [\underline{\mathbb{G}_a^2 \times (\mathbf{A}^4_{\mathbf{Z}} \backslash O)}/H] = [\underline{V}/H]$.

The complement of $\mathscr{V}$ in $\mathscr{A}$ is $[\underline{\mathbb{G}_a^2 \times O}/H]$, which is of dimension $2 - 6 = -4$, so it is of codimension 4 in $\mathscr{A}$. $\square$

Chapter 3

The algebraic stack of twisted cubic forms

3.1 Classification over algebraically closed fields

Suppose that k is an algebraically closed field. A twisted cubic form over k is a pair $(V,\, p)$, where V is a vector space of dimension 2 over k and $p \in \mathrm{Sym}^3(V) \otimes_k (\wedge^2 V)^{-1}$. Two twisted cubic forms $(V_1,\, p_1)$ and $(V_2,\, p_2)$ are said to be isomorphic if there exists an isomorphism of vector spaces $f : V_1 \to V_2$ such that $p_1 = f^* p_2$.

Because all the 2-dimensional vector spaces over k are isomorphic, we only need to classify the cubic forms $(V,\, p)$ for a fixed vector space V. Fix a basis $(v_1,\, v_2)$ of V.

Theorem 3.1.1. *A twisted cubic form $(V,\, p)$ over k is isomorphic to one of the following:*

(1) $p_1 = v_1 v_2 (v_1 - v_2) \otimes_k (v_1 \wedge v_2)^{-1}$.

(2) $p_2 = v_1^2 v_2 \otimes_k (v_1 \wedge v_2)^{-1}$.

(3) $p_3 = v_1^3 \otimes_k (v_1 \wedge v_2)^{-1}$.

(4) $p_4 = 0$.

Proof. Obviously $(V,\, 0)$ is a twisted cubic form. Suppose that $p \in \mathrm{Sym}^3(V) \otimes_k (\wedge^2 V)^{-1}$, $p \neq 0$. Let $p = (a v_1^3 + b v_1^2 v_2 + c v_1 v_2^2 + d v_2^3) \otimes_k (v_1 \wedge v_2)^{-1},\ a, b, c, d \in k$. By assumption,

k is algebraically closed, so p can be written as

$$p = (x_1v_1 - y_1v_2)(x_2v_1 - y_2v_2)(x_3v_1 - y_3v_2) \otimes_k (v_1 \wedge v_2)^{-1},$$

for some $x_i, y_i \in k,\ i = 1, 2, 3$. Let $a_i = [x_i : y_i],\ i = 1, 2, 3$ be three points on $\mathbf{P}^1_k$.

(1) Suppose that $a_i,\ i = 1, 2, 3$ are distinct to each other. It is well known that there exists an element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PGL}_2(k)$ such that

$$\gamma(a_1) = [1 : 0],\ \gamma(a_2) = [0 : 1],\ \gamma(a_3) = [1 : 1],$$

where $\gamma(z) = \frac{az+b}{cz+d}$, $\forall z \in \mathbf{P}^1_k$. For any $\gamma' \in \mathrm{GL}_2(k)$ whose image in $\mathrm{PGL}_2(k)$ is γ, we have

$$\gamma'^*(v_1v_2(v_1 - v_2) \otimes_k (v_1 \wedge v_2)^{-1}) = m(\gamma')p,$$

for some $m(\gamma') \in k^\times$. Since k is a field, we can choose appropriately an element $\tilde{\gamma} \in \mathrm{GL}_2(k)$ whose image in $\mathrm{PGL}_2(k)$ is γ, such that

$$\tilde{\gamma}^*(v_1v_2(v_1 - v_2) \otimes_k (v_1 \wedge v_2)^{-1}) = p.$$

(2) Suppose that $a_1 = a_2 \neq a_3$. As in the above, there exists one element $\gamma \in \mathrm{PGL}_2(k)$ such that

$$\gamma(a_1) = \gamma(a_2) = [1 : 0],\ \gamma(a_3) = [0 : 1].$$

Similar to the case (1), we can choose an element $\tilde{\gamma} \in \mathrm{GL}_2(k)$ whose image in $\mathrm{PGL}_2(k)$ is γ, such that

$$\tilde{\gamma}^*(v_1^2v_2 \otimes_k (v_1 \wedge v_2)^{-1}) = p.$$

The cases $a_1 = a_3 \neq a_2$ and $a_2 = a_3 \neq a_1$ can be treated in the same way.

(3) Suppose that $a_1 = a_2 = a_3$. There exists one element $\gamma \in \mathrm{PGL}_2(k)$ such that

$$\gamma(a_1) = \gamma(a_2) = \gamma(a_3) = [1 : 0].$$

Similar to case (1), we can choose an element $\tilde{\gamma} \in \mathrm{GL}_2(k)$ whose image in $\mathrm{PGL}_2(k)$ is γ, such that

$$\tilde{\gamma}^*(v_1^3 \otimes_k (v_1 \wedge v_2)^{-1}) = p.$$

□

Theorem 3.1.2. *The automorphism groups of the above twisted cubic forms are*

(1) $\mathrm{Aut}_k((V, p_1)) \cong \mathfrak{S}_3$.

(2) $\mathrm{Aut}_k((V, p_2)) \cong k^\times$.

(3) $\mathrm{Aut}_k((V, p_3)) \cong \left\{ \begin{pmatrix} a & b \\ & a^2 \end{pmatrix} \in \mathrm{GL}_2(k) | \, a \in k^\times, \, b \in k \right\}$.

(4) $\mathrm{Aut}_k((V, p_4)) = \mathrm{GL}_2(k)$.

Proof. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(k)$, then $\gamma(v_1) = av_1 + cv_2$, $\gamma(v_2) = bv_1 + dv_2$.

(1) Suppose that $\gamma \in \mathrm{Aut}_k((V, \, p_1))$. Denote by X the zero section of the cubic form $v_1 v_2 (v_1 - v_2)$ over $\mathbf{P}(V) = \mathbf{P}^1_k$. Let $x_1 = [0 : 1]$, $x_2 = [1 : 0]$, $x_3 = [1 : 1]$, then $X = x_1 + x_2 + x_3$ as a divisor on $\mathbf{P}^1_k$. By assumption $\gamma^* p_1 = p_1$, so γ induces an automorphism of X. Since $\mathrm{Aut}(X) \cong \mathfrak{S}_3$, we obtain a homomorphism $j : \mathrm{Aut}_k((V, \, p_1)) \to \mathfrak{S}_3$.

We claim that j is injective. In fact, for $\gamma \in \mathrm{Aut}_k((V, \, p_1))$ such that $j(\gamma) = 1$, because

$$\begin{aligned}(ad - bc)^{-1}(av_1 + cv_2)(bv_1 + dv_2)[(a - b)v_1 + (c - d)v_2] \otimes_k (v_1 \wedge v_2)^{-1} \\ = v_1 v_2 (v_1 - v_2) \otimes_k (v_1 \wedge v_2)^{-1},\end{aligned}$$

and $\gamma(x_i) = x_i, \; i = 1, 2, 3$, we get equalities

$$\begin{aligned}c = 0, \quad b = 0 \\ av_1 - dv_2 = v_1 - v_2,\end{aligned}$$

i.e. $\gamma = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. So j is injective.

Conversely, for any permutation $\sigma \in \mathfrak{S}_3$, there exists a unique $a_\sigma \in \mathrm{PGL}_2(k) = \mathrm{Aut}_k(\mathbf{P}^1_k)$, such that $a_\sigma(x_i) = x_{\sigma(i)}$. Then for any $A \in \mathrm{GL}_2(k)$ whose image in $\mathrm{PGL}_2(k)$ is a_σ, we have $A^*(p_1) = m(A)p_1$, for some $m(A) \in k^\times$. Since k is a field, we can find an $A_0 \in \mathrm{GL}_2(k)$ whose image in PGL_2 is a_σ such that $A_0^*(p_1) = p_1$. So j is surjective, hence an isomorphism between $\mathrm{Aut}_k((V,\, p))$ and $\mathfrak{S}_3$.

(2) Suppose that $\gamma \in \mathrm{Aut}_k((V,\, p_2))$. Denote by X the zero section of the cubic form $v_1^2 v_2$ over $\mathbf{P}(V) = \mathbf{P}^1_k$. Let $x_1 = [0:1]$, $x_2 = [1:0]$, then $X = 2x_1 + x_2$ as a divisor on $\mathbf{P}^1_k$. Since $\gamma^*(p_2) = p_2$, we get $\gamma(x_1) = x_1$ and $\gamma(x_2) = x_2$. Combining with the equality

$$\begin{aligned}(ad-bc)^{-1}(av_1+cv_2)^2(bv_1+dv_2) \otimes_k (v_1 \wedge v_2)^{-1} \\ = v_1^2 v_2 \otimes_k (v_1 \wedge v_2)^{-1}.\end{aligned}$$

We get $a = 1,\, b = c = 0$, i.e. $\gamma = \begin{pmatrix} 1 & \\ & d \end{pmatrix}$ with $d \in k^\times$. It is easy to check that any such γ defines an automorphism of $(V,\, p_2)$. So $\mathrm{Aut}_k((V,\, p_2)) \cong k^\times$.

(3) $\gamma \in \mathrm{Aut}_k((V,\, p_3)$ if and only if

$$(ad-bc)^{-1}(av_1+cv_2)^3 \otimes_k (v_1 \wedge v_2)^{-1} = v_1^3 \otimes_k (v_1 \wedge v_2)^{-1}.$$

Comparing the two sides of the equality, we get $c = 0,\, a^2 = d$. So

$$\mathrm{Aut}_k((V,\, p_3)) = \left\{ \begin{pmatrix} a & b \\ & a^2 \end{pmatrix} \middle|\, a \in k^\times,\, b \in k \right\}.$$

(4) This case is obvious.

□

3.2 Smoothness and dimension

Definition 3.2.1. Let S be an arbitrary scheme, a *based twisted cubic form* over S is a pair (V, p, φ), where (V, p) is a twisted cubic form over S, $\phi : \mathcal{O}_S^{\oplus 2} \to V$ is an isomorphism of $\mathcal{O}_S$-modules.

Definition 3.2.2. Let $\mathscr{F}'$ be the category of based twisted cubic forms. Its objects are the based twisted cubic forms. A morphism from a based twisted cubic form (V_1, p_1, φ_1) over S_1 to (V_2, p_2, φ_2) over S_2 consists of a morphism $f : S_1 \to S_2$, an isomorphism $h : V_1 \to f^*V_2$ such that $p_1 = h^*(f^*(p_2))$ and $\varphi_1 = h^{-1} \circ (f^*(\varphi_2))$.

Proposition 3.2.3. *The category $\mathscr{F}'$ is a category fibered in groupoids over $\mathcal{S}$, it can be represented by $\mathbf{A}^4_{\mathbf{Z}}$.*

Proof. The first conclusion is a simple consequence of the existence of a well defined pull-back. For the second conclusion, since any based twisted cubic form over S has trivial isomorphism group, we only need to give the 1-morphism over the affine base scheme. Suppose that (V, p, φ) is a based twisted cubic form over $S = \mathrm{Spec}(R)$, R a commutative ring, such that V is a free rank 2 sheaf of $\mathcal{O}_S$-module. Let $e_1 = \varphi((1, 0))$, $e_2 = \varphi((0, 1))$, then p can be written uniquely as $(ae_1^3 + be_1^2e_2 + ce_1e_2^2 + de_2^3) \otimes (e_1 \wedge e_2)^{-1}$, for some $a, b, c, d \in R$, from which the proposition is easily deduced. □

The linear algebraic group $\mathrm{GL}_{2,\mathbf{Z}}$ acts naturally on $\mathscr{F}'$. Suppose that (V, p, φ) is a based twisted cubic form on S. Let $g \in \mathrm{GL}_2(\mathcal{O}_S)$, define $g\varphi : \mathcal{O}_S^{\oplus 2} \to V$ to be $g\varphi(x) = \varphi(gx)$, $\forall x \in \mathcal{O}_S^{\oplus 2}$. Define $g(V, \varphi, p) = (V, g\varphi, g^*(p))$, it is easily verified that this defines an action of GL_2 on $\mathscr{F}'$.

Theorem 3.2.4. *The stack $\mathscr{F}$ is a smooth Artin stack.*

Proof. Obviously we have $\mathscr{F} = [\mathscr{F}'/\,\mathrm{GL}_2] = [\underline{\mathbf{A}^4}/\,\mathrm{GL}_2]$. Since both $\mathbf{A}^4$ and GL_2 are smooth, $[\underline{\mathbf{A}^4}/\,\mathrm{GL}_2]$ is a smooth Artin stack. So $\mathscr{F}$ is also a smooth Artin stack. □

Proposition 3.2.5. *The dimension of $\mathscr{F}$ is* 0.

Proof. Since $\mathscr{F} = [\mathscr{F}'/\,\mathrm{GL}_2] = [\underline{\mathbf{A}^4}/\,\mathrm{GL}_2]$ and $\dim(\mathbf{A}^4) = \dim(\mathrm{GL}_2) = 4$, we have $\dim(\mathscr{F}) = 0$. □

Corollary 3.2.6. *The stack $\mathscr{G}$ of geometric cubic forms is a smooth Artin stack of dimension* 0.

Proof. By theorem 1.2.1, $\mathscr{G} \cong \mathscr{F}$ as stack, so the corollary results from the above two propositions. □

3.3 Primitive geometric cubic forms

Let O the origin of $\mathbf{A}^4_{\mathbf{Z}}$, $V = \mathbf{A}^4_{\mathbf{Z}} \backslash O$. Obviously $[\underline{V}/\operatorname{GL}_2]$ is an open sub algebraic stack of $\mathscr{F} = [\underline{\mathbf{A}^4_{\mathbf{Z}}}/\operatorname{GL}_2]$, with complement of codimension 4. The objects of $[\underline{V}/\operatorname{GL}_2]$ will be called *primitive twisted cubic forms*. By theorem 1.2.1, $\mathscr{F} \cong \mathscr{G}$. Denote by $\mathscr{W}$ the open sub algebraic stack of $\mathscr{G}$ that corresponds to $[\underline{V}/\operatorname{GL}_2]$ in $\mathscr{F}$. Obviously the complement of $\mathscr{W}$ is of codimension 4 in $\mathscr{G}$. The objects of $\mathscr{W}$ will be called *primitive geometric cubic forms*.

Proposition 3.3.1. *Suppose that $(P, \mathcal{O}(1), a)$ is a primitive geometric cubic form over S, let $\mathcal{J} := \mathcal{O}(3) \otimes \pi^*(\wedge^2 \pi_* \mathcal{O}(1))^{-1}$. Then the morphism of sheaves $\mathcal{J} \xrightarrow{a} \mathcal{O}$ is injective.*

Proof. By assumption, a is non zero at every point, so the morphism of sheaves $\mathcal{J} \xrightarrow{a} \mathcal{O}$ is injective. ☐

Chapter 4

Proof of the main result

4.1 From geometric cubic forms to cubic algebras

In this section, we will construct a 1-morphism of algebraic stacks $F_1 : \mathscr{G} \to \mathscr{A}$.

Suppose that $(P, \mathcal{O}(1), a)$ is a geometric cubic form, i.e. $\pi : P \to S$ is a family of genus 0 curves, $a \in \Gamma(P,\, \mathcal{O}(3) \otimes \pi^*(\wedge^2 \pi_* \mathcal{O}(1))^{-1})$. Let $\mathcal{J} := \mathcal{O}(-3) \otimes \pi^*(\wedge^2 \pi_* \mathcal{O}(1))$, then we have $\mathcal{J}^{-1} = \mathcal{O}(3) \otimes \pi^*(\wedge^2 \pi_* \mathcal{O}(1))^{-1}$. So $a \in \Gamma(P,\, \mathcal{J}^{-1}) = \Gamma(P, \mathcal{H}om(\mathcal{J},\, \mathcal{O}))$ defines a morphism of sheaves of modules

$$a : \mathcal{J} \to \mathcal{O},$$

which will be regarded as a complex of coherent $\mathcal{O}$-modules with $\deg(\mathcal{J}) = -1$. Define $A := \mathbf{R}^0 \pi_* (\mathcal{J} \xrightarrow{a} \mathcal{O})$, the 0-th hypercohomology of the complex $\mathcal{J} \to \mathcal{O}$. From the exact sequence of complexes

$$0 \to \mathcal{O} \to (\mathcal{J} \xrightarrow{a} \mathcal{O}) \to \mathcal{J}[1] \to 0,$$

we obtain the following long exact sequence by applying π_*,

$$\cdots \to \mathbf{R}^i \pi_* \mathcal{O} = R^i \pi_* \mathcal{O} \to \mathbf{R}^i \pi_* (\mathcal{J} \xrightarrow{a} \mathcal{O}) \to \mathbf{R}^i \pi_* (\mathcal{J}[1]) = R^{i+1} \pi_* (\mathcal{J}) \to \cdots . \quad (4.1)$$

Lemma 4.1.1. *We have*

$$\pi_*\mathcal{O} = \mathcal{O}_S,$$
$$R^i\pi_*\mathcal{O} = 0, \quad \textit{if } i \neq 0.$$

Proof. Firstly, we suppose that S is affine and $P = \mathbf{P}^1_S$. Let $S = \mathrm{Spec}(R)$, R a commutative ring. By [Hart], chapter III, proposition 8.5, $R^i\pi_*\mathcal{O} = H^i(\mathbf{P}^1_R, \mathcal{O})^\sim$. So by [Hart], chapter III, theorem 5.1,

$$\pi_*\mathcal{O} = \Gamma(\mathbf{P}^1_R, \mathcal{O})^\sim = \mathcal{O}_S,$$

and for $i \neq 0$,

$$R^i\pi_*\mathcal{O} = H^i(\mathbf{P}^1_R, \mathcal{O})^\sim = 0.$$

In general case we can cover S by small enough affine open subsets U such that $\pi^{-1}(U) = \mathbf{P}^1_U$. Since the above equalities are canonical, we get the same equalities in general case. □

Lemma 4.1.2. *We have*

$$R^i\pi_*(\mathcal{J}) = 0, \quad \textit{if } i \neq 1,$$
$$R^1\pi_*(\mathcal{J}) = (\pi_*\mathcal{O}(1))^\vee.$$

Proof. First of all, observe that

$$\mathcal{J} = \mathcal{O}(-3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}(1)) = \mathcal{O}(-1) \otimes \Omega^1_{P/S}.$$

Suppose that S is affine and $P = \mathbf{P}^1_S$. Let $S = \mathrm{Spec}(R)$, R a commutative ring. By [Hart], chapter III, proposition 8.5, $R^i\pi_*\mathcal{J} = H^i(\mathbf{P}^1_R, \mathcal{J})^\sim$. So by [Hart], chapter III, theorem 5.1,

$$\pi_*\mathcal{J} = \pi_*\mathcal{O}(-3) \otimes (\wedge^2\pi_*\mathcal{O}(1)) = H^0(\mathbf{P}^1_R, \mathcal{O}(-3))^\sim \otimes (\wedge^2\pi_*\mathcal{O}(1)) = 0.$$

By Serre's duality,

$$R^1\pi_*\mathcal{J} = H^1(\mathbf{P}^1_R, \mathcal{O}(-1) \otimes \Omega^1_{P/S})^\sim = \mathrm{Hom}_R(H^0(\mathbf{P}^1_R, \mathcal{O}(1)), R)^\sim = (\pi_*\mathcal{O}(1))^\vee.$$

For $i \neq 0, 1$, $R^i\pi_*\mathcal{J} = 0$ since the relative dimension of P over S is 1.

In the general case we can cover S by small enough affine open subsets U such that $\pi^{-1}(U) = \mathbf{P}^1_U$. Since the above equalities and isomorphisms are canonical, we get the same equalities as claimed in the lemma. □

Proposition 4.1.3. *We have the exact sequence*

$$0 \to \mathcal{O}_S \to \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O}) \to (\pi_*\mathcal{O}(1))^\vee \to 0,$$

and

$$\mathbf{R}^i\pi_*(\mathcal{J} \to \mathcal{O}) = 0, \quad \textit{if } i \neq 0.$$

Proof. This is the result of the long exact sequence 4.1 and lemma 4.1.1, 4.1.2. □

Corollary 4.1.4. $A = \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O})$ *is a locally free* $\mathcal{O}_S$*-module of rank* 3*.*

Proof. This is because of proposition 4.1.3 and the fact that $\pi_*\mathcal{O}(1)$ is a locally free $\mathcal{O}_S$-module of rank 2, □

Now we want to define a commutative unital $\mathcal{O}_S$-algebra structure on A.

First of all, by definition

$$\begin{aligned}
&(\mathcal{J} \xrightarrow{a} \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{J} \xrightarrow{a} \mathcal{O}) \\
&= \mathcal{J} \otimes \mathcal{J} \xrightarrow{\mathrm{id}\otimes a \oplus (-a)\otimes \mathrm{id}} (\mathcal{J} \otimes \mathcal{O}) \oplus (\mathcal{O} \otimes \mathcal{J}) \xrightarrow{a\otimes\mathrm{id} + \mathrm{id}\otimes a} \mathcal{O} \otimes \mathcal{O} \\
&= \mathcal{J} \otimes \mathcal{J} \xrightarrow{\mathrm{id}\otimes a \oplus (-a)\otimes \mathrm{id}} \mathcal{J} \oplus \mathcal{J} \xrightarrow{a+a} \mathcal{O}.
\end{aligned} \tag{4.2}$$

Define $r : \mathcal{J} \oplus \mathcal{J} \to \mathcal{J}$ by $r((i, j)) = i + j$, $\forall i, j \in \mathcal{J}$. It is easy to check that the following diagram is commutative

$$\begin{array}{ccccc}
\mathcal{J} \otimes \mathcal{J} & \xrightarrow{\mathrm{id}\otimes a \oplus (-a)\otimes \mathrm{id}} & \mathcal{J} \oplus \mathcal{J} & \xrightarrow{a+a} & \mathcal{O} \\
\downarrow 0 & & \downarrow r & & \| \\
0 & \longrightarrow & \mathcal{J} & \xrightarrow{a} & \mathcal{O},
\end{array}$$

in particular it defines a morphism θ_1 of the two horizontal complexes. By identity (4.2), θ_1 defines a homomorphism of complexes

$$\theta_1 : (\mathcal{J} \xrightarrow{a} \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{J} \xrightarrow{a} \mathcal{O}) \to (\mathcal{J} \xrightarrow{a} \mathcal{O}),$$

which induces a morphism

$$\vartheta_1 : \mathbf{R}^0\pi_*[(\mathcal{J} \xrightarrow{a} \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{J} \xrightarrow{a} \mathcal{O})] \to \mathbf{R}^0\pi_*(\mathcal{J} \xrightarrow{a} \mathcal{O}).$$

Consider the fiber product $\pi \times_S \pi : P \times_S P \to S$, and the complex of sheaves

$$(\mathcal{J} \to \mathcal{O}) \boxtimes (\mathcal{J} \to \mathcal{O}) := \mathrm{pr}_1^*(\mathcal{J} \to \mathcal{O}) \otimes_{\mathcal{O}_{P \times_S P}} \mathrm{pr}_2^*(\mathcal{J} \to \mathcal{O})$$

on $P \times_S P$.

Proposition 4.1.5. *There is a natural isomorphism*

$$\vartheta_2 : \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O}) \otimes_{\mathcal{O}_S} \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O}) \cong \mathbf{R}^0(\pi \times_S \pi)_*[(\mathcal{J} \to \mathcal{O}) \boxtimes (\mathcal{J} \to \mathcal{O})].$$

Proof. Since both $\mathcal{J}$ and $\mathcal{O}$ are invertible sheaves on P and the morphism $\pi : P \to S$ is flat, $\mathcal{J}$ and $\mathcal{O}$ are both flat over S. By proposition 4.1.3, $\mathbf{R}^i\pi_*(\mathcal{J} \to \mathcal{O})$ are locally free $\mathcal{O}_S$-modules for all $i \in \mathbf{Z}$, in particular they are flat $\mathcal{O}_S$-modules. So the hypothesis of [EGA III], theorem 6.7.8 holds, and we have the isomorphism stated in the proposition. □

Let $\Delta : P \to P \times_S P$ be the diagonal morphism, it is a closed immersion. Because

$$\Delta^*((\mathcal{J} \to \mathcal{O}) \boxtimes (\mathcal{J} \to \mathcal{O})) = (\mathcal{J} \to \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{J} \to \mathcal{O}),$$

Δ induces a homomorphism

$$\Delta^* : \mathbf{R}^0(\pi \times_S \pi)_*[(\mathcal{J} \to \mathcal{O}) \boxtimes (\mathcal{J} \to \mathcal{O})] \to \mathbf{R}^0\pi_*[(\mathcal{J} \to \mathcal{O}) \otimes_{\mathcal{O}} (\mathcal{J} \to \mathcal{O})].$$

Define $\Theta = \vartheta_1 \circ \Delta^* \circ \vartheta_2 : \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O}) \otimes_{\mathcal{O}_S} \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O}) \to \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O})$, it is obvious that this is a bilinear homomorphism. Because $\vartheta_1, \Delta^*, \vartheta_2$ are symmetric in the two variables, Θ is also symmetric.

Now we will describe Θ more concretely. Because $A = \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O})$ is a coherent sheaf on S, we only need to consider the case when S is affine, i.e. $S = \operatorname{Spec}(R)$ for some commutative ring R. By [EGA 0] §12.4.3,

$$\mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O}) = \mathbf{H}^0(P, \mathcal{J} \to \mathcal{O})^{\sim},$$

where $\mathbf{H}^i$ denotes the i-th hypercohomology group.

Suppose that $\mathscr{U} = \{U_0, U_1\}$ is the standard covering of $P = \mathbf{P}^1_R$. The Čech complex $C^\bullet(\mathscr{U}, \mathcal{J} \to \mathcal{O})$ is

$$\begin{array}{ccc} \mathcal{J}(U_{01}) & \xrightarrow{a} & \mathcal{O}(U_{01}) \\ \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} \\ \mathcal{J}(U_0) \oplus \mathcal{J}(U_1) & \xrightarrow{a \oplus a} & \mathcal{O}(U_0) \oplus \mathcal{O}(U_1), \end{array}$$

where the degree of $\mathcal{O}(U_{01})$ is $(0, 1)$.

By [EGA 0] §12.4.7, $\mathbf{H}^0(P, \mathcal{J} \to \mathcal{O}) = H^0(C^\bullet(\mathscr{U}, \mathcal{J} \to \mathcal{O}))$, i.e. it is the 0-th cohomology group of the following complex, in which $\mathcal{O}(U_{01})$ is of degree 1,

$$\mathcal{J}(U_0) \oplus \mathcal{J}(U_1) \xrightarrow{(-d)\oplus(a\oplus a)} \mathcal{J}(U_{01}) \oplus \mathcal{O}(U_0) \oplus \mathcal{O}(U_1) \xrightarrow{a \oplus d} \mathcal{O}(U_{01}).$$

Let $c_1, c_2 \in \mathbf{H}^0(P, \mathcal{J} \to \mathcal{O})$, suppose that c_1, c_2 have representatives $(i, f_0, f_1), (j, g_0, g_1) \in \mathcal{J}(U_{01}) \oplus \mathcal{O}(U_0) \oplus \mathcal{O}(U_1)$ respectively. Then $\Theta(c_1,\, c_2)$ has representative $(f_0 j + g_0 i + aij,\ f_0 g_0,\ f_1 g_1) \in \mathcal{J}(U_{01})) \oplus \mathcal{O}(U_0) \oplus \mathcal{O}(U_1)$.

Calculating by Čech cocycle, it is easy to see that the injective morphism of sheaves $\mathcal{O}_S \hookrightarrow \mathbf{R}^0\pi_*(P, \mathcal{J} \to \mathcal{O})$ in the exact sequence of proposition 4.1.3 defines the unital structure of A.

Proposition 4.1.6. *Let $c_1,\, c_2,\, c_3 \in \mathbf{H}^0(P, \mathcal{J} \to \mathcal{O})$, then*

$$\Theta((c_1,\, c_2),\, c_3) = \Theta(c_1,\, (c_2,\, c_3)).$$

Proof. Suppose that $c_i \in \mathbf{H}^0(P, \mathcal{J} \to \mathcal{O})$ have representatives by Čech cocyles $(j_i, f_i, g_i) \in \mathcal{J}(U_{01}) \oplus \mathcal{O}(U_0) \oplus \mathcal{O}(U_1)$ for $i = 1, 2, 3$. A direct computation shows that both $\Theta((c_1,\, c_2),\, c_3)$

and $\Theta(c_1,(c_2,c_3))$ can be represented by the cocycle

$$(f_1f_2j_3 + f_2f_3j_1 + f_1f_3j_2 + af_1j_2j_3 + af_2j_1j_3 + af_3j_1j_2 \\ +a^2j_1j_2j_3,\ f_1f_2f_3,\ g_1g_2g_3).$$

So $\Theta((c_1,\, c_2),\, c_3) = \Theta(c_1,\, (c_2,\, c_3))$. □

In conclusion Θ defines a commutative unital $\mathcal{O}_S$-algebra structure on $A = \mathbf{R}^0\pi_*(\mathcal{J} \to \mathcal{O})$ and A is thus a cubic algebra over S.

Example 4.1.1. Suppose that $a \in \Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}(1))^{-1})$ is not a zero divisor, then we have the exact sequence

$$0 \to \mathcal{J} \xrightarrow{a} \mathcal{O} \to \mathcal{O}/\mathcal{J}' \to 0,$$

where $\mathcal{J}'$ is the image of $\mathcal{J}$ in $\mathcal{O}$ under the morphism a. So $A = \mathbf{R}^0\pi_*(\mathcal{J} \xrightarrow{a} \mathcal{O}) = \pi_*(\mathcal{O}/\mathcal{J}')$, and the ring structure on A defined by Θ comes down to the one induced by the ring structure of $\mathcal{O}/\mathcal{J}'$.

Example 4.1.2. Suppose that $a = 0$, then the complex $\mathcal{J} \to \mathcal{O}$ equals $\mathcal{O} \oplus \mathcal{J}[1]$. So $A = \mathbf{R}^0\pi_*(P,\, \mathcal{O} \oplus \mathcal{J}[1]) = \pi_*\mathcal{O} \oplus R^1\pi_*\mathcal{J} = \mathcal{O}_S \oplus (\pi_*\mathcal{O}(1))^\vee$. Denote $\mathfrak{m} = (\pi_*\mathcal{O}(1))^\vee$. Calculating by Čech cocycle, it is easy to find that $\mathfrak{m}^2 = \Theta(\mathfrak{m},\, \mathfrak{m}) = 0$.

In this way, we have constructed a cubic algebra A over S from a geometric cubic form $(P, \mathcal{O}(1), a)$ over S, the construction commutes with arbitrary base change by corollary 4.1.4 and the expressions with Čech cocycles. So in fact we have constructed a 1-morphism of algebraic stacks $F_1 : \mathscr{G} \to \mathscr{A}$.

4.2 From cubic algebras to geometric cubic forms

In this section, we will construct a 1-morphism $F_2 : \mathscr{A} \to \mathscr{G}$ of algebraic stacks. In the following, we will always assume the base scheme S to be noetherian. In fact, since $\mathscr{A} = [\mathbf{A}^6_{\mathbf{Z}}/H]$, we will lose no generality with this assumption.

Suppose that A is a cubic algebra over S, i.e. A is a sheaf of $\mathcal{O}_S$-algebras, locally free of rank 3 as $\mathcal{O}_S$-module. Let $X = \mathrm{Spec}(A)$, let $f : X \to S$ be the structure morphism. Then f is a finite flat morphism of degree 3 and $A = f_*\mathcal{O}_X$.

Proposition 4.2.1. *The sheaf $f_*\mathcal{O}_X/\mathcal{O}_S$ is locally free of rank 2 as a sheaf of $\mathcal{O}_S$-modules.*

Proof. Applying $\mathcal{H}om_{\mathcal{O}_S}(\bullet, \mathcal{O}_S)$ to the morphism $\mathcal{O}_S \hookrightarrow f_*\mathcal{O}_X$, we get a morphism $\mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{O}_X, \mathcal{O}_S) \to \mathcal{O}_S$. We claim that it is surjective.

This is a local question, we can assume that $S = \mathrm{Spec}(R)$ for a noetherian local ring R. Suppose that x is the only closed point of S. Because the morphism $R \hookrightarrow A$ comes from the unity of A as an R-algebra, we have the exact sequence

$$0 \to R \otimes_R k(x) = k(x) \to A \otimes_R k(x).$$

So the homomorphism

$$\mathrm{Hom}_{k(x)}(A \otimes_R k(x), k(x)) \to k(x)$$

is surjective. Because A is a free R-module,

$$\mathrm{Hom}_{k(x)}(A \otimes_R k(x), k(x)) = \mathrm{Hom}_R(A, R) \otimes_R k(x).$$

By Nakayama's lemma, the homomorphism $\mathrm{Hom}_R(A, R) \to R$ is surjective.

Now that $\mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{O}_X, \mathcal{O}_S) \to \mathcal{O}_S$ is surjective, the exact sequence

$$0 \to \mathcal{O}_S \to f_*\mathcal{O}_X \to f_*\mathcal{O}_X/\mathcal{O}_S \to 0$$

is locally split. So $f_*\mathcal{O}_X/\mathcal{O}_S$ is a locally free $\mathcal{O}_S$-module. Obviously it is of rank 2. ☐

Denote $V = (f_*\mathcal{O}_X/\mathcal{O}_S)^\vee$, define $P = \mathbf{P}(V)$. Let $\pi : P \to S$ be the structure morphism. Then P is a family of genus 0 curves on S.

Now we begin the construction of the 1-morphism $F_2 : \mathscr{A} \to \mathscr{G}$. First of all, we restrict ourselves to the case that A is Gorenstein over S. Recall that in §2.2, we have defined a coherent sheaf ω on X such that $f_*\omega = \mathcal{H}om_{\mathcal{O}_S}(f_*\mathcal{O}_X, \mathcal{O}_S)$, X is said to be Gorenstein if ω is an invertible sheaf on X.

By the exact sequence

$$0 \to \mathcal{O}_S \to f_*\mathcal{O}_X \to f_*\mathcal{O}_X/\mathcal{O}_S \to 0, \tag{4.3}$$

we get a morphism of sheaves $\psi : V = (f_*\mathcal{O}_X/\mathcal{O}_S)^\vee \hookrightarrow (f_*\mathcal{O}_X)^\vee = f_*\omega$. Because f^* is the left adjoint of f_*, we get a morphism of sheaves $\varphi : f^*V \to \omega$.

Proposition 4.2.2. *The morphism of sheaves $\varphi : f^*V \to \omega$ is surjective.*

Proof. Because the problem is local, we can assume that $S = \mathrm{Spec}(R)$, R a noetherian local ring. Then A is a R-algebra, free of rank 3 as R-module. We need to prove that the following morphism is surjective:

$$\theta : \mathrm{Hom}_R(A/R,\, R) \otimes_R A \to \mathrm{Hom}_R(A,\, R),$$

with $\theta(f \otimes a)(b) = f(\overline{ab})$, $\forall a, b \in A,\ f \in \mathrm{Hom}_R(A/R,\, R)$, where $\overline{ab}$ denotes the image of ab in A/R under the projection.

Let $y \in S$ be the unique closed point of S. By Nakayama's lemma, we only need to prove that the morphism

$$\theta \otimes \mathrm{id}_{k(y)} : (\mathrm{Hom}_R(A/R,\, R) \otimes_R A) \otimes_R k(y) \to \mathrm{Hom}_R(A,\, R) \otimes_R k(y)$$

is surjective. Since A is a free R-module, we have

$$(\mathrm{Hom}_R(A/R,\, R) \otimes_R A) \otimes_R k(y) = \mathrm{Hom}_{k(y)}((A \otimes_R k(y))/k(y),\, k(y))$$

and

$$\mathrm{Hom}_R(A,\, R) \otimes_R k(y) = \mathrm{Hom}_{k(y)}(A \otimes_R k(y),\, k(y)).$$

So we are reduced to the case that R is a field. In fact we can even assume that R is algebraically closed since this is a homomorphism of vector spaces. By the classification result of theorem 2.1.2 and example 2.3.1, we need to check the surjectivity of θ for $A = A_1,\ A_2$ and A_3.

(1) For $A_1 = R \times R \times R$, $e_1 = (1, 1, 1)$ is the unity of A_1. Let $e_2 = (0, 1, 0)$, $e_3 =$

$(0,0,1)$, $V = Re_2 \oplus Re_3$. We identify V with A/Re_1. Let (f_1, f_2, f_3) be the basis of $A^\vee$ dual to (e_1, e_2, e_3). It is obvious that $\theta(f_2|_V) = f_2$, $\theta(f_3|_V) = f_3$, where $f_i|_V$ denotes the restriction of f_i to V, for $i = 2, 3$. By calculation,

$$\theta(f_2 \otimes (e_2 - e_1))(e_1) = f_2(e_2 - e_1) = 1,$$
$$\theta(f_2 \otimes (e_2 - e_1))(e_2) = f_2(e_2 - e_2) = 0,$$
$$\theta(f_2 \otimes (e_2 - e_1))(e_3) = f_2(0) = 0.$$

So $\theta(f_2 \otimes (e_2 - e_1)) = f_1$ and θ is hence surjective.

(2) For $A_2 = R \times (R[\epsilon]/(\epsilon^2))$, $e_1 = (1,1)$ is the unity of A_1. Let $e_2 = (1,0)$, $e_3 = (0,\epsilon)$, $V = Re_2 \oplus Re_3$. We identify V with A/Re_1. Let (f_1, f_2, f_3) be the basis of $A^\vee$ dual to (e_1, e_2, e_3). It is obvious that $\theta(f_2|_V) = f_2$, $\theta(f_3|_V) = f_3$. A simple calculation as above tells us that $\theta(f_2 \otimes (e_2 - e_1)) = f_1$, so θ is surjective.

(3) For $A_3 = R[\epsilon]/(\epsilon^3)$, $e_1 = 1$ is the unity of A_1. Let $e_2 = \epsilon$, $e_3 = \epsilon^2$, $V = Re_2 \oplus Re_3$. We identify V with A/Re_1. Let (f_1, f_2, f_3) be the basis of $A^\vee$ dual to (e_1, e_2, e_3). It is obvious that $\theta(f_2|_V) = f_2$, $\theta(f_3|_V) = f_3$. A simple calculation as above tells us that $\theta(f_2 \otimes e_2) = f_1$, so θ is surjective.

□

By [Hart], Chap II, proposition 7.12, the surjective morphism of sheaves $\varphi : f^*V \to \omega$ determines a morphism $\phi : X \to P = \mathbf{P}(V)$ of S-schemes.

Lemma 4.2.3. *Let $h : Y \to Z$ be a finite morphism of schemes, then h is a closed immersion if and only if the induced morphism of structure sheaves $h^\sharp : \mathcal{O}_Z \to h_*\mathcal{O}_Y$ is surjective.*

Proof. The necessity is obvious. For the sufficiency, since this is a local question, we can assume that Z is affine. Since h is finite, Y is also affine, and replacing Z by the scheme-theoretic image of h, we can assume that h is surjective. Let $Y = \mathrm{Spec}(R)$, $Z = \mathrm{Spec}(T)$. Since h is surjective, the induced morphism of structure sheaves $h^\sharp : \widetilde{T} \to h_*\widetilde{R}$ is injective. By assumption, it is also surjective, so $h^\sharp$ is an isomorphism. In particular, $h^\sharp(Z) : T = \widetilde{T}(Z) \to h_*\widetilde{R}(Z) = S$ is an isomorphism, so h is an isomorphism. □

Proposition 4.2.4. *The S-morphism $\phi : X \to P = \mathbf{P}(V)$ is a closed immersion.*

Proof. The morphism $\pi \circ \phi = f : X \to S$ is finite, in particular it is proper. Since π is separated, ϕ must be proper by [Hart], chap II, corollary 4.8(e). In fact, ϕ is a finite morphism since it is obviously quasi-finite. So to prove that ϕ is a closed immersion, we only need to prove that the induced morphism of sheaves $\phi^\sharp : \mathcal{O}_P \to \phi_* \mathcal{O}_X$ is surjective by lemma 4.2.3. For this purpose, it is enough to check it for every geometric point on S by Nakayama's lemma. So we can assume that $S = \mathrm{Spec}(k)$, k an algebraically closed field. By the classification result of theorem 2.1.2 and example 2.3.1, we need to check the cases $A = A_1,\ A_2,\ A_3$.

(1) For $A_1 = k \times k \times k$, take $e_1 = (1,1,1),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$. Let (f_1, f_2, f_3) be the basis of $A_1^\vee$ dual to (e_1, e_2, e_3). Define $h \in A_1^\vee$ by $h(ae_1 + be_2 + ce_3) = a + b + c,\ \forall a, b, c \in k$. A simple calculation shows that $(-e_1 + e_2 + e_3)h = f_1,\ (e_1 - e_3)h = f_2,\ (e_1 - e_2)h = f_3$. Let $x_i = f_i/h,\ i = 2, 3$, we need to prove that $A_{x_2} = k[x_3/x_2]$ and $A_{x_3} = k[x_2/x_3]$. By calculation, $x_2 = e_1 - e_3,\ x_3 = e_1 - e_2$ and $A_{x_2} = k[\frac{e_1}{e_1 - e_3}, \frac{e_1 - e_2}{e_1 - e_3}] = k[\frac{e_3}{e_1 - e_3}, \frac{e_1 - e_2}{e_1 - e_3}]$. But in the ring A_{x_2}, $\frac{e_3}{e_1 - e_3} = 0$ since $e_3(e_1 - e_3) = 0$. So $A_{x_2} = k[x_3/x_2]$. The same proof gives $A_{x_3} = k[x_2/x_3]$.

(2) For $A = A_2 = k \times (k[\epsilon]/(\epsilon^2))$, take $e_1 = (1,1),\ e_2 = (0,1),\ e_3 = (0,\epsilon)$. Let (f_1, f_2, f_3) be the basis of $A_2^\vee$ dual to (e_1, e_2, e_3). Define $h \in A_2^\vee$ by $h(ae_1 + be_2 + ce_3) = 2a + b + c,\ \forall a, b, c \in k$. A simple calculation shows that $(e_1 - e_2)h = f_1,\ (-e_1 + e_2 + e_3)h = f_2,\ (e_2 - e_3)h = f_3$. Let $x_i = f_i/h,\ i = 2, 3$, we need to prove that $A_{x_2} = k[x_3/x_2]$ and $A_{x_3} = k[x_2/x_3]$. By calculation, $x_2 = -e_1 + e_2 + e_3,\ x_3 = e_2 - e_3$ and $A_{x_2} = k[\frac{e_1}{-e_1 + e_2 + e_3}, \frac{e_2 - e_3}{-e_1 + e_2 + e_3}] = k[\frac{e_3}{-e_1 + e_2 + e_3}, \frac{e_2 - e_3}{-e_1 + e_2 + e_3}]$. But in the ring A_{x_2}, $\frac{e_3}{-e_1 + e_2 + e_3} = 0$ since $e_3(-e_1 + e_2 + e_3) = 0$. So $A_{x_2} = k[x_3/x_2]$.

Similarly, $A_{x_3} = k[\frac{e_1}{e_2 - e_3}, \frac{-e_1 + e_2 + e_3}{e_2 - e_3}] = k[\frac{e_3}{e_2 - e_3}, \frac{-e_1 + e_2 + e_3}{e_2 - e_3}]$. But in the ring A_{x_3}, $\frac{e_3}{e_2 - e_3} = 0$ since $e_3(e_2 - e_3) = 0$. So $A_{x_3} = k[x_2/x_3]$.

(3) For $A = A_3 = k[\epsilon]/(\epsilon^3)$, $(1, \epsilon, \epsilon^2)$ is a basis of A_3. Let (f_1, f_2, f_3) be the basis of $A_3^\vee$ dual to $(1, \epsilon, \epsilon^2)$. Define $h \in A_3^\vee$ by $h(a + b\epsilon + c\epsilon^2) = a + b + c,\ \forall a, b, c \in k$. A simple calculation shows that $\epsilon^2 h = f_1,\ (\epsilon - \epsilon^2)h = f_2,\ (1 - \epsilon)h = f_3$. Let $x_i = f_i/h,\ i = 2, 3$, we need to prove that $A_{x_2} = k[x_3/x_2]$ and $A_{x_3} = k[x_2/x_3]$.

By calculation, $x_2 = \epsilon - \epsilon^2$, $x_3 = 1 - \epsilon$ and $A_{x_2} = k[\frac{1}{\epsilon - \epsilon^2}, \frac{1-\epsilon}{\epsilon-\epsilon^2}] = 0 = k[x_3/x_2]$. Similarly, $A_{x_3} = A = k[x_2/x_3]$ since x_3 is invertible in A_3.

□

Let D the image of $\phi : X \to P$. Since ϕ is a closed immersion, D is isomorphic to X. Since X is flat over S, D is an effective relative divisor of P over S of degree 3. Define $\mathcal{O}(1) = \mathcal{O}(D) \otimes \Omega^1_{P/S}$. Then $\mathcal{O}(D) = \mathcal{O}(1) \otimes (\Omega^1_{P/S})^\vee = \mathcal{O}(3) \otimes \pi^*(\wedge^2 \pi_* \mathcal{O}(1))^{-1}$. So the element $1 \in \mathcal{O}(D)$ defines a global section $a \in \Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2 \pi_* \mathcal{O}(1))^{-1})$. In this way we have defined a geometric cubic form $(P, \mathcal{O}(1), a)$ over S from a Gorenstein cubic algebra A over S. It is obvious that this construction commutes with arbitrary base change, so it defines a 1-morphism of algebraic stacks $F_2' : \mathscr{V} \to \mathscr{G}$.

Generally, consider the category fibered in groupoids $\mathscr{B}^1$ of based cubic algebras. By proposition 2.2.6, it is representable by $\mathbf{A}^6_{\mathbf{Z}}$. So there exists a universal based cubic algebra $\widetilde{A}$ over $\mathbf{A}^6_{\mathbf{Z}}$. By theorem 2.3.2, the category of based Gorenstein cubic algebra is represented by the open subscheme $V = \mathbb{G}_a^2 \times (\mathbf{A}^4_{\mathbf{Z}} \backslash O)$ of $\mathbf{A}^6_{\mathbf{Z}}$. The restriction A_1 of $\widetilde{A}$ on V is a Gorenstein cubic algebra. Let $F_2'(A_1) = (P_1, \mathcal{O}_{P_1}(1), a_1)$, which is a geometric cubic form over V.

Proposition 4.2.5. *The geometric cubic form $(P_1, \mathcal{O}_{P_1}(1), a_1)$ over V extends uniquely to a geometric cubic form $(P, \mathcal{O}(1), a)$ over $\mathbf{A}^6_{\mathbf{Z}}$.*

Proof. Define $P = \mathbf{P}((\widetilde{A}/\mathcal{O}_{\mathbf{A}^6_{\mathbf{Z}}} \cdot 1)^\vee)$. Let $\pi : P \to \mathbf{A}^6_{\mathbf{Z}}$ be the structural morphism, then P is a family of genus 0 curves over $\mathbf{A}^6_{\mathbf{Z}}$. Recall that $P_1 = \mathbf{P}((A_1/\mathcal{O}_V \cdot 1)^\vee)$, so P is an extension of P_1 to $\mathbf{A}^6_{\mathbf{Z}}$. Let $\pi_1 : P_1 \to V$ be the structural morphism.

Since the complement of V in $\mathbf{A}^6_{\mathbf{Z}}$ is of codimension 4 and $\mathbf{A}^6_{\mathbf{Z}}$ is smooth, the following lemma shows that $\mathcal{O}_{P_1}(1)$ can be extended uniquely to an invertible sheaf $\mathcal{O}(1)$ of P over $\mathbf{A}^6_{\mathbf{Z}}$ of relative degree 1, and the section $a_1 \in \Gamma(P_1, \mathcal{O}_{P_1}(3) \otimes \pi_1^*(\wedge^2 \pi_{1,*}(\mathcal{O}_{P_1}(1)))^{-1})$ can be extended uniquely to the section $a \in \Gamma(P, \mathcal{O}_P(3) \otimes \pi^*(\wedge^2 \pi_*(\mathcal{O}_P(1)))^{-1})$. In this way we obtain a geometric cubic form $(P, \mathcal{O}(1), a)$. □

Lemma 4.2.6. *Suppose that Y is a noetherian, integral, regular, separated scheme. Let U be an open subscheme of Y with a complement of codimension at least 2. Let $\mathcal{L}$ be an invertible sheaf on U and $s \in \Gamma(U, \mathcal{L})$, then both $\mathcal{L}$ and s can be extended uniquely to Y.*

Proof. Since Y is noetherian, integral, regular and separated, by [Hart], chapter II, proposition 6.11 and proposition 6.15, $\mathrm{Pic}(Y) = \mathrm{Cl}(Y)$ and $\mathrm{Pic}(U) = \mathrm{Cl}(U)$. Since the complement of U has codimension at least 2, by [Hart], chapter II, proposition 6.5(b), $\mathrm{Cl}(U) = \mathrm{Cl}(Y)$. So $\mathrm{Pic}(U) = \mathrm{Pic}(Y)$, i.e. every invertible sheaf on U can be extended uniquely to an invertible sheaf on Y.

Let $\mathcal{L}'$ be the extension of $\mathcal{L}$ to Y. Now that every invertible sheaf on U and Y can be embedded into the constant sheaf $K(U) = K(Y)$, every section $s \in \Gamma(U, \mathcal{L})$ can be extended uniquely to a section $s' \in \Gamma(Y, \mathcal{L}')$. $\square$

Proposition 4.2.7. *The geometric cubic form $(P, \mathcal{O}(1), a)$ over $\mathbf{A}^6_{\mathbf{Z}}$ is H-equivariant.*

Proof. For any point x on $\mathbf{A}^6_{\mathbf{Z}}$, denote by $(P_x, \mathcal{O}_{P_x}(1), a_x)$ the fiber of $(P, \mathcal{O}(1), a)$ at x. We need to prove that the geometric cubic form $(P_{hx}, \mathcal{O}_{P_{hx}}(1), a_{hx})$ on $\mathrm{Spec}(k(hx))$ is isomorphic to the geometric cubic form $(P_x, \mathcal{O}_{P_x}(1), a_x)$ on $\mathrm{Spec}(k(x))$ for any $h \in H$.

Let $h \in H$, $\widetilde{A}' := h^*\widetilde{A}$ is a cubic algebra over $\mathbf{A}^6_{\mathbf{Z}}$. We constructed as above a geometric cubic form $(P', \mathcal{O}_{P'}(1), a')$ on $\mathbf{A}^6_{\mathbf{Z}}$ corresponding to $\widetilde{A}'$. Then the fiber of $(P', \mathcal{O}_{P'}(1), a')$ at x is just $(P_{hx}, \mathcal{O}_{P_{hx}}(1), a_{hx})$.

Recall that the construction of F_2' doesn't depend on the choices of basis, so the restriction of $(P', \mathcal{O}_{P'}(1), a')$ on V and $(P_1, \mathcal{O}_{P_1}(1), a_1)$ are the same. Since $(P, \mathcal{O}(1), a)$ is the unique extension of $(P_1, \mathcal{O}_{P_1}(1), a_1)$ to $\mathbf{A}^6_{\mathbf{Z}}$, we have $(P, \mathcal{O}(1), a) = (P', \mathcal{O}_{P'}(1), a')$, in particular their fibers at x will be the same. $\square$

The geometric cubic form $(P, \mathcal{O}(1), a)$ over $\mathbf{A}^6_{\mathbf{Z}}$ corresponds to a 1-morphism of algebraic stacks $F : \mathscr{B}^1 = \underline{\mathbf{A}^6_{\mathbf{Z}}} \to \mathscr{G}$. By proposition 4.2.7, F factors through the quotient $\mathscr{A} = [\mathscr{B}^1/H]$ and defines a 1-morphism $F_2 : \mathscr{A} \to \mathscr{G}$.

4.3 The correspondence is bijective

In this section we prove that the 1-morphisms $F_1 : \mathscr{G} \to \mathscr{A}$ and $F_2 : \mathscr{A} \to \mathscr{G}$ are inverse to each other. As a corollary, we get the main theorem 1.

Proposition 4.3.1. *The 1-morphism $F_1 : \mathscr{G} \to \mathscr{A}$ is left inverse to $F_2 : \mathscr{A} \to \mathscr{G}$.*

Proof. By theorem 2.2.7 and 2.3.2, $\mathscr{V}$ is an open sub algebraic stack of the smooth Artin stack $\mathscr{A}$ with a complement of codimension 4. Since F_2 is constructed as the unique extension of F_2', we only need to prove that $F_1 \circ F_2$ is isomorphic to the identity on $\mathscr{V}$. Suppose that A is a Gorenstein cubic algebra over S and $F_1(A) = (P, \mathcal{O}(1), a)$. We use the notation of section 4.2. Since ϕ is a closed immersion, $A \cong \pi_*\mathcal{O}_D$. By definition of $\mathcal{O}(1)$, we have $\mathcal{O}(D) = \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}(1))^{-1})$. Let $\mathcal{J} := \mathcal{O}(-3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}(1)) = \mathcal{O}(-D)$. By definition, the section $a \in \Gamma(P, \mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}(1))^{-1})$ correspond to $1 \in \Gamma(P, \mathcal{O}(D))$. So we have the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \xrightarrow{a} & \mathcal{O} & \longrightarrow & \mathcal{O}/\mathcal{J}' & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ 0 & \longrightarrow & \mathcal{O}(-D) & \xrightarrow{1} & \mathcal{O} & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0, \end{array}$$

where $\mathcal{J}'$ is the image of $\mathcal{J}$ in $\mathcal{O}$ under a. Since 1 is obviously not a zero divisor of $\Gamma(P, \mathcal{O}(D))$, a is not a zero divisor. So by example 4.1.1, $F_2((P, \mathcal{O}(1), a)) = \pi_*(\mathcal{O}/\mathcal{J}') = \pi_*\mathcal{O}_D \cong A$ and $F_1 \circ F_2$ is identity on $\mathscr{V}$. □

Proposition 4.3.2. *The 1-morphism $F_1 : \mathscr{G} \to \mathscr{A}$ is right inverse to $F_2 : \mathscr{A} \to \mathscr{G}$.*

Proof. Since $\mathscr{W}$ is an open sub algebraic stack of the smooth Artin stack $\mathscr{G}$ with complement of codimension 4, we only need to prove that $F_2 \circ F_1$ is identity on $\mathscr{W}$. Suppose that $(P, \mathcal{O}_P(1), a)$ is a primitive geometric cubic form over S. Let $\mathcal{J} := \mathcal{O}_P(-3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}_P(1)) = \mathcal{O}_P(-1) \otimes \Omega^1_{P/S}$. By proposition 3.3.1, the morphism of sheaves $\mathcal{J} \xrightarrow{a} \mathcal{O}_P$ is injective. Let $\mathcal{J}'$ be the image of $\mathcal{J}$ in $\mathcal{O}_P$ under the morphism a. $\mathcal{J}'$ is an ideal sheaf in $\mathcal{O}_P$, it defines an effective relative divisor D of P over S of degree 3. In other words, We have the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J} & \xrightarrow{a} & \mathcal{O}_P & \longrightarrow & \mathcal{O}_P/\mathcal{J}' & \longrightarrow & 0 \\ & & \downarrow & & \| & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_P(-D) & \xrightarrow{1} & \mathcal{O}_P & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0, \end{array}$$

in which the first downward arrow is an isomorphism.

Let $A := F_1((P, \mathcal{O}_P(1), a))$. Since a is not a zero divisor, by example 4.1.1, we have

$A = \pi_*(\mathcal{O}/\mathcal{J}') = \pi_*\mathcal{O}_D$. Let $F_2(A) = (P', \mathcal{O}_{P'}(1), a')$. By proposition 4.1.3, we have the exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{1} A \to (\pi_*\mathcal{O}_P(1))^\vee \to 0.$$

By construction in §4.2, $P' = \mathbf{P}((A/\mathcal{O}_S \cdot 1)^\vee) = \mathbf{P}(\pi_*\mathcal{O}_P(1)) = P$ by the above exact sequence. Let $\phi : \mathrm{Spec}(A) \hookrightarrow P' = P$ be the closed immersion constructed in §4.2. Since $\mathrm{Aut}_S(P)$ acts strictly three times transitive on P, we have $D' := \phi(\mathrm{Spec}(A)) \cong D$ after applying an automorphism of P. So $\mathcal{O}_{P'}(1) = \mathcal{O}_{P'}(D') \otimes \Omega^1_{P'/S} \cong \mathcal{O}_P(D) \otimes \Omega^1_{P/S} \cong (\mathcal{O}_P(3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}_P(1))^{-1}) \otimes \Omega^1_{P/S} = \mathcal{O}_P(1)$. Denote the composition of these isomorphisms by η. Let $\mathcal{J}_1 = \mathcal{O}_{P'}(-3) \otimes \pi^*(\wedge^2(\pi_*\mathcal{O}_{P'}(1)))$. By the construction of a', we have the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{J}_1 & \xrightarrow{a'} & \mathcal{O}_{P'} & \longrightarrow & \mathcal{O}_{P'}/\mathcal{J}_1' & \longrightarrow & 0 \\ & & \| & & \| & & & & \\ 0 & \longrightarrow & \mathcal{O}_{P'}(-D') & \xrightarrow{1} & \mathcal{O}_{P'} & \longrightarrow & \mathcal{O}_{D'} & \longrightarrow & 0, \end{array}$$

where $\mathcal{J}_1'$ is the image of $\mathcal{J}_1$ in $\mathcal{O}_{P'}$ under the morphism a'. Combining the above two commutative diagram of exact sequences, we find that the isomorphism $\eta : \mathcal{O}_{P'}(1) \cong \mathcal{O}_P(1)$ induces an isomorphism between $\mathcal{O}_{P'}(3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}_{P'}(1))^{-1}$ and $\mathcal{O}(3) \otimes \pi^*(\wedge^2\pi_*\mathcal{O}(1))^{-1}$ sending a' to a. So $F_2(A) = (P, \mathcal{O}_P(1), a)$ in $\mathscr{G}$ and $F_2 \circ F_1$ is identity on $\mathscr{W}$. □

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