The Weil Representation and Its Character

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Introduction

The Weil representation at a glance

For a local field F ($\mathrm{char}(F) \neq 2$) and a fixed non-trivial additive unitary character ψ of F, let $(W, \langle , \rangle)$ be a symplectic space of finite dimension over F and $\mathrm{Sp}(W)$ be the associated symplectic group, the Weil representation is a projective representation $\bar{\omega}_\psi$ of $\mathrm{Sp}(W)$. In simple terms, $\bar{\omega}_\psi$ is constructed as follows.

Consider the Heisenberg group $H(W)$, it is the space $W \times F$ equipped with the binary operation

$$(w_1, t_1) \cdot (w_2, t_2) = \left(w_1 + w_2, t_1 + t_2 + \frac{\langle w_1, w_2 \rangle}{2} \right).$$

By the Stone-von Neumann Theorem 3.2.1, $H(W)$ has a unique irreducible smooth (or unitary) representation (ρ_ψ, S) over $\mathbb{C}$ of central character ψ. As $\mathrm{Sp}(W)$ operates on $H(W)$ in the obvious way, $\rho_\psi^g := \rho_\psi \circ g$ and ρ_ψ are intertwined by an operator $M[g] : S \to S$, unique up to multiplication by $\mathbb{C}^\times$. That is:

$$M[g] \circ \rho_\psi = \rho_\psi^g \circ M[g].$$

This gives rise to a projective representation

$$\begin{aligned} \bar{\omega}_\psi : \mathrm{Sp}(W) &\longrightarrow \mathrm{PGL}(S) \\ g &\longmapsto M[g] \end{aligned}$$

In fact, one can define the *metaplectic group* $\widetilde{\mathrm{Sp}}_\psi(W) := \mathrm{Sp}(W) \times_{\mathrm{PGL}(S)} \mathrm{GL}(S)$, then there exists a subgroup $\widehat{\mathrm{Sp}}(W)$ of $\widetilde{\mathrm{Sp}}_\psi(W)$ which is a two-fold covering of $\mathrm{Sp}(W)$. The natural projection $\widehat{\mathrm{Sp}}(W) \to \mathrm{GL}(S)$ gives rise to a representation ω_ψ. One can show that ω_ψ decomposes into two irreducible representations: $\omega_\psi = \omega_{\psi,\mathrm{odd}} \oplus \omega_{\psi,\mathrm{even}}$. Moreover, ω_ψ is admissible.

Two problems remain.

1. An explicit description of the group $\widehat{\mathrm{Sp}}(W)$.

2. An explicit model of ω_ψ.

To answer these problems, we must study the models of irreducible smooth representations of Heisenberg group together with their intertwiners. We will mainly rely on *Schrödinger models* associated to lagrangians of W. The *Maslov index* associated to n lagrangians intervenes when we compose n intertwiners cyclically.

It turns out that $\widehat{\mathrm{Sp}}(W)$ is a non-trivial covering of $\mathrm{Sp}(W)$ when $F \neq \mathbb{C}$. Since the group scheme $\mathbb{S}\mathrm{p}(2n)$ is simply-connected, $\widehat{\mathrm{Sp}}(W)$ is non-algebraic (or equivalently: nonlinear). Since $\pi_1(\mathrm{Sp}(2n, \mathbb{R}), *) \simeq \mathbb{Z}$, such a two-fold covering for $\mathrm{Sp}(W)$ is unique when $F = \mathbb{R}$. For p-adic local fields, the uniqueness follows from the work of C. Moore in [14].

A brief history, motivations

The Weil representation was originally motivated by theoretical physics, namely by *quantization*. It was firstly defined on the level of Lie algebra by L. van Hove in 1951, then on the level of Lie group by I. E. Segal and D. Shale in the 1960's. On the arithmetical side, A. Weil generalized this machinery to include all local fields in [19]; this is the main ingredient of Weil's representation-theoretic approach to theta functions. In fact, the theta functions can be interpreted as automorphic forms of $\widehat{\mathrm{Sp}}(W)$ once the group of adélic points $\widehat{\mathrm{Sp}}(W, \mathbb{A})$ is properly defined.

One of the relevance's of Weil representation to number theory is the *Howe correspondence*. Roughly speaking, it predicts a bijection between irreducible representations of a *dual reductive pair* in $\mathrm{Sp}(W)$ which are quotients of the restriction of (ω_ψ, S). When $\dim W = 6$, this includes the *Shimura correspondence* between modular forms of half-integral weight and that of integral weight.

As the title suggests, the other aspect of this thesis is the *character* Θ_{ω_ψ} of ω_ψ. The role of characters in the representation theory of compact groups is well-known. The character theory for reductive algebraic groups (or *almost algebraic groups* in the sense of [12] p.257) was initiated by Harish-Chandra. He showed that one can define the character of an admissible representation as a distribution. A deep *regularity theorem* of Harish-Chandra ([7]) asserts that the character of a reductive p-adic group is a locally integrable function which is smooth on the dense subset of semi-simple regular elements.

The character Θ_{ω_ψ} in the non-archimedean case is computed first by K. Maktouf in [12], then T. Thomas gave a somewhat shorter proof [18]. We will follow the latter in our calculations of the character.

One of the reasons to study Θ_{ω_ψ} is the *endoscopy* theory of metaplectic groups. The relation between Langlands' functoriality and the Howe correspondence is an interesting question. However, L-groups can only be defined for reductive algebraic groups. Thus the framework of Langlands-Shelstad cannot be copied verbatim.

In [1], J. Adams defined the notion of stability on $\widehat{\mathrm{Sp}}(2n, \mathbb{R})$, an explicit correspondence of stable conjugacy classes $g \leftrightarrow g'$ between $\mathrm{SO}(n+1, n)$ and $\widehat{\mathrm{Sp}}(2n, \mathbb{R})$ by matching eigenvalues, then he defined a lifting of *stably invariant eigendistributions*[1] from $\mathrm{SO}(n+1, n)$ to $\widehat{\mathrm{Sp}}(2n, \mathbb{R})$ (dual to the usual picture of matching *orbital integrals*) by

$$\begin{aligned} \Gamma : & \Theta \mapsto \Theta' \\ & \Theta'(g') = \Phi(g')\Theta(g) \end{aligned}$$

where $\Phi(g') := \Theta_{\omega_{\psi,\mathrm{even}}}(g') - \Theta_{\omega_{\psi,\mathrm{odd}}}(g')$. Thus the character Θ_{ω_ψ} plays a role similar to transfer factors. Adams' map Γ satisfies some desirable properties; for example Γ restricts to a bijection of stable virtual characters.

It would be interesting as well as important to consider an analogous picture for matching orbital integrals when F is non-archimedean. Our study of Θ_{ω_ψ} may be regarded as a first step towards this topic.

Excursus

My policy is prove only what is needed; as a result, many important aspects of Heisenberg groups and the Weil representations are omitted. There is a far more complete treatment in [13].

1. To keep the thesis at a moderate size, the basics of harmonic analysis on locally profinite groups are assumed.
2. Some properties of the Maslov index are omitted, for example: local constancy, uniqueness, the self-dual measure on T, etc. A possible reference is [17].

[1] That is, a distribution which is invariant under stable conjugation and is an eigenfunction of the commutative algebra of bi-invariant differential operators.

3. We will not touch on the adélic aspect of Weil representations.

4. Although the representation theory of real groups is technically more complicated, the study of Heisenberg groups and metaplectic groups actually predate their p-adic counterparts, and is well-known. The character formula in the real case is computed by many authors, using various methods. See [12, 18] for a short bibliography.

   Our exposition works with few modification for the case $F = \mathbb{R}$. In particular, Thomas' calculation of characters works identically for $F = \mathbb{R}$, and even for F a finite field. In order to keep things simple, we will restrict ourselves to non-archimedean F of characteristic not equal to 2 in this thesis.

Finally, the main ideas of this thesis already exist in [15, 13, 17, 18]. The only (possible) improvements are some technical or expository details.

Organization of thesis

This thesis is organized as follows.

Chapter 1: This chapter covers our conventions on densities and measures, Fourier transforms, quadratic spaces, and generalities of the symplectic group and lagrangians.

Chapter 2: We will follow [17] to define the Maslov index associated to n lagrangians ($n \geq 3$) as a canonically defined quadratic space (T, q). We will also use A. Beilinson's nice approach to interpret $(T, -q)$ as the H^1 of some constructible sheaf on a solid n-gon, the quadratic form being induced by cup-products. This enables one to "see through" the basic properties of Maslov index. Its dimension and discriminant will be calculated. We will also record a dual form which is used in the next chapter.

Chapter 3: The rudiments of Heisenberg group and Stone-von Neumann theorem are stated and proven. We will define Schrödinger models and their canonical intertwiners. The canonical intertwiner will be expressed as an integral operator against a kernel, then we will relate Maslov indices to cyclic compositions of canonical intertwiners, in which the above-mentioned dual form will appear naturally.

Chapter 4: The metaplectic group is defined in this chapter. Using Schrödinger models, the two-fold covering of $\mathrm{Sp}(W)$ can be constructed using the *Maslov cocycles*.

Chapter 5: We will follow [18] closely to calculate the character of ω_ψ. A formula of Θ_{ω_ψ} as the pull-back of a function on $\widehat{\mathrm{Sp}}(\overline{W} \oplus W)$ is also obtained.

In the appendix, we will collect some basic facts about trace class operators. A variant of *Mercer's theorem* will also be stated. This is the main tool for computation of traces.

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Conventions and notations

Unless otherwise specified, the following conventions are followed throughout this thesis:

- The arrow $\rightarrowtail$ stands for injections; $\twoheadrightarrow$ stands for surjections.
- F denotes a non-archimedean local field of characteristic not equal to 2. Its ring of integers is denoted by $\mathcal{O}_F$. A chosen uniformiser is denoted by ϖ.
- A non-trivial continuous additive unitary character $\psi : F \to \mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\}$ is fixed once and for all.
- By a topological group, we mean a *Hausdorff* topological space equipped with a group structure compatible with its topology.
- Since we are working with a non-archimedean local field, the adjective *smooth* for functions means *locally constant*.
- For algebraic groups, we will use boldface letters (e.g. $\mathbb{S}\mathbb{p}$) to denote the scheme, and use roman letters (e.g. Sp) to denote the topological group of its F-points.
- The dual group of a commutative locally compact group G is denoted by $\hat{G}$.
- A representation of a group always acts on a *complex* vector space. By a unitary representation, we mean a *continuous* representation $\pi : G \times V \to V$ where V is a Hilbert space, such that $\pi(g) : V \to V$ is a unitary operator for all $g \in G$.
- The Schwarz-Bruhat functions on a group G is denoted by $\mathscr{S}(G)$. Since F is assumed to be non-archimedean, this is just the locally constant functions with compact support on G.
- The constant sheaf determined by an abelian group A on a space is denoted by $\underline{A}$.

Chapter 1

Preliminaries

1.1 Densities

1.1.1 Densities and measures

Densities serve as a bookkeeping tool for measures. One can formulate canonical versions of various integral constructions in harmonic analysis by means of densities. Here is a simplified version for F-vector spaces.

Definition 1.1.1. Let V be a finite-dimensional F-vector space. For $\alpha \in \mathbb{R}$, the $\mathbb{R}$-vector space of α-densities on V is defined to be

$$\Omega_\alpha^{\mathbb{R}}(V) := \left\{ \nu : \bigwedge^{\max} V \to \mathbb{R} : \forall x \in \bigwedge^{\max} V,\ t \in F^\times,\ \ \nu(tx) = |t|^\alpha \nu(x) \right\}$$

In particular, when $\alpha = 1$, we may identify $\Omega_1^{\mathbb{R}}(V)$ with the $\mathbb{R}$-vector space of real invariant measures (possibly zero) on V by sending $\nu \in \Omega_1^{\mathbb{R}}(V)$ to the invariant measure that assigns $\nu(v_1 \wedge \cdots \wedge v_n)$ to the set $\{a_1 v_1 + \cdots a_n v_n : |a_i| \leq 1\}$.

Remark 1.1.2. Although $\Omega_\alpha^{\mathbb{R}}$ is always 1-dimensional, there is usually no canonical non-zero element. However, when a non-trivial additive continous character ψ of F is prescribed and V comes with with a non-degenerate bilinear form B, the map $\psi \circ B : V \times V \to \mathbb{S}^1$ then yields a self-duality for V, hence we can take the self-dual Haar measure as the distinguished element in $\Omega_1^{\mathbb{R}}(V)$.

Some basic operations on densities are listed below.

- **Functoriality**. Let $f : V \to W$ be an isomorphism, then f induces $f_* : \bigwedge^{\max} V \to \bigwedge^{\max} W$, hence $f^* : \Omega_\alpha^{\mathbb{R}}(W) \to \Omega_\alpha^{\mathbb{R}}(V)$.
- **Product**. Let $\alpha, \beta \in \mathbb{R}$, we can define a product operation $\otimes : \Omega_\alpha^{\mathbb{R}}(V) \otimes \Omega_\beta^{\mathbb{R}}(V) \to \Omega_{\alpha+\beta}^{\mathbb{R}}(V)$ by $(\nu \otimes \omega)(x) := \nu(x)\omega(x)$.
- **Duality**. One can identify $\Omega_\alpha(V)^* \simeq \Omega_{-\alpha}(V) \simeq \Omega_\alpha(V^*)$. The first $\simeq$ comes from above product pairing and the canonical isomorphism $\Omega_0^{\mathbb{R}}(V) = \mathbb{R}$. As for the second $\simeq$, given $\nu^* \in \Omega_\alpha(V^*)$, define $\nu \in \Omega_{-\alpha}(V)$ by $\nu(e_1 \wedge \cdots \wedge e_n) = \nu^*(e_1^* \wedge \cdots \wedge e_n^*)$, it is clearly well-defined.
- **Additivity**. Given a short exact sequence $0 \to V' \to V \to V'' \to 0$, there is a canonical isomorphism $\cdot : \Omega_\alpha^{\mathbb{R}}(V') \otimes_{\mathbb{R}} \Omega_\alpha^{\mathbb{R}}(V'') \xrightarrow{\sim} \Omega_\alpha^{\mathbb{R}}(V)$, given by $(\nu' \cdot \nu'')(x \wedge \bar{y}) := \nu'(x)\nu''(y)$, where $\bar{y}$ denote any lifting of y to $\bigwedge^{\dim V''} V$.

- **Square root**. For any $\nu \in \Omega_1^{\mathbb{R}}(V)$, define $\nu^{1/2} : x \mapsto |\nu(x)|^{1/2}$. If ν corresponds to a positive measure, then $\nu^{1/2} \otimes \nu^{1/2} = \nu$. Half-densities are especially useful in formulating dualities.

We can also define complex densities by $\Omega_\alpha(V) := \Omega_\alpha^{\mathbb{R}}(V) \otimes_{\mathbb{R}} \mathbb{C}$. As $\Omega_1^{\mathbb{R}}(V)$ corresponds to real invariant measures on V, $\Omega_\alpha(V)$ corresponds to complex invariant measures on V as well. The basic operations above are also valid for $\Omega_\alpha(-)$.

Let f be an element of $L^1(V) \otimes_{\mathbb{R}} \Omega_1(V)$, we can integrate such a density-valued function by choosing any $\nu \in \Omega_1(V)$, $\nu \neq 0$, write $f = \bar{f} \otimes \nu$ and integrate $\bar{f}$ with respect to the complex measure $\mathrm{d}\nu$ corresponding to ν:

$$\int_{v \in V} f(v) = \int_V \bar{f}(v)\, \mathrm{d}\nu(v).$$

This is cleary independent of the choice of ν.

For an isomorphism $\phi : V \xrightarrow{\sim} W$, the formula of change of variables reads

$$\int_V \phi^* f = \int_W f$$

1.1.2 Distributions

One can now reformulate the theory of distributions on F-vector spaces.

Definition 1.1.3. Let V be a finite-dimensional vector space over a local field F. The space of *distributions* on V is the dual space of $\mathscr{S}(V) \otimes \Omega_1(V)$, where $\mathscr{S}(V)$ denotes the collection of *Schwartz-Bruhat functions* on V. Since the local field F is assumed to be non-archimedean, $\mathscr{S}(V) = C_c^\infty(V)$ is just the compactly supported, locally constant functions on V. There is no need to distinguish distributions and tempered distributions, and the space $\mathcal{D}(V)$ is simply the algebraic dual with discrete topology.

We rephrase now the standard operations on distributions using densities:

1. **Locally integrable functions as distributions**. Let f be a locally integrable function on V, then f defines a distribution via

$$\phi \in \mathscr{S}(V) \otimes \Omega_1(V) \mapsto \int_V f\phi.$$

2. **Push-forward and pull-back**. Consider a short exact sequence of finite-dimensional F-vector spaces

$$0 \longrightarrow W \xrightarrow{i} V \xrightarrow{p} U \longrightarrow 0$$
$$\Omega_1^{\mathbb{R}}(V) = \Omega_1^{\mathbb{R}}(W) \otimes \Omega_1^{\mathbb{R}}(U)$$

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p_* on test functions	$p_* : \mathscr{S}(V) \otimes \Omega_1^{\mathbb{R}}(V) \to \mathscr{S}(U) \otimes \Omega_1^{\mathbb{R}}(U)$
$\updownarrow$	$(p_*\phi)(u) = \displaystyle\int_{p(v)=u} \phi(v)$
p^* on distributions	$p^* : \mathcal{D}(U) \to \mathcal{D}(V)$

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i^* on test functions	$i^* : \mathscr{S}(V) \otimes \Omega_1^{\mathbb{R}}(V) \to \mathscr{S}(W) \otimes \Omega_1^{\mathbb{R}}(V)$
$\updownarrow$	$(i^*\phi)(w) = \phi(i(w))$
i_* on distributions	$i_* : \mathcal{D}(W) \to \mathcal{D}(V) \otimes \Omega_1^{\mathbb{R}}(U)$

3. **Dirac measures** The same notations as above. As a special case of push-forward of distributions, the Dirac measure concentrated on $W \subset V$ can be defined by the map

$$\mathscr{S}(V) \otimes \Omega_1^{\mathbb{R}}(V) \longrightarrow \Omega_1^{\mathbb{R}}(U)$$
$$\phi \longmapsto \int_W i^*\phi$$

By taking dual, we get a map $\Omega_1^{\mathbb{R}}(U)^* \longrightarrow \mathcal{D}(V)$.

4. **Fourier transform**. Fix a nontrivial additive character ψ of F. Define the Fourier transform of functions in the familiar way:

Fourier on test functions	$\mathscr{S}(V^*) \otimes \Omega_1^{\mathbb{R}}(V^*) \to \mathscr{S}(V)$
$\updownarrow$	$\phi^\wedge(v) := \int_{v^* \in V^*} \phi(v^*)\psi(\langle v^*, v\rangle)$
Fourier on distributions	$\mathcal{D}(V) \otimes \Omega_1^{\mathbb{R}}(V) \to \mathcal{D}(V^*)$
	$\langle (f\nu)^\wedge, \phi\rangle = \langle f, \phi^\wedge \nu\rangle$

When f comes from a Schwartz-Bruhat function, a simple application of Fubini's theorem shows that the Fourier transform of f as a distribution coincides that of f as a Schwartz-Bruhat function.

Observing that $\Omega_1(V) = \Omega_{1/2}(V) \otimes \Omega_{1/2}(V)$ and that $\Omega_{1/2}(V^*) = \Omega_{1/2}(V)^*$, the Fourier transform can be put in a more symmetric form:

$$\mathcal{D}(V) \otimes \Omega_{1/2}(V) \longmapsto \mathcal{D}(V^*) \otimes \Omega_{1/2}(V^*).$$

For any positive measure $\nu \in \Omega_1^{\mathbb{R}}(V)$, there exists a unique positive measure $\hat{\nu} \in \Omega_1^{\mathbb{R}}(V)$, called the *dual measure*, such that the Fourier inversion formula holds for Schwartz-Bruhat functions:

$$\forall v \in V, \quad \int_{v^* \in V^*} (\phi\nu)^\wedge(v^*)\psi(\langle v^*, v\rangle) \cdot \hat{\nu} = \phi(-v).$$

Set $c_\psi := \langle \nu, \hat{\nu}\rangle^{-1}$ under the pairing $\langle , \rangle : \Omega_1^{\mathbb{R}}(V) \otimes \Omega_1^{\mathbb{R}}(V^*) \to \mathbb{R}$. It is a positive constant depending only on the conductor of ψ, and equals 1 when the conductor of ψ is $\mathcal{O}_F$. Then the Fourier inversion formula reads:

$$(f\nu)^{\wedge\wedge} = c_\psi \cdot \tau^*(f\nu), \qquad \tau \text{ being the function } x \mapsto -x \text{ on } V.$$

One can rephrase the Plancherel formula in a similar manner:

$$\int_V s\bar{t} = c_\psi \int_{V^*} \hat{s}\bar{\hat{t}}, \quad s, t \in \mathscr{S}(V) \otimes \Omega_{1/2}(V).$$

Fourier transforms and pull-back/push-forwards are compatible in the sense below:

Proposition 1.1.4. *Consider the short exact sequence of finite-dimensional F-vector spaces*

$$0 \to W \xrightarrow{\iota} V \xrightarrow{\pi} U \to 0$$

and its dual

$$0 \to U^* \xrightarrow{i} V^* \xrightarrow{p} W^* \to 0$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}(U) & \xrightarrow{\pi^*} & \mathcal{D}(V) \\ \Big\downarrow{\scriptstyle Fourier} & & \Big\downarrow{\scriptstyle Fourier} \\ \mathcal{D}(U^*) \otimes \Omega_1(U^*) & \xrightarrow[c_\psi \cdot i_*]{} & \mathcal{D}(V^*) \otimes \Omega_1(V^*) \end{array}$$

$$\begin{array}{ccc} \mathcal{D}(W) & \xrightarrow{\iota_*} & \mathcal{D}(V) \otimes \Omega_1(U) \\ \Big\downarrow{\scriptstyle Fourier} & & \Big\downarrow{\scriptstyle Fourier} \\ \mathcal{D}(W^*) \otimes \Omega_1(W^*) & \xrightarrow[p^*]{} & \mathcal{D}(V^*) \otimes \Omega_1(W^*) \end{array}$$

Proof. For the first diagram, it suffices to consider its dual (i.e. for test functions):

$$\begin{array}{ccc} \mathscr{S}(V^*) & \xrightarrow{c_\psi \cdot i^*} & \mathscr{S}(U^*) \\ \Big\downarrow{\scriptstyle \text{Fourier}} & & \Big\downarrow{\scriptstyle \text{Fourier}} \\ \mathscr{S}(V) \otimes \Omega_1(V) & \xrightarrow[\pi_*]{} & \mathscr{S}(U) \otimes \Omega_1(U) \end{array}$$

Let $\phi \in \mathscr{S}(V^*)$, then the top-right composition transforms ϕ to

$$u \mapsto c_\psi \cdot \int_{u^* \in U^*} \phi(u^*)\psi(\langle u^*, u\rangle)$$

in which one should insert some element of $\Omega_1(U^*) \otimes \Omega_1(U^*)^*$ to make the integral meaningful. The bottom-left composition transforms ϕ to

$$u \to \int_{\substack{v \in V \\ \pi(v)=u}} \int_{v^* \in V^*} \phi(v^*)\psi(\langle v^*, v\rangle)$$

Fix $v_0 \in V$ such that $\pi(v_0) = u$. Choose a complement of $U^* \subset V^*$ and identify it with W^*. We can now unfold the last integral as

$$\int_{u^* \in U^*} \int_{\substack{w \in W \\ w^* \in W^*}} \phi(w^* + u^*)\psi(\langle w^* + u^*, v_0 + w\rangle)$$

Set $\Phi_{u^*}(w^*) = \phi(u^* + w^*)\psi(\langle u^*, v_0\rangle)\psi(\langle w^*, v_0\rangle)$ to write the integral as

$$\int_{u^* \in U^*} \int_{\substack{w \in W \\ w^* \in W^*}} \Phi_{u^*}(w^*)\psi(\langle w^*, w\rangle) \underbrace{\psi(\langle u^*, w\rangle)}_{=1}$$

A Haar measure α on W and its dual measure $\hat{\alpha}$ must be inserted to integrate over $W \times W^*$. Recall that $c_\psi := \langle \alpha, \hat{\alpha}\rangle^{-1}$. Then Fourier inversion formula implies that the inner integral is

$$c_\psi \Phi_{u^*}(0) = c_\psi \phi(u^*)\psi(\langle u^*, v_0\rangle) = c_\psi \phi(u^*)\psi(\langle u^*, u\rangle)$$

Hence the top-right and the bottom-left compositions are equal.

The commutativity of the second diagram is even easier. □

1.2 Quadratic spaces

1.2.1 Basic definitions

Definition 1.2.1. A *quadratic space* over F is a pair (V, q), where V is a finite-dimensional F-vector space and q is a *non-degenerate* quadratic form on V.

Let (V, q), (V', q') be two quadratic spaces. A F-linear map $\phi : V \to V'$ is called an isometry if it preserves quadratic forms: $q'(\phi(x)) = q(x)$.

Since the characteristic of F is not equal to 2, a quadratic form q on V can also be described by a non-degenerate symmetric bilinear form B such that $q(x) = B(x, x)$. We will use the same symbol q to denote both a quadratic form $q(-)$ or its associated bilinear form $q(-,-)$.

We will often abuse notations to denote a quadratic space (V, q) by V or q.

Quadratic spaces over F and their isometries form a category. We can define orthogonal sums and tensor products [1] as follows: Let (V, q), (V', q') be two quadratic spaces, set

$$\begin{aligned}(q \oplus q')(x + x') &:= q(x) + q(x') && \text{on } V \oplus V' \\ (q \otimes q')(x \otimes x') &:= q(x)q'(x') && \text{on } V \otimes V'\end{aligned}$$

Example 1.2.2. Let $a \in F^\times$, set $\langle a \rangle$ to be the quadratic space defined by F with the quadratic form $x \mapsto ax^2$.

Example 1.2.3 (Hyperbolic planes)**.** Define H to be the quadratic space F^2 with the quadratic form $(x, y) \mapsto xy$. Since $\mathrm{char}(F) \neq 2$, it is also isometric to the same space with quadratic form $(x, y) \mapsto x^2 - y^2$. A quadratic space isometric to H will be called a *hyperbolic plane*.

Example 1.2.4 (The dual form)**.** A non-degenerate bilinear form q on V can be described by an isomorphism $\rho : V \to V^*$ such that

$$\langle \rho(x), y \rangle = q(x, y)$$

Define a bilinear form on V^* by $q^*(x^*, y^*) := \langle x^*, \rho^{-1}(y^*) \rangle$. If q is symmetric, so is q^*. In this case, q^* is called the *dual form* of q. Note that $\rho : V \to V^*$ defines an isometry of quadratic spaces.

The following elementary fact says that quadratic spaces can be diagonalized.

Proposition 1.2.5. *Every quadratic space V can be decomposed into an orthogonal sum:*

$$V \simeq \langle d_1 \rangle \oplus \cdots \oplus \langle d_n \rangle$$

Proof. See [9] Chapter I, 2.4. □

We will make use of the following invariants:

Definition 1.2.6 (Discriminant)**.** The *discriminant* $D(V)$ of a quadratic space V is an element of $F^\times / F^{\times 2}$, defined in the following way: Let $\rho : V \to V^*$ be the homomorphism such that $\langle x, \phi(y) \rangle = q(x, y)$. Fix a basis $e_1, \ldots, e_n$ for V and its dual basis $e_1^*, \ldots e_n^*$ for V^*. Then $D(q)$ is defined to be $\det \rho$ with respect to these basis. It is well-defined up to $F^{\times 2}$.

If $V = \langle d_1 \rangle \oplus \cdots \oplus \langle d_n \rangle$, then $D(q) = \prod_i d_i \mod F^{\times 2}$.

Definition-Proposition 1.2.7 (Hasse invariant)**.** If $V = \langle d_1 \rangle \oplus \cdots \oplus \langle d_n \rangle$, define its *Hasse invariant* by $\epsilon(V) := \prod_{i<j} (d_i, d_j)$, where $(-,-)$ is the quadratic Hilbert symbol for F. This number is independent of the chosen diagonalization of V.

Proof. This is a corollary of Witt's chain-equivalence theorem. See [9] Chapter 5, 3.18. □

[1] Also known as the Kronecker product.

1.2.2 Witt group

The Grothendieck construction leads to the following structure, called the *Witt group*.

Definition 1.2.8. Let $\hat{W}(F)$ be the Grothendieck group of isometry classes of quadratic forms over F. This is called the Witt-Grothendieck group. Define the Witt group $W(F)$ as

$$W(F) := \hat{W}(F)/\mathbb{Z}H.$$

Recall that H stands for the hyperbolic plane. Addition in $W(F)$ comes from orthogonal sums of quadratic spaces. In fact, tensor product equips $W(F)$ with a ring structure, called the *Witt ring*. We will not use the multiplication operation on $W(F)$ in this thesis.

Remark 1.2.9. We will also consider the classes of possibly degenerate quadratic forms in $W(F)$, by dividing out its kernel to get a non-degenerate one. The compatibility with addition in $W(F)$ is ultimately settled by Witt's decomposition theorem ([9] Chapter 1, 4.1).

Two quadratic spaces over F are called *Witt equivalent* if they have the same class in $W(F)$.

Proposition 1.2.10. *Two quadratic spaces V_1, V_2 are Witt equivalent if and only if there exist $n \in \mathbb{N}$ such that $V_1 \oplus nH \simeq V_2$ or $V_1 \simeq V_2 \oplus nH$.*

Proof. This is a consequence of Witt's cancellation theorem ([9] Chapter 1, 4.1). □

We will also use the technique of *sublagrangian reductions*.

Proposition 1.2.11. *Let (V, q) be a possibly degenerate quadratic space, $I \subset V$ an isotropic subspace (also known as sublagrangian). Then $I^\perp/I$ has the same class in $W(F)$ as V.*

Proof. We will start with the non-degenerate case. The quadratic form q restricts to another non-degenerate quadratic form $\bar{q}$ on $I^\perp/I$. Set $\overline{I^\perp/I}$ to be the quadratic space equipped with the form $-\bar{q}$. Then the diagonal embedding $\delta : I^\perp \rightarrowtail V \oplus \overline{I^\perp/I}$ has its image $\delta(I^\perp)$ as an isotropic subspace. Moreover, the quadratic form $(q, -\bar{q})$ induces an isomorphism

$$\frac{V \oplus \overline{I^\perp/I}}{\delta(I^\perp)} \xrightarrow{\sim} (I^\perp)^*.$$

Hence $V \oplus \overline{I^\perp/I}$ is a sum of hyperbolic planes, which amounts to that V and $I^\perp/I$ are Witt equivalent.

If V is degenerate, let $\pi : V \to V'$ be the isometry onto its non-degenerate quotient. It is clear that $\pi(I^\perp) = \pi(I)^\perp$, thus $I^\perp/I$ has $\pi(I)^\perp/\pi(I)$ as its non-degenerate quotient. The latter was known to be Witt equivalent to V'. □

1.2.3 Weil character

For a fixed non-trivial continuous additive character ψ of F, Weil defined in [19] §14 a character $\gamma : W(F) \to \mathbb{S}^1$. The description of γ is as follows: Let (V, q) be a quadratic space. Set $f_q(x) = \psi(\frac{q(x,x)}{2})$. Weil proved the following result.

Theorem 1.2.12 (Weil, [19] §14 Théorème 2)**.** *Let $\mathrm{d}q$ be the self-dual measure with respect to the duality $\psi \circ q : V \times V \to \mathbb{S}^1$. Then there exists a constant $\gamma(q) \in \mathbb{S}^1$ such that*

$$(f_q \,\mathrm{d}q)^\wedge = \gamma(q) f_{-q^*}$$

as distributions on V^*.

Moreover, $\gamma(-)$ induces a character $W(F) \to \mathbb{S}^1$.

In down to earth terms, take an arbitrary Schwartz-Bruhat function ϕ on V, then

$$(\phi * f_q\, \mathrm{d}q)^\wedge(0) = ((\phi\, \mathrm{d}q)^\wedge \cdot (f_q\, \mathrm{d}q)^\wedge)(0) = \gamma(q)(\phi\, \mathrm{d}q)^\wedge(0) \cdot f_{-q^*}(0)$$

Comparing the left and right hand sides yields

$$\int\int \phi(x-y) f_q(y)\, \mathrm{d}y\, \mathrm{d}x = \gamma(q) \int \phi(x)\, \mathrm{d}x$$

where the integrals are taken with respect to the self-dual measure $\mathrm{d}q$. Since ϕ can be chosen so that $\int \phi(x)\, \mathrm{d}x \neq 0$, this formula characterizes $\gamma(q)$.

We will make use of another recipe to compute the Weil character, as follows:

Proposition 1.2.13. *Let (V, q) be a quadratic space. Let $\mathrm{d}q$ be the self-dual measure. Choose h to be a Schwartz-Bruhat function on V such that its Fourier transform $h^\wedge$ is a positive measure and that $h(0) = 1$. Set $h_s(x) := h(sx)$, then*

$$\gamma(q) = \lim_{s \to 0} \int\limits_{x \in V} h_s(x) \psi\left(\frac{q(x,x)}{2}\right)\, \mathrm{d}q.$$

Moreover, $|\int_{x\in V} h_s(x)\psi\left(\frac{q(x,x)}{2}\right)\, \mathrm{d}q| \leq 1$ for all s.

Proof. Identify V and V^* by q. By the Plancherel formula and the positivity of $h^\wedge$,

$$\begin{aligned}\int\limits_{x\in V} h_s(x)\psi\left(\frac{q(x,x)}{2}\right)\, \mathrm{d}q &= \int\limits_{y\in V} (h_s\, \mathrm{d}q)^\wedge(y)(f_q\, \mathrm{d}q)^\wedge(y)\, \mathrm{d}q \\ &= \gamma(q) \int\limits_{y\in V} f_{-q^*}(y) h_s^\wedge(y)\end{aligned}$$

The hypothesis $h(0) = 1$ is equivalent to that $\int_V (h\, \mathrm{d}q)^\wedge\, \mathrm{d}q = 1$, and the same holds for h_s. Since $|\gamma(q)| = 1$, the second assertion follows. As $s \to 0$, $(h_s\, \mathrm{d}q)^\wedge(y)$ converges weakly to the Dirac measure at $y = 0$, this establishes the first assertion. □

If $(V, q) = \langle a \rangle$, $a \in F^\times$, we will set $\gamma(a) := \gamma(q)$. Note that $\gamma(a)$ only depends on $a \mod F^{\times 2}$. Weil also proved the following properties of γ:

Proposition 1.2.14. *For all $a, b \in F^\times$, we have*

$$\frac{\gamma(ab)\gamma(1)}{\gamma(a)\gamma(b)} = (a, b)$$

Corollary 1.2.15. *The function $x \mapsto \frac{\gamma(x)^2}{\gamma(1)^2}$ is a character of $F^\times$.*

Corollary 1.2.16. *For any quadratic space (V, q), we have*

1. $\gamma(q)^8 = 1$
2. $\gamma(q) = \gamma(1)^{\dim V - 1}\gamma(D(q))\epsilon(q)$.

Proof.

1. Observe that $\gamma(H) = \gamma(1) \cdot \gamma(-1) = 1$, thus $\gamma(1) = \gamma(-1)^{-1}$. Put $a = b = -1$ in the preceding proposition to get $\gamma(1)^4 = (-1, -1) = \pm 1$, hence $\gamma(1)^8 = 1$. Put $a = b$ to get $\gamma(a)^2 = \gamma(1)^2(a, a)$, hence $\gamma(a)^8 = 1$. As for general quadratic forms: diagonalize.

2. Take a diagonalisation $V \simeq \bigoplus_i \langle d_i \rangle$, then $\gamma(q) = \prod_i \gamma(d_i)$. It follows by induction that

$$\prod_i \gamma(d_i) = \gamma(1)^{\dim V - 1} \gamma(\prod_i d_i) \prod_{i<j} (d_i, d_j),$$

in which the last product is just the Hasse invariant.

□

Example 1.2.17. The Weil character has great significance in number theory. Some basic examples are listed below.

1. Weil's theory also works when F is a finite field and $\mathrm{char}(F) \neq 2$. Fix a non-trivial additive character ψ as before. The self-dual measure $\mathrm{d}q$ is just the counting measure divided by $\sqrt{|V|} = |F|^{-\frac{1}{2} \dim V}$. Then

$$(f_q \, \mathrm{d}q)^\wedge(0) = |F|^{\frac{1}{2} \dim V} \sum_{x \in V} \psi(q(x)/2)$$
$$f_{-q^*}(0) = 1$$

Hence $\gamma(q) = |F|^{-\frac{1}{2} \dim V} \sum_{x \in V} \psi(\frac{1}{2} q(x))$, the link with Gauß sums is then obvious.

2. For $F = \mathbb{R}$, take ψ to be the character $x \mapsto e^{-2\pi i x}$. Weil showed in [19] that $\gamma(a) = e^{\frac{-i\pi}{4} \cdot \mathrm{sgn}(a)}$.

3. For $F = \mathbb{C}$, take ψ to be the character $z \mapsto e^{-2\pi i \cdot \mathrm{Re}(z)}$, then $\gamma \equiv 1$. Hence Weil's theory over $\mathbb{C}$ is more or less trivial.

4. When F is a non-archimedean local field, the formulas of Weil characters are more complicated. Consult [15] A.4-A.5 for a complete calculation.

1.3 Symplectic spaces

1.3.1 Basic definitions

Definition 1.3.1. A symplectic space over F is a pair $(V, \langle , \rangle)$ where V is a finite-dimensional F-vector space and $\langle , \rangle$ is a non-degenerate alternating form on V.

As in the case of quadratic spaces, there is an obvious notion of symplectic equivalence between symplectic spaces. We will follow the same abuse to denote a symplectic space $(V, \langle , \rangle)$ by V.

Definition 1.3.2. Let V be a symplectic space. A subspace $\ell \subset V$ is called a *lagrangian* in V if ℓ is a maximal isotropic subspace (that it, the spaces on which $\langle , \rangle$ is identically zero).

The structure of symplectic spaces and their lagrangians is englobed in the following result.

Proposition 1.3.3. *Let ℓ_1, ℓ_2 be two lagrangians of V. Then there exists a basis $p_1, \ldots, p_n, q_1, \ldots, q_n$ of V such that*

1. *$\langle p_i, p_j \rangle = \langle q_i, q_j \rangle = 0$ for all i, j. $\langle p_i, q_j \rangle = \delta_{ij}$. Such a basis is called a symplectic basis for V.*

2. *$\ell_1 \cap \ell_2 = Fp_1 \oplus \cdots \oplus Fp_s$, where $s = \dim \ell_1 \cap \ell_2$.*

3. *$\ell_1 = Fp_1 \oplus \cdots \oplus Fp_n$.*

4. *$\ell_2 = Fp_1 \oplus \cdots \oplus Fp_s \oplus Fq_{s+1} \oplus \cdots \oplus Fq_n$.*

Proof. Elementary linear algebra, see [11] (1.4.6). □

Corollary 1.3.4. *In particular:*

1. *Every symplectic space has even dimension.*
2. *Every lagrangian of a symplectic space has the same dimension.*
3. *Every two symplectic spaces of the same dimension are symplectically equivalent.*

Definition 1.3.5. Let V be a symplectic space. Define $\mathrm{Sp}(V)$ to be the linear algebraic group over F of automorphisms of V preserving the symplectic form on V.

As shown by the preceding proposition, every symplectic space of dimension $2n$ is equivalent to $\bigoplus_{i=1}^n Fp_i \oplus \bigoplus_{i=1}^n Fq_i$ with the prescribed $\langle,\rangle$. The corresponding symplectic group is denoted by $\mathrm{Sp}(2n, F)$. In terms of the ordered basis $p_1, \ldots, p_n, q_n, \ldots, q_1$, $\mathrm{Sp}(2n, -)$ can be expressed as

$$\mathrm{Sp}(2n, -) = \left\{ X \in \mathrm{GL}(2n, -) : X^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} X = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

Remark 1.3.6. Assume that $n > 0$. The group scheme $\mathbb{Sp}(2n)$ is a geometrically connected, simply connected semi-simple group scheme of dimension $n(2n+1)$. It is split. The center of $\mathrm{Sp}(2n, F)$ is ± 1. The group $\mathrm{Sp}(2n, F)$ is equal to its derived group unless $n = 1, F = \mathbb{F}_2, \mathbb{F}_3$ or $n = 2, F = \mathbb{F}_2$ (see [3] 1.3), this includes all the cases in this thesis.

Remark 1.3.7. When $n = 1$, $\mathbb{Sp}(2)$ is just $\mathbb{SL}(2)$.

1.3.2 Lagrangians

Let $\Lambda(V)$ be the set of lagrangians of V. Let $2n = \dim V$, then $\Lambda(V)$ embeds into the Grassmannian variety of n-dimensional linear subspaces in a $2n$-dimensional space, denoted by $G(2n, n)(F)$, as the closed subvariety

$$\{\ell \in G(2n, n)(F) : \langle -, - \rangle = 0 \text{ on } \ell \times \ell\}.$$

Corollary 1.3.8. $\mathrm{Sp}(V, F)$ *acts transitively on* $\Lambda(V)$.

Proof. This follows immediately from our proposition. □

For a fixed lagrangian $\ell \subset V$, we have a surjective morphism $\mathrm{Sp}(V) \to \Lambda(V)$ defined by $g \mapsto g\ell$. The stabilizer of ℓ is a maximal parabolic subgroup; when V takes the standard form and $\ell = \bigoplus_i Fp_i$, the elements X stabilizing ℓ are of the form

$$X = \begin{pmatrix} A & B \\ 0 & (A^t)^{-1} \end{pmatrix} \in \mathrm{Sp}(2n, F)$$

$\Lambda(V)$ admits a cellular decomposition into locally closed subvarieties:

$$\Lambda(V) = \bigcup_{i=0}^{n} L_{\ell,i} \tag{1.1}$$

$$L_{\ell,i} := \{\ell' \in \Lambda(V) : \dim \ell' \cap \ell = i\} \tag{1.2}$$

Taking preimages yields a cellular decomposition of $\mathrm{Sp}(V)$:

$$\mathrm{Sp}(V) = \bigcup_{i=0}^{n} N_{\ell,i} \tag{1.3}$$

$$N_{\ell,i} := \{g \in \mathrm{Sp}(V) : \dim g\ell \cap \ell = i\} \tag{1.4}$$

Among all $L_{\ell,i}$ [resp. $N_{\ell,i}$], the cell $L_{\ell,0}$ [resp. $N_{\ell,0}$] is the unique Zariski open and dense one; it is called the *big cell* in the literature.

1.3.3 Oriented lagrangians

We will also need the notion of *oriented lagrangians*. Firstly, we will define the orientation of a vector space.

Definition 1.3.9. Let V be a finite dimensional F-vector space. An orientation of V is an element in

$$(\bigwedge^{\max} V \setminus \{0\})/F^{\times 2},$$

or equivalently, in

$$\{\text{Basis of } V\}/(\text{automorphisms whose determinant lies in } F^{\times 2}).$$

Here we adopt the usual convention that $\bigwedge^0\{0\} = F$, hence $\mathfrak{o}(\{0\}) = F^\times/F^{\times 2}$.

We will write $\mathfrak{o}(V)$ as the set of orientations of V. The group $F^\times/F^{\times 2}$ acts freely and transitively on $\mathfrak{o}(V)$.

From the second description, it clearly coincides with the usual notion of orientation when $F = \mathbb{R}$. When $F = \mathbb{C}$, there is only one orientation for every space.

Definition-Proposition 1.3.10. For finite dimensional F-vector spaces, we define the following pairings.

1. If $0 \to V' \to V \to V'' \to 0$ is a short exact sequence of finite dimensional F-vector spaces, then the exterior product induces a map compatible with $F^\times/F^{\times 2}$-action:

$$\wedge : \mathfrak{o}(V') \times \mathfrak{o}(V'') \to \mathfrak{o}(V)$$

$$(\xi \mod F^{\times 2}, \eta \mod F^{\times 2}) \mapsto \xi \wedge \tilde{\eta} \mod F^{\times 2},$$

   where $\tilde{\eta} \in \bigwedge^{\dim V''} V$ is an arbitrary preimage of η.

2. Let $\beta : V_1 \times V_2 \to F$ be a perfect pairing between finite dimensional F-vector spaces, it induces a map compatible with $F^\times/F^{\times 2}$-actions:

$$\beta : \mathfrak{o}(V_1) \times \mathfrak{o}(V_2) \to F^\times/F^{\times 2}$$

   as follows: Let $\xi = e_1 \wedge \cdots \wedge e_n \in \bigwedge^{\max} V_1 \setminus \{0\}$ and $\eta = f_1 \wedge \cdots \wedge f_n \in \bigwedge^{\max} V_2 \setminus \{0\}$, let $f_1^*, \ldots, f_n^*$ be the dual basis of $f_1, \ldots, f_n$. Define

$$\beta(\xi \mod F^{\times 2}, \eta \mod F^{\times 2}) := \det(V_1 \xrightarrow{\beta} V_2^*) \mod F^{\times 2}$$

   where the determinant is taken with respect to basis $e_1, \ldots, e_n$ for V_1 and $f_1^*, \ldots, f_n^*$ for V_2. In particular, $\beta(e_i, f_j) = \delta_{ij}$ implies $\beta(e, f) = 1$.

   By convention, β becomes the multiplication map $(F^\times/F^{\times 2})^2 \to F^\times/F^{\times 2}$ when $V_1 = V_2 = \{0\}$.

Proof. The mapping $(\xi, \eta) \mapsto \xi \wedge \tilde{\eta}$ in the first assertion is a well-defined map from $\bigwedge^{\max} V' \times \bigwedge^{\max} V''$ to $\bigwedge^{\max} V$: it is independent of the choice of $\tilde{\eta}$. Observe that the map is compatible with multiplication by $F^\times$, hence it induces a map $\mathfrak{o}(V') \times \mathfrak{o}(V'') \to \mathfrak{o}(V)$ compatible with $F^\times/F^{\times 2}$-actions.

Similarly, the second mapping only depends on $e \in \bigwedge^{\max} V_1$, $f \in \bigwedge^{\max} V_2$ and respects the action of $F^\times$. This suffices to conclude. □

Remark 1.3.11. In particular, take $V_2 = V_1^*$ in the second pairing yields a bijection $\mathfrak{o}(V) \to \mathfrak{o}(V^*)$ by sending a basis to its dual basis.

Definition 1.3.12 (Oriented lagrangians)**.** An *oriented lagrangian* of a symplectic space V is a pair (ℓ, e), usally written as ℓ^e, where $\ell \in \Lambda(V)$ and $e \in \mathfrak{o}(\ell)$. We will write $\Lambda(V)^{\mathrm{or}}$ as the set of oriented lagrangians of V.

When there is no worry of confusion, the superscript e will be omitted.

The construction below will be used to define metaplectic groups.

Definition-Proposition 1.3.13. Let $\ell_1^{e_1}, \ell_2^{e_2} \in \Lambda(V)^{\mathrm{or}}$. Then the symplectic form gives rise to a perfect pairing $\langle , \rangle$ on $(\ell_1/\ell_1 \cap \ell_2) \times (\ell_2/\ell_1 \cap \ell_2)$.

Choose any $e \in \mathfrak{o}(\ell_1 \cap \ell_2)$, then there exists unique orientations $\bar{e}_i \in \mathfrak{o}(\ell_i/\ell_1 \cap \ell_2)$ such that $e \wedge \bar{e}_i = e_i$ $(i = 1, 2)$. Set

$$A_{\ell_1^{e_1}, \ell_2^{e_2}} := \langle \bar{e}_1, \bar{e}_2 \rangle$$

this is independent of the choice of e. Indeed, the assertions on uniqueness and independence follows immediately from the fact that $F^\times/F^{\times 2}$ acts freely and transitively on orientations, and that the operations in the preceding proposition respect those actions.

The following observation will be useful later.

Proposition 1.3.14.

$$A_{\ell_1^{e_1}, \ell_2^{e_2}} = (-1)^{\frac{\dim V}{2} - \dim \ell_1 \cap \ell_2} \cdot A_{\ell_2^{e_2}, \ell_1^{e_1}}$$

Proof. We may suppose that $\ell_1 \neq \ell_2$. If $e_1, \ldots, e_n,\ f_1, \ldots, f_n$ are dual basis for the pairing $(\ell_1/\ell_1 \cap \ell_2) \times (\ell_2/\ell_1 \cap \ell_2) \to F$, then $f_1, \ldots, f_n, -e_1, \ldots, -e_n$ are dual basis for the transposed pairing $(\ell_2/\ell_1 \cap \ell_2) \times (\ell_1/\ell_1 \cap \ell_2) \to F$. Here $n = \frac{\dim V}{2} - \dim \ell_1 \cap \ell_2$. □

Chapter 2

The Maslov index

The *Maslov index* originates from Maslov's work on partial differential equations and is extensively used in symplectic geometry. There exists several definitions sharing the same name: some are geometric, some are analytic, and some are algebraic. The Maslov indices to be introduced in this chapter is a generalization of Kashiwara's algebraic definition. See [2] for the relations between different definitions.

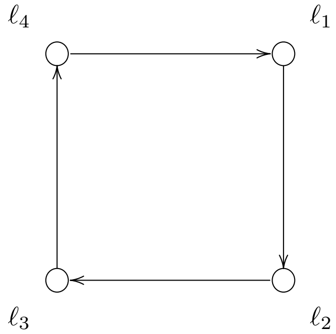
The theory actually applies to any field F of characteristic not 2, not just non-archimedean local fields.

2.1 Basic properties

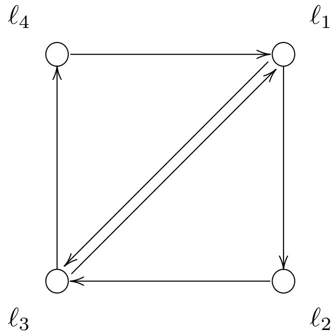
Let $(W, \langle , \rangle)$ be a symplectic space over F. Given n lagrangians $\ell_1, \ldots, \ell_n$ where $n \geq 3$, we are going to associate a class $\tau(\ell_1, \ldots, \ell_n) \in W(F)$. The classes $\tau(\ell_1, \ldots, \ell_n)$ will satisfy the following properties.

1. **Symplectic invariance**. For any $g \in \mathrm{Sp}(W)$,
$$\tau(\ell_1, \ldots, \ell_n) = \tau(g\ell_1, \ldots, g\ell_n).$$
2. **Symplectic additivity**. Let W_1, W_2 be symplectic spaces, $W := W_1 \oplus W_2$. If $\ell_1, \ldots, \ell_n$ are lagrangians of W_1 and $\ell'_1, \ldots, \ell'_n$ are lagrangians of W_2, then
$$\tau(\ell_1 \oplus \ell'_1, \ldots, \ell_n \oplus \ell'_n) = \tau(\ell_1, \ldots, \ell_n) + \tau(\ell'_1, \ldots, \ell'_n).$$
3. **Dihedral symmetry**.
$$\tau(\ell_1, \ldots, \ell_n) = \tau(\ell_2, \ldots, \ell_n, \ell_1),$$
$$\tau(\ell_1, \ell_2, \ldots, \ell_n) = -\tau(\ell_n, \ell_{n-1}, \ldots, \ell_1).$$
4. **Chain condition**. For any $3 \leq k < n$,
$$\tau(\ell_1, \ldots, \ell_n) = \tau(\ell_1, \ldots, \ell_k) + \tau(\ell_1, \ell_k, \ldots, \ell_n).$$

The situation can be visualized by identifying $\{1, \ldots, n\}$ with $\mathbb{Z}/n\mathbb{Z}$, so that the lagrangians are viewed as vertices of a n-gon. Here is an illustration for $n = 4$:



The etymology of *dihedral symmetry* is then clear. The chain condition corresponds to decomposition of polygons. The case $n = 4, k = 3$ is illustrated below:



$$\tau(\ell_1, \ell_2, \ell_3, \ell_4) = \tau(\ell_1, \ell_2, \ell_3) + \tau(\ell_1, \ell_3, \ell_4)$$

2.2 Maslov index as a quadratic space

We are going to associate a canonically defined quadratic space (T, q) to n given lagrangians $\ell_1, \ldots, \ell_n$.

Identify $\{1, \ldots, n\}$ and $\mathbb{Z}/n\mathbb{Z}$ as before. Given n lagrangians $\ell_1, \ldots, \ell_n$ of W. The first step is to construct the sum map $\tilde{\Sigma}$ and the backward difference map $\tilde{\partial}$

$$\tilde{\Sigma} : \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W \to W$$
$$w = (w_i) \mapsto \sum_{i \in \mathbb{Z}/n\mathbb{Z}} w_i$$

$$\tilde{\partial} : \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W \to \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W$$
$$w = (w_i) \mapsto (\tilde{\partial} w)_i = w_i - w_{i-1}$$

where the addition of subscripts is that in $\mathbb{Z}/n\mathbb{Z}$.

Lemma 2.2.1. *The image of $\tilde{\partial}$ is equal to the kernel of $\tilde{\Sigma}$.*

Proof. It is clear that $\operatorname{Im} \tilde{\partial} \subset \operatorname{Ker} \tilde{\Sigma}$. Conversely, if $\tilde{\Sigma}(w) = 0$, we can take

$$\hat{w}_i := \sum_{j=1}^{i} w_j \tag{2.1}$$

It is straightforward to check that $\hat{w}_i - \hat{w}_{i-1} = w_i$ for $i = 1, \ldots, n$. □

Consider the complex

$$\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i \cap \ell_{i+1} \xrightarrow{\partial} \bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i \xrightarrow{\Sigma} W \tag{2.2}$$

where ∂, Σ are the restrictions of $\tilde{\partial}, \tilde{\Sigma}$ on the relevant subspaces. Observe that

$$\operatorname{Ker}\partial = \bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i$$

$$\operatorname{Im}\Sigma = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i$$

$$(\bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i)^\perp = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i^\perp = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i$$

Definition-Proposition 2.2.2. Define a bilinear form q on $\operatorname{Ker}\Sigma$ by the formula

$$q(v, w) := \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle v_i, \hat{w}_i\rangle \tag{2.3}$$

where $\hat{w} = (\hat{w}_i)$ is any element in $\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} W$ satisfying $\tilde{\partial}(\hat{w}) = w$. The formula is independent of choice of $\hat{w}$. Moreover, q is symmetric.

Proof. The existence of $\hat{w}$ such that $\tilde{\partial}(\hat{w}) = w$ is already established. If $\partial(\hat{w} - \hat{w}') = 0$, then $\hat{w}_i - \hat{w}'_i = c \in W$ is independent of i, and

$$\sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle v_i, \hat{w}_i\rangle - \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle v_i, \hat{w}'_i\rangle = \langle \sum_{i\in\mathbb{Z}/n\mathbb{Z}} v_i, c\rangle = 0$$

since $v \in \operatorname{Ker}\Sigma$, hence this bilinear form is well-defined.

To show that q is symmetric, we do a summation by parts

$$\begin{aligned} q(v, w) &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle v_i, \hat{w}_i\rangle \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle \hat{v}_i - \hat{v}_{i-1}, \hat{w}_i\rangle \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle \hat{v}_i, \hat{w}_i\rangle - \langle \hat{v}_{i-1}, \hat{w}_i\rangle) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle \hat{v}_i, \hat{w}_i - \hat{w}_{i+1}\rangle \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle w_i, \hat{v}_{i-1}\rangle \end{aligned}$$

It remains to show that the last sum is equal to $\sum_{i\in\mathbb{Z}/n\mathbb{Z}}\langle w_i, \hat{v}_i\rangle = q(w, v)$. Indeed, their difference is

$$\sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle w_i, \hat{v}_i - \hat{v}_{i-1}\rangle = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \langle w_i, v_i\rangle = 0$$

since $v_i, w_i \in \ell_i$. □

Lemma 2.2.3. *We have $q(v, w) = 0$ if $v \in \operatorname{Im}(\partial)$ or $w \in \operatorname{Im}(\partial)$.*

Proof. Since q is symmetric, it suffices to consider the case $w = \partial(w')$ for some $w' = (w'_i)_i \in \bigoplus_n \ell_i \cap \ell_{i+1}$. Then we may take $\hat{w}_i = w'_i$ in (2.3), and then $\langle v_i, w'_i\rangle = 0$ for all i since $\ell_i = \ell_i^\perp$. □

Remark 2.2.4. Using (2.1), the formula (2.3) has the explicit (but less symmetric) form

$$q(v,w) = \sum_{n\geq i>j\geq 1} \langle v_i, w_j\rangle = \sum_{n\geq i>j>1} \langle v_i, w_j\rangle. \tag{2.4}$$

Definition 2.2.5 (The Maslov index)**.** Set $T := \operatorname{Ker}\Sigma/\operatorname{Im}\partial$. The class of the quadratic space (T, q) in $W(F)$ will be denoted by $\tau(\ell_1, \dots, \ell_n)$, it is called the *Maslov index* associated to the lagrangians $\ell_1, \dots, \ell_n$.

If our construction is applied to the case $n = 2$, then $\operatorname{Ker}\Sigma = \operatorname{Im}\partial$, and the quadratic space (T, q) is trivial.

The following assertion is taken for granted for the moment; the proof is postponed to §2.6.

Theorem 2.2.6. (T, q) *is non-degenerate. Moreover,* τ *satisfies all the properties listed in section 2.1.*

2.3 Relation with Kashiwara index

M. Kashiwara defines the Maslov index associated to 3 lagrangians ℓ_1, ℓ_2, ℓ_3 by the following explicit formula.

Definition 2.3.1. Let ℓ_1, ℓ_2, ℓ_3 be 3 lagrangians of W. Set $K := \ell_1 \oplus \ell_2 \oplus \ell_3$ and define the quadratic form q^{Kash} on K as follows

$$q^{\text{Kash}}(v,w) := \frac{1}{2}(\langle v_1, w_2 - w_3\rangle + \langle v_2, w_3 - w_1\rangle + \langle v_3, w_1 - w_2\rangle)$$

Its class in $W(F)$ is denoted by $\tau^{\text{Kash}}(\ell_1, \ell_2, \ell_3)$.

Remark 2.3.2. For $(v_1, v_2, v_3) \in K$, we have

$$q^{\text{Kash}}((v_1, v_2, v_3)) = \langle v_1, v_2\rangle + \langle v_2, v_3\rangle + \langle v_3, v_1\rangle$$

This is the usual formula for q^{Kash} in the literature.

The goal of this section is to prove the following

Proposition 2.3.3.

$$\tau^{Kash}(\ell_1, \ell_2, \ell_3) = \tau(\ell_1, \ell_2, \ell_3)$$

Proof. The proof is based on the easy observations below.

- $I := \ell_1 \subset \ell_1 \oplus \ell_2 \oplus \ell_3$ is an isotropic subspace for K.
- $I^\perp = \{(v_1, v_2, v_3) : v_2 - v_3 \in \ell_1\}$.
- The map $(v_1, v_2, v_3) \mapsto (v_2 - v_3, -v_2, v_3)$ defines an isometric surjection from $I^\perp$ onto $\operatorname{Ker}\Sigma$. Indeed, the surjectivity is evident, while

  $$q^{\text{Kash}}((v_1, v_2, v_3)) = \langle v_2, v_3\rangle = q((v_2 - v_3, -v_2, v_3))$$

  for all $(v_1, v_2, v_3) \in I^\perp$ by using formula (2.4). Note that I is mapped to 0.

From those observations, we have an isometry from the non-degenerate quotient of $I^\perp/I$ onto T. However $I^\perp/I$ is Witt equivalent to K by Proposition 1.2.11. □

2.4 Dimension and discriminant

The dimension and discriminant of the quadratic space (T, q) associated to $\ell_1, \ldots, \ell_n$ can be explicitly determined.

Proposition 2.4.1.

$$\dim T = \frac{(n-2)\dim W}{2} - \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \dim(\ell_i \cap \ell_{i+1}) + 2\dim \bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i. \tag{2.5}$$

Proof. Consider the complex (2.2); its Euler-Poincaré characteristic is

$$\dim(\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i \cap \ell_{i+1}) - \dim(\bigoplus_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i) + \dim W$$

which is equal to that of its cohomology

$$\dim(\bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i) - \dim T + \dim(W/\sum_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i)$$

where we have used Theorem 2.2.6. Recall that $(\bigcap_i \ell_i)^\perp = \sum_i \ell_i$, it follows that

$$\dim T = \frac{(n-2)\dim W}{2} - \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \dim(\ell_i \cap \ell_{i+1}) + 2\dim \bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i.$$

□

Let $\ell_1, \ldots, \ell_n$ be lagrangians of W equipped with arbitrary orientations ($n \geq 3$). Let $A_{\ell_i,\ell_{i+1}}$ be the element in $F^\times/F^{\times 2}$ defined in Definition-Proposition 1.3.13. Note that we have omitted the superscripts of orientations of $\ell_1, \ldots, \ell_n$.

Proposition 2.4.2. *Notations as above. We have*

$$D(q) = (-1)^{\frac{\dim W}{2} + \dim \bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} \ell_i} \prod_{i\in\mathbb{Z}/n\mathbb{Z}} A_{\ell_i,\ell_{i+1}}. \tag{2.6}$$

We will proceed by several reduction steps. For any subsequence $i_1, \ldots, i_s$ of $1, \ldots, n$, let $(T_{i_1,\ldots,i_s}, q_{i_1,\ldots,i_s})$ be the quadratic space constructed in §2.2.

1. **Reduction to $n = 3$ lagrangians**. Given n lagrangians $\ell_1, \ldots, \ell_n$ ($n > 3$), the chain condition asserts that

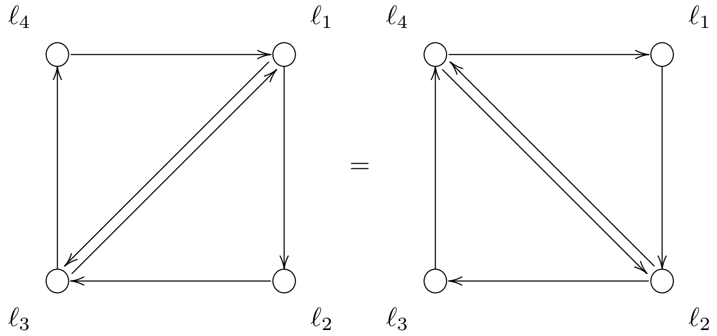
   $$\tau(\ell_1, \ldots, \ell_n) = \tau(\ell_1, \ldots, \ell_k) + \tau(\ell_1, \ell_k, \ldots, \ell_n) \text{ for any } 3 \leq k < n.$$

   Hence the spaces $T_{1,\ldots,n}$ and $T_{1,\ldots,k} \oplus T_{1,k,\ldots,n+1}$ becomes isometric after taking direct product with some copies of the hyperbolic plane H. Each copy of H has dimension 2 and contributes -1 to the discriminant. Hence

   $$D(q_{1,\ldots,n}) = (-1)^{\frac{\dim q_{1,\ldots,n} - \dim q_{1,\ldots,k} - \dim q_{1,k,\ldots,n}}{2}} \cdot D(q_{1,\ldots,k}) \cdot D(q_{1,k,\ldots,n})$$

   Using the dimension formula (2.5), it follows that formula (2.6) holds for any two among $q_{1,\ldots,n}, q_{1,\ldots,k}, q_{1,k,\ldots,n}$ if and only if it holds for all the three. Therefore our problem can be reduced to the case $n = 3$.

2. **Reduction to 3 transversal lagrangians**. Suppose that $n = 3$. If $\bigcap \ell_i \neq \{0\}$, choose another lagrangian ℓ_4 transversal to ℓ_1, ℓ_2, ℓ_3. The chain condition and dihedral symmetry implies that



$$\tau(\ell_1, \ell_2, \ell_3) + \tau(\ell_1, \ell_3, \ell_4) = \tau(\ell_1, \ell_2, \ell_3, \ell_4) = \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_2, \ell_4, \ell_1)$$

By virtue of the first reduction step, it suffices to consider the case of 3 lagrangians where ℓ_3 is transversal to ℓ_1 and ℓ_2.

3. **Reduction to a simpler quadratic space**. Given three lagrangians ℓ_1, ℓ_2, ℓ_3 with ℓ_3 transversal to ℓ_1, ℓ_2, then $\tau(\ell_1, \ell_2, \ell_3)$ is Witt equivalent to the quadratic form on ℓ_2

$$p_{1,2,3}(x_2) = \langle \pi_1(x_2), x_2 \rangle, \tag{2.7}$$

in which $\pi_1 : \ell_2 \to \ell_1$ is the projection with respect to the splitting $W = \ell_1 \oplus \ell_3$. P. Perrin gave a proof by explicit formulas (see [15] 1.4.2). Beware that $p_{1,2,3}$ could degenerate; its kernel is evidently $\ell_1 \cap \ell_2$ by inspecting formula (2.7). Hence the non-degenerate quotient has dimension $\frac{\dim W}{2} - \dim \ell_1 \cap \ell_2$, which is just $\dim T_{1,2,3}$. The same argument in the first reduction step reduces the problem to showing that

$$D(p_{1,2,3}) = (-1)^{\frac{\dim W}{2}} A_{\ell_1,\ell_2} A_{\ell_2,\ell_3} A_{\ell_3,\ell_1}.$$

Proof. By 1.3.14, it amounts to

$$D(p_{1,2,3}) = A_{\ell_1,\ell_2} A_{\ell_2,\ell_3} A_{\ell_1,\ell_3}$$

Let $a_{j,i} : \ell_i/\ell_i \cap \ell_j \xrightarrow{\sim} (\ell_j/\ell_i \cap \ell_j)^*$ be the isomorphism induced by the symplectic pairing $\langle -, - \rangle$, where $i, j = 1, 2, 3$. Recall that $A_{\ell_j,\ell_i} = \det a_{j,i} \mod F^{\times 2}$.

It follows immediately from formula (2.7) that $p_{1,2,3}$ corresponds to the symmetric isomorphism

$$a_{2,1} \circ \Phi_{1,2} : \ \ell_2/\ell_1 \cap \ell_2 \xrightarrow{\sim} (\ell_2/\ell_1 \cap \ell_2)^*$$

where $\Phi_{1,2} : \ell_2/\ell_1 \cap \ell_2 \to \ell_1/\ell_1 \cap \ell_2$ is induced by $(\pi_1)|_{\ell_2}$. Since $(\pi_1)|_{\ell_2}$ is identity on $\ell_1 \cap \ell_2$, we have $\det \Phi_{1,2} = \det(\pi_1)|_{\ell_2}$.

On the other hand, $(\pi_1)|_{\ell_2} = a_{3,1}^{-1} \circ a_{3,2}$ by its definition. Since $\det a_{3,1}^{-1} = \det a_{3,1} \mod F^{\times 2}$, the proof is now complete. □

For any $\ell, \ell' \in \Lambda(W)^{\mathrm{or}}$, define

$$m(\ell, \ell') := \gamma(1)^{\frac{\dim W}{2} - \dim \ell \cap \ell' - 1} \gamma(A_{\ell,\ell'}). \tag{2.8}$$

Using (2.6) and the properties of Weil character $\gamma(-)$ in §1.2.3, the square of $\gamma(\tau(\ell_1, \ldots, \ell_n))$ can be expressed via orientated lagrangians

Theorem 2.4.3 (M. Vergne for $F = \mathbb{R}$, P. Perrin for general case)**.** *Fix arbitrary orientations on n lagrangians $\ell_1, \ldots, \ell_n$ of W ($n \geq 3$), then*

$$\gamma(\tau(\ell_1, \ldots, \ell_n))^2 = \prod_{i \in \mathbb{Z}/n\mathbb{Z}} m(\ell_i, \ell_{i+1})^2,$$

or equivalently

$$\gamma(\tau(\ell_1, \ldots, \ell_n)) = \pm \prod_{i \in \mathbb{Z}/n\mathbb{Z}} m(\ell_i, \ell_{i+1}).$$

Proof. Recall from §1.2.3 that $x \mapsto \frac{\gamma(x)^2}{\gamma(1)^2}$ is a character. Using the previous proposition, the right-hand side of the first equation can be written as

$$\begin{aligned}
&\prod_{i \in \mathbb{Z}/n\mathbb{Z}} \left(\gamma(1)^{\frac{\dim W}{2} - \dim \ell_i \cap \ell_{i+1}} \cdot \frac{\gamma(A_{\ell, \ell_{i+1}})}{\gamma(1)} \right)^2 \\
=& \gamma(1)^{n \dim W - 2 \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \dim \ell_i \cap \ell_{i+1}} \cdot \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \frac{\gamma(A_{\ell_i, \ell_{i+1}})^2}{\gamma(1)^2} \\
=& \gamma(1)^{n \dim W - 2 \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \dim \ell_i \cap \ell_{i+1}} \cdot \frac{\gamma((-1)^{\frac{\dim W}{2} - \dim \bigcap_i \ell_i} D(q))^2}{\gamma(1)^2} \\
=& \gamma(1)^{n \dim W - 2 \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \dim \ell_i \cap \ell_{i+1}} \cdot \left(\frac{\gamma(-1)}{\gamma(1)} \right)^{\dim W - 2 \dim \bigcap_i \ell_i} \cdot \frac{\gamma(D(q))^2}{\gamma(1)^2}
\end{aligned}$$

Recall that $\gamma(1) = \gamma(-1)^{-1}$, the last term is also

$$\gamma(1)^{(n-2) \dim W - 2 \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \dim \ell_i \cap \ell_{i+1} + 4 \dim \bigcap_i \ell_i} \cdot \frac{\gamma(D(q))^2}{\gamma(1)^2}.$$

The dimension formula (2.5) and Corollary 1.2.16 (2) identify this with $\gamma(\tau(\ell_1, \ldots, \ell_n))^2$. □

Finally, here are some general properties of the scalars $m(\ell, \ell')$,

Proposition 2.4.4. *Let $\ell, \ell' \in \Lambda(W)^{\mathrm{or}}$, then*

1. *$m(g\ell, g\ell') = m(\ell, \ell')$ for every $g \in \mathrm{Sp}(W)$, if $g\ell, g\ell'$ carry the transported orientations.*
2. *$m(\ell, \ell) = 1$.*
3. *$m(\ell, \ell') = \pm m(\ell', \ell)^{-1}$.*
4. *If $W = W_1 \oplus W_2$, $\ell_i, \ell_i' \in \Lambda(W_i)^{\mathrm{or}}$ ($i = 1, 2$), then*

$$m(\ell_1 \oplus \ell_2, \ell_1', \oplus \ell_2') = m(\ell_1, \ell_1') m(\ell_2, \ell_2') \cdot (A_{\ell_1, \ell_1'}, A_{\ell_2, \ell_2'}).$$

where the $(,)$ in the last assertion stands for the quadratic Hilbert symbol.

Proof. The first assertion is evident. The second follows directly from definition,

$$m(\ell, \ell) = \gamma(1)^{\frac{\dim W}{2} - \dim \ell - 1} \gamma(1) = \gamma(1)^{-1} \gamma(1) = 1.$$

As for the third one [1], note that $\gamma(\tau(\ell, \ell, \ell')) = 1$ since $\tau(\ell, \ell, \ell')$ is represented by a vector space of dimension 0 (use equation (2.5)). Then Theorem 2.4.3 implies that

$$1 = \gamma(\tau(\ell, \ell, \ell'))^2 = m(\ell, \ell)^2 m(\ell, \ell')^2 m(\ell', \ell)^2,$$

hence $m(\ell, \ell') = \pm m(\ell', \ell)^{-1}$.

The last assertion follows from a direct computation using definition and properties of Weil character listed in §1.2.3. □

[1]See [15] 1.5.2 for a direct proof.

2.5 The dual form

Granting the truth of Theorem 2.2.6, this section is devoted to the description of the dual form of (T, q).

Consider the dual of the complex (2.2). Using the symplectic form $\langle -, - \rangle$, it can be identified with

$$W \xrightarrow{\Sigma^*} \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W/\ell_i \xrightarrow{\partial^*} \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W/(\ell_i + \ell_{i+1}) \tag{2.9}$$

$$\Sigma^*(w) = (w + \ell_i)_{i \in \mathbb{Z}/n\mathbb{Z}}$$
$$\partial^*((w_i)_{i \in \mathbb{Z}/n\mathbb{Z}}) = (w_i - w_{i+1})_{i \in \mathbb{Z}/n\mathbb{Z}}$$

Set $S := \operatorname{Ker} \partial^* = \{(\bar{x}_i) \in \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W/\ell_i : \bar{x}_{i+1} - \bar{x}_i \in \ell_i + \ell_{i+1}\}$.

The quadratic form q yields an isomorphism $\Phi : T \xrightarrow{\sim} T^*$ such that $\Phi(\bar{v})(\bar{w}) = q(\bar{v}, \bar{w})$ for all $\bar{v}, \bar{w} \in T$. By inspecting (2.3), the following diagram describes Φ.

$$\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \ell_i \rightarrowtail \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W \underset{\partial}{\overset{w \mapsto \hat{w}}{\rightleftarrows}} \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W \twoheadrightarrow \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W/\ell_i$$

Namely, given $\bar{v} \in T$, choose a representative $v \in \operatorname{Ker} \Sigma \subset \bigoplus_i \ell_i$, we can choose $w \in \bigoplus_i W$ such that $\partial w = v$. Then the projection $\bar{w}$ of w in $(\bigoplus_i W/\ell_i)/\operatorname{Im} \Sigma^*$ gives $\Phi(\bar{v})$.

Formulas of the dual form. The dual form q^* on T^* is characterized by

$$q^*(\Phi(\bar{v}), \Phi(\bar{v}')) = q(\bar{v}, \bar{v}') \qquad \text{for all } \bar{v}, \bar{v'} \in T.$$

Given $\bar{v}, \bar{v}' \in T$, let $\bar{w} := \Phi(\bar{v}), \bar{w}' := \Phi(\bar{v}')$ and take representatives $v, v' \in \operatorname{Ker} \Sigma$. We can assume $w = \hat{v}, w' = \hat{v'} \in S$ as in the construction above, then

$$\begin{aligned} q^*(\bar{w}, \bar{w}') &= q(\bar{v}, \bar{v}') = q(\bar{v}', \bar{v}) \\ &= -\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \langle \hat{v}_i, v'_i \rangle \\ &= -\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \langle w_i, v'_i \rangle \\ &= -\sum_{i \in \mathbb{Z}/n\mathbb{Z}} \langle w_i, w'_i - w'_{i-1} \rangle \end{aligned} \tag{2.10}$$

We proceed to lift this form to S.

Definition 2.5.1. Given $x_i, x_{i+1} \in W^2$ such that $x_{i+1} - x_i \in \ell_i + \ell_{i+1}$, define a linear functional $\epsilon_{i,i+1}(x_i, x_{i+1})$ on $\ell_i + \ell_{i+1}$ as follows. For $v \in \ell_i + \ell_{i+1}$, suppose that $v = a + b$ with $a \in \ell_i, b \in \ell_{i+1}$, then define

$$\epsilon_{i,i+1}(x_i, x_{i+1})(v) := \langle a, x_i \rangle + \langle b, x_{i+1} \rangle.$$

This functional is well-defined. Indeed, if $v = a + b = a' + b'$ with $a, a' \in \ell_i, b, b' \in \ell_{i+1}$, then $a - a' = b' - b \in \ell_i \cap \ell_{i+1}$, hence

$$\langle a, x_i \rangle + \langle b, x_{i+1} \rangle - \langle a', x_i \rangle - \langle b', x_{i+1} \rangle = 2\langle a - a', x_i - x_{i+1} \rangle = 0.$$

Proposition 2.5.2. *The bilinear form q^* on S defined by*

$$q^*(\bar{w}, \bar{w'}) = \sum_{i \in \mathbb{Z}/n\mathbb{Z}} \epsilon_{i,i+1}(w_i, w_{i+1})(w'_{i+1} - w'_i)$$

where $w, w' \in \bigoplus_i W$ are representatives of $\bar{w}, \bar{w'}$, satisfies

1. q^* is symmetric.

2. q^* lifts q^* to S.

Proof. First of all, note that the $\sum_{i\in\mathbb{Z}/n\mathbb{Z}} \epsilon_{i,i+1}(w_i, w_{i+1})(w'_{i+1} - w'_i)$ is independent of choice of representatives w, w'.

1. **Symmetry:** For any $\bar{x}, \bar{y} \in S_{1,\dots,n}$, choose representatives $x, y \in \bigoplus_i W$. Suppose that

$$x_{i+1} - x_i = r_i + s_i, \qquad r_i \in \ell_i, s_i \in \ell_{i+1}$$
$$y_{i+1} - y_i = a_i + b_i, \qquad a_i \in \ell_i, b_i \in \ell_{i+1}$$

Hence

$$\begin{aligned} q^*(x,y) &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} \epsilon_{i,i+1}(x_i, x_{i+1})(y_{i+1} - y_i) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle a_i, x_i\rangle + \langle b_i, x_{i+1}\rangle) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle a_i, x_i\rangle + \langle b_i, x_i + r_i + s_i\rangle) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle a_i, x_i\rangle + \langle b_i, x_i + r_i\rangle) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle a_i + b_i, x_i\rangle + \langle b_i, r_i\rangle) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle y_{i+1} - y_i, x_i\rangle + \langle b_i, r_i\rangle) \end{aligned}$$

Interchanging x, y gives

$$q^*(y,x) = \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle x_{i+1} - x_i, y_i\rangle + \langle s_i, a_i\rangle)$$

Observe that $\langle y_{i+1} - y_i, x_{i+1} - x_i\rangle = \langle a_i + b_i, r_i + s_i\rangle = \langle a_i, s_i\rangle + \langle b_i, r_i\rangle$, thus

$$\begin{aligned} q^*(x,y) - q^*(y,x) &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle y_{i+1} - y_i, x_i\rangle - \langle x_{i+1} - x_i, y_i\rangle + \langle b_i, r_i\rangle - \langle s_i, a_i\rangle) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle y_{i+1} - y_i, x_i\rangle - \langle x_{i+1} - x_i, y_i\rangle + \langle y_{i+1} - y_i, x_{i+1} - x_i\rangle) \\ &= \sum_{i\in\mathbb{Z}/n\mathbb{Z}} (\langle y_{i+1} - y_i, x_i\rangle + \langle y_{i+1}, x_{i+1} - x_i\rangle) \\ &= 0 \end{aligned}$$

2. **Lifting**: Retain the notations used in deriving equations (2.10). For $\bar{w} = \Phi(\bar{v}), \bar{w}' = \Phi(\bar{v}') \in T^*$, and their representatives $w, w' \in S.$, since $w'_{i+1} - w'_i = v'_{i+1} \in \ell_{i+1}$, we have

$$\epsilon_{i,i+1}(w_i, w_{i+1})(w'_{i+1} - w'_i) = \langle w'_{i+1} - w'_i, w_{i+1}\rangle = -\langle w_{i+1}, w'_{i+1} - w'_i\rangle$$

The last terms sum to $-\sum_i \langle w_i, w'_i - w'_{i-1}\rangle = q^*(\bar{w}, \bar{w}')$ as derived in equation (2.10).

□

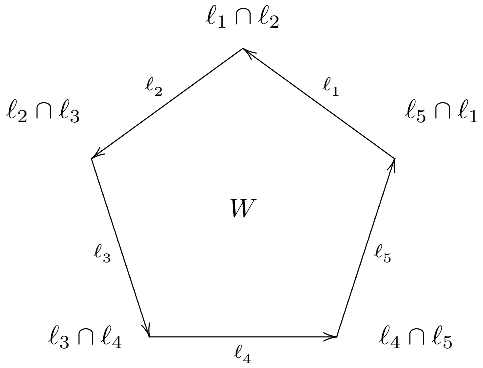
2.6 An interpretation via sheaf cohomology

A sheaf-theoretic construction of minus (T, q), due to A. Beilinson, will be given in this section. Theorem 2.2.6 can then be easily deduced. Although it can also be proved by more elementary means (see [17]), the sheaf-theoretic approach is less tricky, and makes it possible to *see through* the dihedral symmetry and chain condition.

Setting. Let W be a symplectic space and $\ell_1, \ldots, \ell_n$ be n lagrangians ($n \geq 3$). Consider the solid n-gon with vertices labelled by $\mathbb{Z}/n\mathbb{Z}$ and with the counter-clockwise orientation. Denote it by D. Let U be the interior of D and $j : U \to X$ the inclusion. Define a constructible subsheaf P of the constant sheaf $\underline{W}$ on D by requiring

$$P_x := \begin{cases} \ell_{i-1} \cap \ell_i & \text{, if } x = \text{the i-th vertex.} \\ \ell_i & \text{, if } x \in \text{the edge } (i, i+1) \\ W & \text{, if } x \in U \end{cases}$$

The case $n = 5$ is depicted below.



Roughly speaking, $(T, -q)$ is somehow the simplicial cohomology with coefficients in P, and there is a version of Poincaré duality. We will cast everything in terms of sheaf cohomology and employ Verdier duality theorem, to be recalled below.

Theorem 2.6.1 (Verdier duality, adapted for our present setting. cf. [8])**.** *Let X be a connected compact manifold of dimension d with boundary, and let $D^b(X)$ be the derived category of bounded complexes of sheaves of F-modules. Then there exists $\omega_X \in D^b(X)$ such that*

1. *If we define*
$$\mathbb{D}\mathcal{F} := R\mathscr{H}om(\mathcal{F}, \omega_X) \quad \textit{(Verdier's duality operator)}$$
then there exists a functorial isomorphism
$$R\Gamma(X, \mathbb{D}(-)) \simeq R\Gamma(X, -)^*.$$

2. *In particular, $R^0\Gamma(X, \omega_X) = H^0(R\Gamma(X, \omega_X)) = F$.*

3. *F-orientations for X correspond to isomorphisms $\omega_X \simeq j_!\underline{F}[d]$, where $j : X \setminus \partial X \to X$ is the inclusion map.*

4. *For any sheaf $\mathcal{F}$, regard it as a complex concentrated at degree 0. In view of the last identification, the isomorphism is compatible with cup products*
$$H^p(X, \mathcal{F}) \times H^{d-p}(X, \mathscr{H}om(\mathcal{F}, j_!\underline{F})) \xrightarrow{\smile} H^d(X, j_!\underline{F}) \xrightarrow{\sim} F.$$

Lemma 2.6.2. *The symplectic form induces an isomorphism*

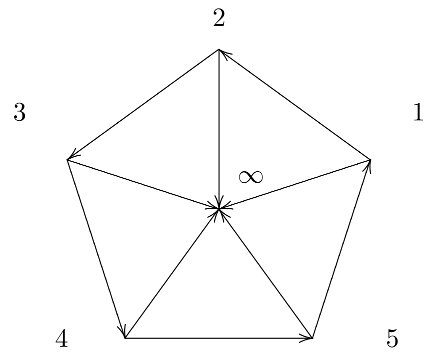
$$P \xrightarrow{\sim} H^0(\mathbb{D}P[-2]) = \mathscr{H}om(P, j_!\underline{F}).$$

Proof. Let V be any nonempty open subset of D such that $V \cap U$ is connected, then $V \cap U \neq \emptyset$ and $\Gamma(V, P) \to \Gamma(V \cap U, P) = W$ is injective. Hence $\mathscr{H}om(P, j_! \underline{F})$ can be identified as a subsheaf of $\underline{W}$ as

$$(\mathscr{H}om(P, j_!\underline{F}))_x \simeq \begin{cases} \ell_{i-1}^{\perp} \cap \ell_i^{\perp} = \ell_{i-1} \cap \ell_i & \text{, if } x = \text{the i-th vertex.} \\ \ell_i^{\perp} = \ell_i & \text{, if } x \in \text{the edge } (i, i+1) \\ W & \text{, if } x \in U \end{cases}$$

This completes the proof. ☐

Relation with $(T, -q)$. Triangulate D by adding the center ∞ and connect it to the other vertices to obtain edges (i, ∞), $i \in \mathbb{Z}/n\mathbb{Z}$. The vertices in each simplex are ordered by $i < i+1 < \infty$. The case $n = 5$ is depicted below:



The cohomologies of P become trivial if we remove at least one edge from ∂D, hence the sheaf cohomology $H^\bullet(D, P) = H^\bullet_{\check{\mathrm{C}}\mathrm{ech}}(D, P) = H^\bullet_{\mathrm{simp}}(D, P)$; by $H^\bullet_{\mathrm{simp}}(D, P)$ we mean the cohomology defined by the cochain complex with coefficients in P described below.

- A 2-cochain with coefficients in P is a datum $\alpha = \{\alpha_{i,i+1,\infty} \in W : i \in \mathbb{Z}/n\mathbb{Z}\}$.
- A 1-cochain with coefficients in P is a datum $\alpha = \{\alpha_{i,i+1} \in \ell_i,\ \alpha_{i,\infty} \in W, i \in \mathbb{Z}/n\mathbb{Z}\}$. Coboundary map is $(d\alpha)_{i,i+1,\infty} = \alpha_{i+1,\infty} - \alpha_{i,\infty} + \alpha_{i,i+1}$.

  In particular, $d\alpha = 0$ if and only if $\alpha_{i,i+1} = \alpha_{i,\infty} - \alpha_{i+1,\infty}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. This condition implies $\sum_i \alpha_{i,i+1} = 0$.
- A 0-cochain with coefficients in P is a datum $\alpha = \{\alpha_i \in \ell_{i-1} \cap \ell_i, i \in \mathbb{Z}/n\mathbb{Z};\ \alpha_\infty \in W\}$. Coboundary map is $(d\alpha)_{i,i+1} = \alpha_{i+1} - \alpha_i$, $(d\alpha)_{i,\infty} = \alpha_\infty - \alpha_i$.

Lemma 2.6.3. *Recall the notation in (2.2), we have $H^1(D, P) \simeq T$ via $[\alpha] \mapsto (\alpha_{i,i+1})_{i \in \mathbb{Z}/n\mathbb{Z}}$ mod $\operatorname{Im} \partial$.*

Proof. Define a map $g : Z^1(D, P) \to \operatorname{Ker} \Sigma$ by setting $g(\alpha) = (\alpha_{i,i+1})_{i \in \mathbb{Z}/n\mathbb{Z}}$. Using the description of cochains above, it is routine to show that g induces $H^1(D, P) \simeq T = \operatorname{Ker} \Sigma / \operatorname{Im} \partial$. ☐

We proceed to study the quadratic form on $H^1(D, P)$.

- A 2-cochain with coefficients in $j_!\underline{F}$ is a datum $\{\gamma_{i,i+1,\infty} \in F,\ i \in \mathbb{Z}/n\mathbb{Z}\}$.
- A 1-cochain with coefficients in $j_!\underline{F}$ is a datum $\{\gamma_{i,\infty} \in F,\ i \in \mathbb{Z}/n\mathbb{Z}\}$. Coboundary map is $(d\gamma)_{i,i+1,\infty} = \gamma_{i+1,\infty} - \gamma_{i,\infty}$.
- The isomorphism $H^2(D, j_!\underline{F}) \xrightarrow{\sim} F$ is given by $[(\gamma_{i,i+1,\infty})_i] \mapsto \sum_i \gamma_{i,i+1,\infty}$.
- Given α, α' two 1-cocycles with coefficients in P. Their cup-product is given by the usual formula

  $$(\alpha \smile \alpha')_{i,i+1,\infty} = \langle \alpha_{i,i+1}, \alpha'_{i+1,\infty} \rangle. \tag{2.11}$$

  After composing with the previous isomorphism, it takes the value $\sum_i \langle \alpha_{i,i+1}, \alpha'_{i+1,\infty} \rangle$.

Lemma 2.6.4. *The bilinear form induced by* $\smile: H^1(D,P) \times H^1(D,P) \to F$ *coincides with* $-q$.

Proof. Observe that $\alpha'_{i+1,\infty} - \alpha'_{i,\infty} = -\alpha'_{i,i+1}$ since $d\alpha' = 0$, then compare (2.11) and (2.3). $\square$

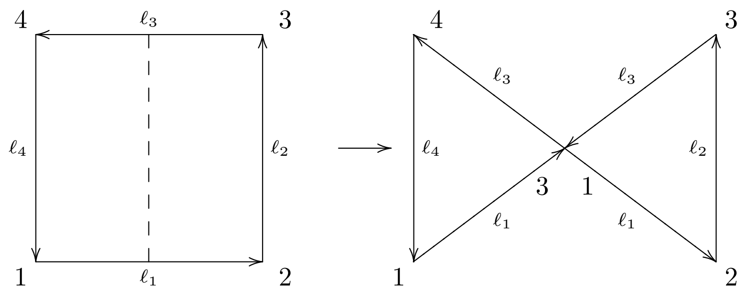
Now we can prove Theorem 2.2.6.

Proof of Theorem 2.2.6. In view of the previous lemma, it suffices to deal with $(H^1(D,P), \smile)$. First of all, $(H^1(D,P), \smile)$ is non-degenerate by Theorem 2.6.1.

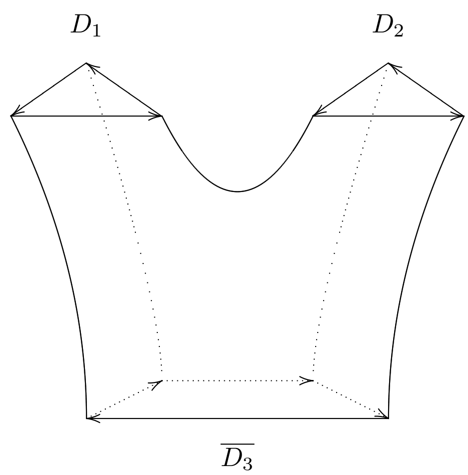
1. **Symplectic invariance/additivity** Obvious.
2. **Non-degeneracy**. This is a part of Theorem 2.6.1.
3. **Dihedral symmetry**. The invariance under cyclic permutation of $\ell_1, \ldots, \ell_n$ is clear, while $\ell_1, \ldots, \ell_n \mapsto \ell_n, \ldots, \ell_1$ flips the orientation, thus gives the cup-product a minus sign.
4. **Chain condition**. This can be interpreted by a bordism. Given $3 \le k < n$. Let
   - (D_1, P_1) be the datum constructed from $\ell_1, \ldots, \ell_k$.
   - (D_2, P_2) be the datum constructed from $\ell_1, \ell_k, \ldots, \ell_n$.
   - (D_3, P_3) be the datum constructed from $\ell_1, \ldots, \ell_n$.

   and let $(T_{1,\ldots,k}, q_{1,\ldots,k})$, $(T_{1,k,\ldots,n}, q_{1,k,\ldots,n})$ and $(T_{1,\ldots,n}, q_{1,\ldots,n})$ be the corresponding quadratic spaces.

   Denote pictographically by Y the space obtained from gluing $D_1 \times [0,1]$, $D_2 \times [0,1]$ and $D_3 \times [0,1]$ via collapsing the line joining the midpoints of edges $(1,2)$ and $(k,k+1)$ in $D_3 \times \{1\}$. The sheaves $P_i \boxtimes \underline{F}$ $(i = 1,2,3)$ glue in the natural way to give a constructible sheaf $\hat{P}$ on Y. A picture of $n = 4, k = 3$ is depicted below



   So the result is a bordism-like object after some juggling.



Let $\hat{U}$ be the interior of Y, $\bar{D}_3$ be D_3 with reversed (i.e. clockwise) orientation, and set $\hat{D} := D_1 \sqcup D_2 \sqcup \bar{D}_3$, which is visually the "caps" of Y. Define the following immersions

$$\begin{aligned} i : \hat{D} &\longrightarrow Y \\ j_{\hat{U}} : \hat{U} &\longrightarrow Y \\ j : Y \setminus \hat{D} &\longrightarrow Y \end{aligned}$$

Then the dualizing sheaf is $(j_{\hat{U}})_! \underline{F}$. By the same reasoning as in Lemma 2.6.2, the symplectic form on W induces an isomorphism

$$j_! j^* \hat{P} \xrightarrow{\sim} H^0(\mathbb{D}\hat{P}[-3]).$$

The following lemma will conclude the proof.

□

Lemma 2.6.5. *The image of* $\pi : H^1(Y, \hat{P}) \to H^1(Y, i_* i^* \hat{P}) = H^1(\hat{D}, i^* \hat{P})$ *is isotropic of half the dimension of* $H^1(\hat{D}, i^* \hat{P})$.

Proof. There is a short exact sequence of sheaves

$$0 \longrightarrow j_! j^* \hat{P} \longrightarrow \hat{P} \longrightarrow i_* i^* \hat{P} \longrightarrow 0.$$

By various compatibilities of Verdier duality, there is a commutative diagram with commutative rows.

$$\begin{array}{ccccc} H^1(Y, \hat{P}) & \xrightarrow{\pi} & H^1(\hat{D}, i^* \hat{P}) & \longrightarrow & H^2(Y, j_! j^* \hat{P}) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ H^2(Y, \mathbb{D}\hat{P})^* & \longrightarrow & H^1(\hat{D}, i^* \hat{P})^* & \xrightarrow{\pi^*} & H^1(Y, \hat{P})^* \end{array}$$

The middle vertical isomorphism induces the quadratic structure on $H^1(\hat{D}, i^* \hat{P})$, which is the orthogonal sum of

$$(T_{1,\dots,k}, -q_{1,\dots,k}) \oplus (T_{1,k,\dots,n}, -q_{1,k,\dots,n}) \oplus (T_{1,\dots,n}, q_{1,\dots,n})$$

By exactness, $\operatorname{Im} \pi$ is isotropic. On the other hand, the diagram implies that

$$\operatorname{Im} \pi \simeq \operatorname{Ker} \pi^* = (\operatorname{Coker} \pi)^*.$$

Hence $\dim \operatorname{Im} \pi = \frac{\dim H^1(\hat{D}, i^* \hat{P})}{2}$. □

Remark 2.6.6. The proof above is very similar to that of the invariance of signature of manifolds under cobordisms.

Chapter 3

The Heisenberg group

3.1 Basic definitions

Recall our situation that F is a non-archimedean local field of characteristic not equal to 2.

Let $(W, \langle , \rangle)$ be a symplectic space. Then $\psi \circ \langle , \rangle : W \times W \to \mathbb{S}^1$ gives rises to a self-duality for W.

Definition 3.1.1. The *Heisenberg group* associated to W is $H(W) := W \times F$ with group structure

$$(w_1, t_1) \cdot (w_2, t_2) = \left(w_1 + w_2, t_1 + t_2 + \frac{\langle w_1, w_2 \rangle}{2} \right)$$

1. The center Z of $H(W)$ is simply $\{0\} \times F$. $H(W)/Z = W$, hence $H(W)$ is a nilpotent algebraic group.
2. The product measure of any Haar measures on W and F yields a product measure on $H(W)$. From this, one can see explicitly that $H(W)$ is unimodular.
3. If $W = W_1 \oplus W_2$, where W_1, W_2 are symplectic spaces. There is a natural homomorphism

$$\begin{aligned} H(W_1) \times H(W_2) &\to H(W) \\ ((w_1, t_1), (w_2, t_2)) &\mapsto (w_1 + w_2, t_1 + t_2) \end{aligned}$$

3.2 Stone-von Neumann Theorem

Fix a symplectic space W and put $H := H(W)$. For any irreducible representation ρ of H, $\rho|_{\{0\} \times F}$ is character of F, called the *central character* of ρ. The irreducible representations of H can be classified by their central characters.

This section will be devoted to the following result.

Theorem 3.2.1 (Stone-von Neumann, version on the level of groups)**.** *There exists a unique smooth irreducible representation ρ of H (up to isomorphism) with central character ψ.*

3.2.1 Existence

The main concern is to construct *models* for such a representation, that is, a particular concrete space with an action of H with the properties prescribed above. Our models come from some special subgroups $A \subset W$.

For any subgroup $A \subset W$, put

$$A^{\perp} := \{ w \in W : \forall a \in A, \ \psi(\langle w, a \rangle) = 1 \}$$

It is a closed subgroup of W. If A is closed, then Pontryagin's duality implies that

1. $A^{\perp\perp} = A$
2. $A^{\perp} \xrightarrow{\sim} \widehat{W/A}$ via $w \mapsto \psi(\langle w, - \rangle)$, where $\widehat{W/A}$ denotes the dual group of W/A.
3. $(A_1 + A_2)^{\perp} = A_1^{\perp} \cap A_2^{\perp}$
4. Suppose that $A_1^{\perp} + A_2^{\perp}$ is closed, then $A_1^{\perp} + A_2^{\perp} = (A_1 \cap A_2)^{\perp}$. This condition is always verified when A_1 and A_2 are vector subspaces or when $\mathrm{char}(F) = 0$. Indeed, the first case is trivial and in the latter case F is a finite extension of some $\mathbb{Q}_p$. A subgroup of a finite-dimensional $\mathbb{Q}_p$-vector space is closed if and only if it is also a $\mathbb{Z}_p$-submodule; this condition is preserved under addition.

Fix a subgroup $A \subset W$ such that $A = A^{\perp}$. Such subgroup exists, for example one can take A to be a lagrangian of W. Set $A_H := A \times F$. Let $\psi_A : A_H \to \mathbb{S}^1$ be a character such that $\psi_A|_{\{0\} \times F} = \psi$ (for example, one can take $\psi_A = 1 \times \psi$ when A is a lagrangian of W or when $\mathrm{char}(\mathcal{O}_F/(\varpi)) \neq 2$). Define (ρ, S_A) to be the smooth induction of ψ_A.

$$
\begin{aligned}
(\rho, S_A) &:= \mathrm{Ind}_{A_H}^H \psi_A \\
S_A &= \left\{ f : H \to \mathbb{C} : \begin{array}{l} \forall a \in A_H, h \in H,\ f(ah) = \psi_A(a) f(h) \\ f \text{ is fixed by some compact open } L \subset W \end{array} \right\} \\
\rho &: \text{ acts by right translation.}
\end{aligned}
$$

We can also consider the compact induction. In our case, they are equal.

Lemma 3.2.2. (ρ, S_A) *coincides with the compact induction c-*$\mathrm{Ind}_{A_H}^H \psi_A$.

Proof. Let $f \in S_A$ be right-invariant under a compact open subgroup $L \subset W$. It suffices to show the compactness of $\mathrm{Supp} f$ in $A_H \backslash H = A \backslash W$.

Suppose that $w \in W, f((w, 0)) \neq 0$. For any $l \in L \cap A$, we have

$$
f((w,0)) = f((w,0)(l,0)) = f((l, \langle w, l \rangle)(w,0)) = \psi(\langle w, l \rangle) \psi_A((l,0)) f((w,0))
$$

Thus $\psi(\langle w, l \rangle) = \psi_A((-l, 0))$. This pinned down the image of w modulo $(L \cap A)^{\perp}$. However $(L \cap A)^{\perp} = L^{\perp} + A^{\perp} = L^{\perp} + A$, and $L^{\perp}$ is compact since W/L is discrete, hence f is supported on the compact subset $A \backslash (L^{\perp} + A)$. ☐

Lemma 3.2.3. *Let* $w \in W$, L *a compact open subgroup of* W. *Suppose that* $\psi_A = 1$ *on* $A_H \cap (w,0)(L \times \{0\})(w,0)^{-1}$ *(this is always possible by taking* L *small enough). We can define a function*

$$
f_{w,L}(h) = \begin{cases} \psi_A(a) & , \text{if } h = a(w,0)(l,0), \quad h \in A_H(w,0)(L \times \{0\}) \\ 0 & , \text{otherwise} \end{cases}
$$

As w *runs over* W *and* L *runs over small enough compact open subgroups, these functions generate* S_A. *In particular,* $S_A \neq \{0\}$.

Proof. The hypothesis that $\psi_A = 1$ on $A_H \cap (w,0)(L \times \{0\})(w,0)^{-1}$ guarantees that $f_{w,L}$ is well-defined and lies in S_A. Note that $S_A = \bigcup_L S_A^L$, the functions in S_A^L are determined by their values on representatives of the double cosets $A_H \backslash H / (L \times \{0\})$, which are zero for all but finitely many representatives by the preceding lemma. Our assertion follows at once. ☐

Proposition 3.2.4. (ρ, S_A) *is irreducible.*

Proof. Fix $f \in S_A, f \neq 0$, we want to show that f generates all $f_{w,L}$ under the action of ρ. Fix $w \in W$. By translating f on the right, we may assume that $f((w,0)) \neq 0$. Let L be a compact open subgroup fixing f. Fix a Haar measure on A and consider the action of $\mathscr{S}(A)$. For any $\phi \in \mathscr{S}(A)$, $w' \in W$, we have

$$\begin{aligned}(\rho|_A(\phi)(f))(w',0) &= \int_A f((w',0)(a,0))\phi(a)\,\mathrm{d}a \\ &= \int_A \psi(\langle w',a\rangle)\psi_A((a,0))\phi(a)\,\mathrm{d}a \cdot f((w',0))\end{aligned}$$

This resembles a Fourier transform; write $\phi(a) = \psi_A(-a,0)\phi'(a)$, the last term becomes

$$\int_A \psi(\langle w',a\rangle)\phi'(a)\,\mathrm{d}a \cdot f((w',0)) = (\phi'\mu)^\wedge(w'+A)\cdot f((w',0))$$

where μ is some Haar measure on A. Choose ϕ' so that $(\phi'\mu)^\wedge$ is the characteristic function of $w+L+A \subset W/A = \hat{A}$, then $\rho|_H(\phi)f$ is f multiplied by the characteristic function of $(A+w+L)\times F$. By taking L small enough, it will be a multiple of $f_{w,L}$. □

This established the existence part of 3.2.1.

Let's consider some choices of A which will be used later.

Example 3.2.5. Take $A = \ell$ to be a lagrangian of W. Then the restriction from H to $W \times \{0\}$ identifies S_ℓ with the space

$$\left\{ f : W \to \mathbb{C} : \begin{array}{l} \forall x \in W, a \in \ell,\ f(x+a) = \psi\left(\frac{1}{2}\langle x,a\rangle\right) f(x) \\ \text{fixed by some open compact subgroup } L \subset W \end{array} \right\}$$

The action of H is given by

$$(\rho_\ell(x,t)f)(x') = \psi\left(\frac{1}{2}\langle x',x\rangle + t\right) f(x'+x).$$

Such models are called *Schrödinger representations.*

Example 3.2.6. Let ℓ, ℓ' be transversal lagrangians, so that there is a splitting $W = \ell \oplus \ell'$. Take $A = \ell$ and choose $\psi_A = 1 \times \psi$. By Lemma 3.2.2, we have

$$\begin{aligned} S_\ell &\xrightarrow{\sim} \mathscr{S}(\ell') \\ f &\mapsto f|_{\ell' \oplus \{0\}} \end{aligned}$$

H acts on $\mathscr{S}(\ell')$ by

$$\forall x \in \ell, y \in \ell', \quad (\rho((x+y,t))f)(y') = \psi\left(\langle y',x\rangle + \frac{\langle y,x\rangle}{2} + t\right) f(y'+y)$$

The representations on $\mathscr{S}(\ell')$ and $\mathscr{S}(\ell)$ are intertwined by a Fourier transform

$$\begin{aligned} \mathscr{S}(\ell') &\to \mathscr{S}(\ell) \\ f &\mapsto f^\wedge(x) = \int_{\ell'} f(y)\psi(\langle y,x\rangle)\,\mathrm{d}y \end{aligned}$$

A generalization as well as a reformulation in half densities will be given in the next section.

Example 3.2.7. Take A to be a compact open subgroup of W such that $A^\perp = A$. This is always possible. Indeed, by taking a symplectic basis of W, it suffices to consider the case $W = Fp \oplus Fq$ with $\langle p,q\rangle = 1$. Set $c = \inf\{t \in \mathbb{Z} : \varpi^t \mathcal{O}_F \subset \operatorname{Ker}\psi\}$ to be the conductor of ψ. Then $A = \mathcal{O}_F\varpi^a \oplus \mathcal{O}_F\varpi^b$ satisfies $A^\perp = A$ if and only if $a+b = c$.

Choose a character ψ_A of A_H such that $\psi_A|_{\{0\}\times F} = \psi$. The corresponding representation acts on the space S_A of functions $f : W \to \mathbb{C}$ such that

- f is locally constant.
- Supp$f$ is compact.
- $f(a+w) = \psi_A\left(a + \frac{\langle a,w\rangle}{2}\right) f(w)$ for all $w \in W, a \in A$.

H acts via

$$(\rho_A(x,t)f)(x') = \psi\left(\frac{1}{2}\langle x', x\rangle + t\right) f(x'+x).$$

Such a choice of A is particular to the case F non-archimedean.

3.2.2 Uniqueness

Let $\mathscr{S}_\psi(H)$ be the space consisting of smooth functions f such that $f(zh) = \psi(z)f(h)$ for all $z \in Z = \{0\} \times F$ and $h \in H$, and that $|f|$ is compactly supported modulo Z. The restriction map gives rise to an isomorphism

$$\begin{aligned} \mathscr{S}_\psi(H) &\xrightarrow{\sim} \mathscr{S}(W) \\ f &\mapsto f|_{W\times\{0\}} \end{aligned}$$

$\mathscr{S}_\psi(H)$ [resp. $\mathscr{S}(W)$] admits two representations:

$$\begin{array}{lll} \rho_d & = \text{right translation} & , \rho_d(g) : f(-) \mapsto f(-g) \\ \rho_s & = \text{left translation} & , \rho_s(g) : f(-) \mapsto f(g^{-1}-) \end{array}$$

so that the $H \times H$-representation $\rho_s \times \rho_d$ on $\mathscr{S}_\psi(H)$ [resp. $\mathscr{S}(W)$] can be defined.

Lemma 3.2.8. *Let (ρ, S) be a representation of H satisfying the hypothesis of 3.2.1. Let $S^\vee$ denote the smooth dual of S, then taking matrix coefficients*

$$\begin{aligned} S^\vee \otimes S &\to \mathscr{S}_\psi(H) \\ s^\vee \otimes s &\mapsto f_{s^\vee,s}(h) := \langle s^\vee, \rho(h)s\rangle \end{aligned}$$

gives rise to an intertwining operator $c : \rho^\vee \otimes \rho \to \rho_s \times \rho_d$ as representations of $H \times H$.

Proof. The crux is to show that for all $s^\vee, s$, the matrix coefficient $f_{s^\vee,s}$ is compactly supported modulo Z. Take a compact open subgroup $L \subset W$ fixing $s^\vee, s$, then for all $l \in L$ we have

$$\begin{aligned} f_{s^\vee,s}((w,0)) &= \langle s^\vee, \rho((w,0))s\rangle = \langle s^\vee, \rho((w,0)(l,0))s\rangle \\ &= \psi(\langle w,l\rangle)\langle \rho((-l,0))s^\vee, \rho((w,0))s\rangle \\ &= \psi(\langle w,l\rangle)\langle s^\vee, \rho((w,0))s\rangle = \psi(\langle w,l\rangle) f_{s^\vee,s}((w,0)) \end{aligned}$$

Hence $f_{s^\vee,s}((w,0)) \neq 0$ implies that $w \in L^\perp$, which is compact. □

Lemma 3.2.9. *ρ_d is isotypic.*

Proof. Take two transversal lagrangians ℓ, ℓ' of W. According to 3.2.6, we have two representations $(\rho, \mathscr{S}(\ell'))$, $(\rho', \mathscr{S}(\ell))$ with central character ψ. Apply the same construction to get $(\bar\rho, \mathscr{S}(\ell'))$, $(\bar\rho', \mathscr{S}(\ell))$, but this time with central character $\bar\psi$.

Fix Haar measures on ℓ, ℓ'. Recall the general fact that $\rho \simeq \mathrm{Ind}_{\ell_H}^H(1\times\psi)$ and $\bar\rho \simeq \mathrm{Ind}_{\ell_H}^H(1\times\bar\psi) = \text{c-}\mathrm{Ind}_{\ell_H}^H(1\times\bar\psi)$ are in duality via the pairing

$$(s', s) \mapsto \int_{\ell'} s'(y')s(y')\,\mathrm{d}y'$$

Now, use Fourier transform to identify $\bar{\rho}$ and $\bar{\rho}'$, the above duality pairing becomes

$$(s', s) \mapsto \int\limits_{\ell \times \ell'} s'(x')s(y')\psi(\langle y', x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y'$$

The matrix coefficients now take the following form: let $x \in \ell, y \in \ell'$,

$$\begin{aligned} f_{s',s}((x+y,0)) &= \int\limits_{\ell \times \ell'} s'(x')\psi\left(\langle y', x\rangle + \frac{\langle y, x\rangle}{2}\right) s(y+y')\psi(\langle y', x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y' \\ &= \int\limits_{\ell \times \ell'} s'(x')s(y')\psi\left(\langle y'-y, x\rangle + \frac{\langle y, x\rangle}{2}\right)\psi(\langle y'-y, x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y' \\ &= \psi\left(\frac{\langle x, y\rangle}{2}\right) \int\limits_{\ell \times \ell'} s'(x')s(y')\psi(\langle y', x\rangle - \langle y, x'\rangle)\psi(\langle y', x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y' \end{aligned}$$

This is the tensor product of two Fourier transforms multiplied by a bicharacter. Hence $\bar{\rho}' \otimes \rho \simeq \rho_s \times \rho_d$. Since ρ is irreducible, restriction to $1 \times H$ shows that ρ_d is a direct sum of copies of ρ. □

Now we can complete the proof of the uniqueness part. Let (σ, S) be any representation of H which is smooth and irreducible. Using the intertwining operator $c : \sigma^\vee \otimes \sigma \to \rho_s \times \rho_d$, we can fix $s^\vee \in S^\vee, s^\vee \neq 0$ to embed σ into $\rho_s \times \rho_d$. The lemma implies that $\sigma \simeq \rho$.

Remark 3.2.10. From now on, we will use ρ_ψ to denote a representation satisfying Theorem 3.2.1.

Proposition 3.2.11. $\rho_{\bar{\psi}} \otimes \rho_\psi \simeq \rho_s \times \rho_d$.

Proof. Take transversal lagrangians ℓ, ℓ'. By 3.2.1, we may assume that $\rho_\psi = \rho$, $\rho_{\bar{\psi}} = \bar{\rho}'$, where $\rho, \bar{\rho}'$ are those constructed in the proof of 3.2.9. This proposition is then a byproduct of 3.2.9 □

3.2.3 Passing to unitary representations

By now the smooth irreducible representations of H are fully classified. However, in order to apply the character theory, we will also consider *unitary representations* and the unitary equivalences between them in Hilbert spaces. Theorem 3.2.1 remains valid for *unitary representations*. It suffices to note that

- The representations (ρ, S_A) are unitarizable: the space S_A can be viewed as *smooth vectors*[1] in $L^2(A_H\backslash H, \psi_A)$. The members of the latter space are measurable functions[2] f on H such that

$$\forall a \in A_H, h \in H,\ f(ah) = \psi_A(a)f(h) \tag{3.1}$$

$$\int\limits_{A_H\backslash H} |f|^2\,\mathrm{d}\dot{h} < +\infty \tag{3.2}$$

  where we use $\mathrm{d}\dot{h}$ to denote an arbitrary invariant measure on $A_H\backslash H$. It admits a structure of Hilbert space defined by the inner product

$$(f|g) := \int\limits_{A_H\backslash H} f\bar{g}\,\mathrm{d}\dot{h}$$

  H acts on $L^2(A_H\backslash H, \psi_A)$ by right translation, which is obviously unitary.

[1]Recall that $s \in S_A$ is called *smooth* if it is stabilized by a compact open subgroup of H.
[2]Strictly speaking, two such functions are identified if they coincide almost everywhere modulo A_H.

- In the proof of the uniqueness part of 3.2.1, all the intertwining operators are unitary equivalences, at least up to some constant factor. Indeed, it suffices to show that

  $$s \mapsto f_{s^\vee, s}, \quad s^\vee \neq 0 \text{ fixed.}$$

  has image in

  $$L^2_\psi(H/Z) := \left\{ f : H \to \mathbb{C} : \begin{array}{l} \forall z \in Z, h \in H, \ f(zh) = \psi(z) f(h), \\ \int_{H/Z} |f|^2 \, \mathrm{d}\ddot{h} < +\infty \end{array} \right\}$$

  and is an isometry up to a constant factor. Here $\mathrm{d}\ddot{h}$ is a chosen Haar measure on H/Z.

  Let (ρ, S) be the concerned unitary representation of central character ψ and let $(,)$ be the inner product on S. Let $\bar{S}$ be the space S with the "twisted" complex structure: $z * s := \bar{z}s$, then $(\rho, \bar{S})$ can be identified with the contragredient representation $(\rho^\vee, S^\vee)$ by setting $\langle s^\vee, s\rangle := (s, s^\vee)$ for every $s^\vee \in \bar{S}, s \in S$.

  If $s^\vee, s$ are smooth and nonzero, then we have seen that $f_{s^\vee,s}$ is compactly supported and continuous. This shows the square-integrability mod Z of (ρ, S).

  Recall *Schur orthogonality relations* for square-integrable representations mod Z ([6], Part I §1): there exists a constant $d_\rho > 0$ depending on ρ and $\mathrm{d}\ddot{h}$, called the *formal degree* of ρ, such that for every $s^\vee, t^\vee \in \bar{S}, s^\vee, t \in S$ we have

  $$\begin{aligned} (f_{s^\vee,s} | f_{t^\vee,t}) &= \int_{H/Z} \langle s^\vee, \rho(h)(s)\rangle \overline{\langle t^\vee, \rho(h)t\rangle} \, \mathrm{d}\ddot{h} \\ &= \int_{H/Z} (\rho(h)(s), s^\vee) \overline{(\rho(h)t, t^\vee)} \, \mathrm{d}\ddot{h} \\ &= d_\rho^{-1} \cdot (s,t)\overline{(s^\vee, t^\vee)}. \end{aligned}$$

  This completes the proof.

The main case to be considered is the case where $A = \ell$ is a lagrangian of W. We will reformulate the constructions above in a canonical way via the language of densities.

Definition 3.2.12. Let ℓ be a lagrangian of a symplectic space W. Define [3] $\mathcal{H}_\ell = \mathcal{H}_\ell(\psi)$ to be the vector space of measurable functions $f : W \to \Omega_{1/2}(W/\ell)$ such that

$$f(v + a) = \psi\left(\frac{\langle v, a\rangle}{2}\right) f(v) \tag{3.3}$$

$$\int_{W/\ell} f\bar{f} < +\infty \tag{3.4}$$

Remark 3.2.13. $\mathcal{H}_\ell(\psi)^\vee \simeq \mathcal{H}_\ell(\bar{\psi})$ via the pairing

$$\begin{aligned} \mathcal{H}_\ell(\psi) \times \mathcal{H}_\ell(\bar{\psi}) &\to \mathbb{C} \\ (f, g) &\mapsto \int_{W/\ell} fg \end{aligned}$$

[3] T. Thomas use the same symbol in [18] to denote a sheaf over W/ℓ. Our space corresponds to its L^2-sections.

3.3 Calculation of characters

Fix a lagrangian ℓ of W and form the corresponding Schrödinger representation ρ_ℓ. Recall that the representation ρ_ℓ on $\mathcal{H}_\ell$ induces a map

$$\rho_\ell : \mathscr{S}(H) \otimes \Omega_1(W \times F) \to \operatorname{End}(\mathcal{H}_\ell)$$

defined by the formula

$$\rho_\ell(h)s = \int_{(w,t)\in H} h((w,t)) \cdot \rho_\ell((w,t))s, \qquad h \in \mathscr{S}(H) \otimes \Omega_1(W \times F), s \in \mathcal{H}_\ell \tag{3.5}$$

(Recall that the Haar measures on H are the same as those of the product vector space $W \times F$.)

As $\mathcal{H}_\ell$ is admissible, $\rho_\ell(h)$ is a bounded operator of finite rank, hence is a *trace class operator* for any $h \in \mathscr{S}(H) \otimes \Omega_1(W \times F)$. Therefore, we obtain a linear functional (the *character* of ρ_ℓ) on $\mathscr{S}(H) \otimes \Omega_1(W \times F)$

$$\Theta_{\rho_\ell} : h \mapsto \operatorname{Tr}(\rho_\ell(h)).$$

It is in fact a distribution on H. This section is devoted to a direct calculation for it.

Proposition 3.3.1. $\Theta_{\rho_\ell} = \delta \otimes \psi \in \Omega_1(W \times F) = \Omega_1(W) \otimes \Omega_1(F)$*, where* δ *is the Dirac measure associated to* $\{0\} \subset W$ *together with the self-dual measure* $\mu_W \in \Omega_1(W)$*.*

Proof. Recall that for $x \in W$, we have

$$\begin{aligned}(\rho_\ell(h)s)(x) &= \int_{(w,t)\in H} s(x+w)\psi\left(\frac{\langle x,w\rangle}{2} + t\right) h(w,t) \\ &= \int_{(w,t)\in H} s(w)\psi\left(\frac{\langle x,w\rangle}{2} + t\right) h(w-x,t)\end{aligned}$$

Choose a complement of $\ell \subset W$ to identify W/ℓ with a subspace of W. Recall that $s(w+a) = \psi(\langle w,a\rangle/2)s(w)$ if $a \in \ell$ (by 3.2.5), the last integral can be unfolded to get

$$\int_{w'\in W/\ell} s(w') \cdot \int_{(a,t)\in \ell\times F} \psi\left(\frac{\langle x+w', w'+a\rangle}{2} + t\right) h(w'+a-x,t)$$

Since a complement of $\ell \subset W$ is chosen, $\mathcal{H}_\ell$ can be identified with $L^2(W/\ell)$. Extract the integral kernel in the formula above

$$K(x,w') := \int_{(a,t)\in \ell\times F} \psi\left(\frac{\langle x+w', w'+a\rangle}{2} + t\right) h(w'+a-x,t).$$

$K(x,w')$ is a smooth function on $W/\ell \times W/\ell$ taking value in $\Omega_1(W/\ell)$. In view of Theorem A.0.6, we set out to do an integration along the diagonal in order to calculate the character.

$$\begin{aligned}\operatorname{Tr}(\rho_\ell(h)) &= \int_{x\in W/\ell} K(x,x) \\ &= \int_{x\in W/\ell} \int_{(a,t)\in\ell\times F} \psi\left(\frac{\langle 2x, x+a\rangle}{2} + t\right) h(a,t) \\ &= \int_{t\in F} \psi(t) \cdot \int_{x\in W/\ell} \int_{a\in\ell} \psi(\langle x,a\rangle) h(a,t)\end{aligned}$$

Write $h = \bar{h} \cdot \mu_W$, where $\mu_W \in \Omega_1(W)$ is the self-dual measure. Observe that $\mu_W = \mu_\ell \cdot \mu_{W/\ell}$, where μ_ℓ and $\mu_{W/\ell}$ are dual measures. By the Fourier inversion formula, the last inner integral is thus $\bar{h}(0,t)$, and

$$\Theta_{\rho_\ell}(h) = \int_{t \in F} \bar{h}(0,t)\psi(t) = \int_{t \in F} \frac{h(0,t)}{\mu_W}\psi(t)$$

In other words, $\Theta_{\rho_\ell} = \delta \otimes \psi$. □

3.4 The canonical intertwiners

According to 3.2.1, the Schrödinger representations associated to different lagrangians are unitarily equivalent. We will construct canonical intertwiners between Schrödinger representations and represent them by integral kernels.

Lemma 3.4.1. *Let ℓ_1, ℓ_2 be two lagrangians of a symplectic space W, then there is a canonical isomorphism*

$$\Omega_1^{\mathbb{R}}((\ell_1 + \ell_2)/\ell_1 \cap \ell_2) = \Omega_1^{\mathbb{R}}(\ell_1) \otimes \Omega_1^{\mathbb{R}}(\ell_2) \otimes (\Omega_1^{\mathbb{R}}(\ell_1 \cap \ell_2)^*)^{\otimes 2}$$

Proof. This follows from the following observations:

$$(\ell_1 + \ell_2)/\ell_1 \cap \ell_2 = (\ell_1/\ell_1 \cap \ell_2) \oplus (\ell_2/\ell_1 \cap \ell_2)$$
$$\Omega_1^{\mathbb{R}}(\ell_i/\ell_1 \cap \ell_2) = \Omega_1^{\mathbb{R}}(\ell_i) \otimes \Omega_1^{\mathbb{R}}(\ell_1 \cap \ell_2)^* \quad (i = 1, 2)$$

□

Of course, the same assertion continues to hold after tensoring with $\mathbb{C}$.

Let ℓ_i, ℓ_j be two lagrangians. Define

$$\mu_{i,j} = \mu_{\ell_i,\ell_j} \in \Omega_1^{\mathbb{R}}(\ell_i) \otimes \Omega_1^{\mathbb{R}}(\ell_j) \otimes (\Omega_1^{\mathbb{R}}(\ell_i \cap \ell_j)^*)^{\otimes 2} = \Omega_1^{\mathbb{R}}((\ell_i + \ell_j)/\ell_i \cap \ell_j)$$

to be the element in $\Omega_1^{\mathbb{R}}((\ell_i + \ell_j)/\ell_i \cap \ell_j)$ corresponding to the self-dual measure induced by the symplectic form.

Theorem 3.4.2. *Let ℓ_i, ℓ_j be two lagrangians of a symplectic space W. There is a unitary intertwiner $\mathscr{F}_{j,i} : \mathcal{H}_{\ell_i} \to \mathcal{H}_{\ell_j}$ determined by*

$$\begin{aligned}(\mathscr{F}_{j,i}\phi)(y) &= \int_{x \in \ell_j/(\ell_i \cap \ell_j)} \phi((x,0)(y,0))\mu_{i,j}^{1/2} \\ &= \int_{x \in \ell_j/(\ell_i \cap \ell_j)} \phi(x+y)\psi\left(\frac{\langle x, y\rangle}{2}\right)\mu_{i,j}^{1/2}\end{aligned}$$

when $\phi \in \mathcal{H}_{\ell_i}$ is a smooth vector.

Note that we have abused the notation $\psi(-)$ to denote $\psi((-,0))$. The first integral can be written as

$$(\mathscr{F}_{j,i}\phi)(h) = \int_{(\ell_j)_H/((\ell_i)_H \cap (\ell_j)_H)} \phi(ah)\, \mathrm{d}a \qquad (h \in H)$$

From this, it is clear that $\mathscr{F}_{j,i}$ commutes with the action of H. The main problem is to fix a good invariant measure $\mathrm{d}a$ on $(\ell_j)_H/((\ell_i)_H \cap (\ell_j)_H)$.

Proof. The main burden of the proof is the unitarity. By 1.3.3, we may fix a symplectic basis $p_1, q_1 \ldots, p_n, q_n$ of W and assume that

$$\ell_i = Fp_1 \oplus \cdots \oplus Fp_s \oplus Fp_{s+1} \oplus \cdots \oplus Fp_n$$
$$\ell_j = Fp_1 \oplus \cdots \oplus Fp_s \oplus Fq_{s+1} \oplus \cdots \oplus Fq_n$$

so that

$$
\begin{aligned}
\ell_i \cap \ell_j &= Fp_1 \oplus \cdots \oplus Fp_s \\
\frac{\ell_j}{\ell_i \cap \ell_j} &= Fq_{s+1} \oplus \cdots \oplus Fq_n \\
\frac{\ell_i + \ell_j}{\ell_i \cap \ell_j} &= Fp_{s+1} \oplus \cdots \oplus Fp_n \oplus Fq_{s+1} \oplus \cdots \oplus Fq_n \\
W/\ell_i &= Fq_1 \oplus \cdots \oplus Fq_s \oplus Fq_{s+1} \oplus \cdots \oplus Fq_n \\
W/\ell_j &= Fq_1 \oplus \cdots \oplus Fq_s \oplus Fp_{s+1} \oplus \cdots \oplus Fp_n
\end{aligned}
$$

Fix the densities below

$$
\begin{aligned}
&\omega_i \in \Omega_{1/2}(Fp_{s+1} \oplus \cdots \oplus Fp_n), \quad \omega_j \in \Omega_{1/2}(Fq_{s+1} \oplus \cdots \oplus Fq_n) \\
&\lambda \in \Omega_{1/2}(Fq_1 \oplus \cdots \oplus Fq_s)
\end{aligned}
$$

where we require that ω_i^2, ω_j^2 are dual Haar measures with respect to the pairing $\psi(\langle -, - \rangle)$. Therefore,

$$
\mu_{i,j}^{1/2} = \omega_i \omega_j \in \Omega_{1/2}((\ell_i + \ell_j)/\ell_i \cap \ell_j).
$$

Let $\phi = \tilde{\phi}\lambda\omega_j \in \mathcal{H}_{\ell_i}$, where $\tilde{\phi}$ is a Schwartz-Bruhat function on $W/\ell_i = Fq_1 \oplus \cdots \oplus Fq_n$. We will use the following decompositions

$$
\begin{aligned}
&y \in Fq_1 \oplus \cdots \oplus Fq_s \oplus Fp_{s+1} \oplus \cdots \oplus Fp_n \\
&\Rightarrow y = y' + y'' \quad \text{ where } \begin{cases} y' \in Fq_1 \oplus \cdots Fq_s \\ y'' \in Fp_{s+1} \oplus \cdots \oplus Fp_n \end{cases} \\
&z \in Fq_1 \oplus \cdots \oplus Fq_n \\
&\Rightarrow z = y' + x \quad \text{ where } \begin{cases} y' \in Fq_1 \oplus \cdots Fq_s \\ x \in Fq_{s+1} \oplus \cdots \oplus Fq_n \end{cases}
\end{aligned}
$$

Also note that $\langle y', y'' \rangle = \langle y', x \rangle = 0$ in such decompositions.

For a fixed $y' \in Fq_1 \oplus \cdots Fq_s$, set $\tilde{\phi}_{y'}(x) = \tilde{\phi}(y' + x)$ for $x \in Fq_{s+1} \oplus \cdots \oplus Fq_n$, it is a Schwartz-Bruhat function on $Fq_{s+1} \oplus \cdots \oplus Fq_n$. Now

$$
\begin{aligned}
(\mathscr{F}_{j,i}(\phi))(y) &= \int\limits_{x \in Fq_{s+1} \oplus \cdots \oplus Fq_n} \tilde{\phi}((x,0)(y,0))\lambda\omega_j \cdot \omega_j\omega_i \\
&= \left(\int\limits_{x \in Fq_{s+1} \oplus \cdots \oplus Fq_n} \tilde{\phi}((x,0)(y',0)(y'',0))\omega_j^2 \right) \lambda\omega_i \\
&= \left(\int\limits_{x \in Fq_{s+1} \oplus \cdots \oplus Fq_n} \tilde{\phi}((y'',0)(x,0)(y',0))\psi(\langle x, y'' \rangle)\omega_j^2 \right) \lambda\omega_i \\
&= \left(\int\limits_{x \in Fq_{s+1} \oplus \cdots \oplus Fq_n} \tilde{\phi}((x,0)(y',0))\psi(\langle x, y'' \rangle)\omega_j^2 \right) \lambda\omega_i \\
&= \left(\int\limits_{x \in Fq_{s+1} \oplus \cdots \oplus Fq_n} \tilde{\phi}_{y'}(x)\psi(\langle x, y'' \rangle)\omega_j^2 \right) \lambda\omega_i \\
&= \widehat{\tilde{\phi}_{y'}\omega_j^2}(y'') \cdot \lambda\omega_i
\end{aligned}
$$

Here the Fourier transform is applied to the coordinates $p_{s+1}, \ldots, p_n \leftrightarrow q_{s+1}, \ldots, q_n$. Recall that ω_i^2, ω_j^2 are dual, hence $\mathscr{F}_{j,i}$ is an unitary operator by Plancherel's theorem. $\square$

Corollary 3.4.3.

$$\mathscr{F}_{i,j} \circ \mathscr{F}_{j,i} = \mathrm{id}$$

Proof. This is essentially the Fourier inversion formula. $\square$

Corollary 3.4.4. *If $\ell_i \cap \ell_j = \{0\}$, then $\langle -, - \rangle$ induces a duality between ℓ_i, ℓ_j, and $\mathscr{F}_{j,i}$ becomes the Fourier transform after the identifications below:*

$$\begin{array}{ccc} \mathcal{H}_{\ell_i} & \xrightarrow{\mathscr{F}_{j,i}} & \mathcal{H}_{\ell_j} \\ \simeq \downarrow & & \downarrow \simeq \\ L^2(\ell_i) & \xrightarrow[\text{Fourier transform}]{} & L^2(\ell_j) \end{array}$$

3.5 Cyclic composition of canonical intertwiners

Given lagrangians $\ell_1, \ldots, \ell_n \quad (n \geq 2)$ of W, we are concerned about their cyclic composition

$$\mathscr{F}_{1,\ldots,n} := \mathscr{F}_{1,n} \circ \cdots \circ \mathscr{F}_{2,1} : \mathcal{H}_{\ell_1} \to \mathcal{H}_{\ell_1}.$$

It is just multiplication by some constant of absolute value 1, according to Schur's lemma. This constant is 1 when $n = 2$ by Corollary 3.4.3. However, those relevant partial Fourier transforms are entangled in a quite subtle way when $n \geq 3$; this constant is related to the Maslov index τ and Weil's character γ as follows.

Theorem 3.5.1 (G. Lion for $F = \mathbb{R}$, P. Perrin for general case)**.** *Let $\ell_1, \ldots, \ell_n \quad (n \geq 3)$ be lagrangians of W, then*

$$\mathscr{F}_{1,\ldots,n} = \gamma(-\tau(\ell_1, \ldots, \ell_n)) \cdot \mathrm{id}.$$

There is a more elementary way to prove this by reducing to the case $n = 3$ and use the quadratic form (2.7), see [15]. However, we will employ an argument that fits well into our framework in Chapter 2 and works directly for general n. The first step is to represent the intertwiners by integral kernels as follows. This formula will also be used in Chapter 5.

Lemma 3.5.2. *For any smooth ϕ in $\mathcal{H}_{\ell_i}$ and for any $y \in W$, we have*

$$(\mathscr{F}_{i+1,i}\phi)(y) = \int\limits_{\substack{x \in W/\ell_i \\ x - y \in \ell_i + \ell_{i+1}}} \phi(x)\psi\left(\frac{Q_{\ell_i,\ell_{i+1}}((x,y))}{2}\right) \cdot \mu_{i,i+1}^{1/2}.$$

Here $Q_{\ell_i,\ell_{i+1}}$ is the quadratic form on $\{(x,y) \in W^2 : x - y \in \ell_i + \ell_{i+1}\}$ defined by

$$(x,y) \mapsto \epsilon_{i,i+1}(x,y)(x-y),$$

the functional $\epsilon_{i,i+1}(x,y)$ being that defined in 2.5.1.

Proof. Choose $A \subset \ell_{i+1}$ so that $A \oplus (\ell_i \cap \ell_{i+1}) = \ell_{i+1}$. Using the second integral formula in 3.4.2 and make a change of variable $z = x + y$, we get

$$(\mathscr{F}_{i+1,i}\phi)(y) = \int\limits_{z \in A + y} \phi(z)\psi\left(\frac{\langle z, y \rangle}{2}\right) \mu_{i+1,i}^{1/2},$$

where the integral of a density over $A + y$ is interpreted in the obvious way.

Since $z - y \in A \subset \ell_{i+1}$, $\langle z, y\rangle = \langle \epsilon_{i,i+1}(z,y), z-y\rangle$ by the very definition of $\epsilon_{i,i+1}$. Observe that $(\ell_i + \ell_{i+1})/\ell_i \simeq \ell_{i+1}/(\ell_i \cap \ell_{i+1})$, and the quotient map restricts to an isomorphism

$$A + y \xrightarrow{\sim} \{x \in W/\ell_i : x - y \in \ell_i + \ell_{i+1}\}.$$

Thus a further change of variable completes the proof. $\square$

Lemma 3.5.3. *Theorem 3.5.1 holds if and if it holds for lagrangians satisfying* $\bigcap_i \ell_i = \{0\}$.

Proof. Given n lagrangians $\ell_1, \ldots, \ell_n$ ($n \geq 3$), it is always possible to pick another ℓ_{n+1} transversal to $\ell_1, \ldots, \ell_n$. By the chain condition and dihedral symmetry

$$\begin{aligned}\tau(\ell_1, \ldots, \ell_n) + \tau(\ell_1, \ell_n, \ell_{n+1}) &= \tau(\ell_1, \ldots, \ell_n, \ell_{n+1}) \\ \tau(\ell_{n+1}, \ell_1, \ldots, \ell_{n-1}) + \tau(\ell_{n+1}, \ell_{n-1}, \ell_n) &= \tau(\ell_{n+1}, \ell_1, \ldots, \ell_n) = \tau(\ell_1, \ldots, \ell_n, \ell_{n+1}).\end{aligned}$$

The same relation holds for $\mathscr{F}_{\ldots}$ by 3.4.3

$$\begin{aligned}\mathscr{F}_{1,\ldots,n} \cdot \mathscr{F}_{1,n,n+1} &= \mathscr{F}_{1,\ldots,n,n+1} \\ \mathscr{F}_{n+1,1,\ldots,n-1} \cdot \mathscr{F}_{n+1,n-1,n} &= \mathscr{F}_{n+1,1,\ldots,n} = \mathscr{F}_{1,\ldots,n,n+1} \quad \text{(Regarded as scalars).}\end{aligned}$$

This proves the lemma. $\square$

Setting of the proof of Theorem 3.5.1: given the previous lemma, we can assume that $\bigcap_i \ell_i = \{0\}$.

Convention: To simplify matters, we will fix arbitrary positive Haar measures on all relevant vector spaces; this arbitrariness will be cancelled out after taking cyclic composition. Define an equivalence relation $\sim$ on $\mathbb{C}$ by stipulating

$$x \sim y \iff \exists \alpha \in \mathbb{R}_{>0},\ x = \alpha y.$$

The same notation also applies to distributions: $f \sim g$ if they differ by a positive constant.

Recall the constructions in §2.2.

- The space T is the cohomology of the center term of the following complex

$$\bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \ell_i \cap \ell_{i+1} \xrightarrow{\partial} \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \ell_i \xrightarrow{\Sigma} W,$$

  There is then a cartesian square

$$\begin{array}{ccc} \operatorname{Ker}\Sigma & \overset{\iota}{\rightarrowtail} & \bigoplus_i \ell_i \\ {\scriptstyle \pi}\downarrow & \square & \downarrow{\scriptstyle \tilde{\pi}} \\ T & \overset{\tilde{\iota}}{\rightarrowtail} & \operatorname{Coker}\partial \end{array} \tag{3.6}$$

- T^* is the cohomology of the center term of the dual complex

$$W \xrightarrow{\Sigma^*} \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W/\ell_i \xrightarrow{\partial^*} \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W/(\ell_i + \ell_{i+1})$$

  Similarly, there is a diagram

$$\begin{array}{ccc} \operatorname{Ker}\partial^* & \overset{i}{\rightarrowtail} & \bigoplus_i W/\ell_i \\ {\scriptstyle p}\downarrow & & \\ T^* & & \end{array}$$

Set

$$f_q(x) := \psi\left(\frac{q(x)}{2}\right) \in \mathcal{D}(T)$$
$$f_{-q^*}(x) := \psi\left(\frac{-q^*(x)}{2}\right) \in \mathcal{D}(T^*).$$

Since the Haar measures are fixed, we can pull-back then push-forward, without worry about densities, to define

$$Q := \iota_* \pi^*(f_q) \in \mathcal{D}(\bigoplus_i \ell_i)$$
$$Q'^{\vee} := i_* p^*(f_{-q^*}) \in \mathcal{D}(\bigoplus_i W/\ell_i)$$

Consider the Fourier transform $Q' := (Q'^{\vee})^{\wedge}$. Using the compatibilities stated in §1.1, we have

$$\begin{aligned} Q' &= (Q'^{\vee})^{\wedge} = (i_* p^*(f_{-q^*}))^{\wedge} \\ &\sim \tilde{\pi}^* \tilde{\iota}_* \widehat{f_{-q^*}} \\ &\sim \iota_* \pi^* \widehat{f_{-q^*}} \sim \gamma(q)^{-1} \iota_* \pi^* f_q \\ &\sim \gamma(-q) Q \end{aligned}$$

where we have used the facts that the diagram 3.6 is cartesian and that f_q is an even function.

Define $X := (\prod_i \ell_i) \times F$ and let $m : X \to H$ be the multiplication morphism $(g_1, \ldots, g_n, t) \mapsto g_n \cdots g_n \cdot (0, t)$, induced via inclusions $\ell_i \subset W \subset H$. The variety X carries the measure induced from the chosen measures on $\ell_1, \ldots, \ell_n$ and F.

For every $(g_1, \ldots, g_n, t) \in X$, define the operator

$$\begin{aligned} \sigma(g_1, \ldots, g_n, t) &:= \psi(t) \cdot \rho_1(g_1) \circ \mathscr{F}_{1,n} \circ \cdots \circ \rho_2(g_2) \circ \mathscr{F}_{2,1} \\ &= \mathscr{F}_{1,\ldots,n} \circ \rho_1(m(g_1, \ldots, g_n, t)) \end{aligned}$$

Lemma 3.5.4. *Under the assumption that $\bigcap_i \ell_i = \{0\}$, the map m is a submersion. In fact, there is an isomorphism $\phi : X \to X$ preserving measures such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & X \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle (g_1,\ldots,g_n,t)\mapsto(g_1+\cdots+g_n,t)} & \\ & H & \end{array}$$

commutes.

Proof. Note that $\bigcap_i \ell_i = \{0\} \iff \sum_i \ell_i = W$, hence m is surjective. The morphism $\phi : X \to X$ can be defined as

$$\phi(g_1, \ldots, g_n, t) = \left(g_1, \ldots, g_n, t + \frac{\sum_{i>j} \langle g_i, g_j \rangle}{2}\right).$$

Its inverse is simply $(g_1, \ldots, g_n, t) \mapsto (g_1, \ldots, g_n, t - \frac{\sum_{i>j}\langle g_i, g_j\rangle}{2})$. It is clear that ϕ preserves the chosen measure on X. □

Remark 3.5.5. By virtue of this lemma, it makes sense to talk about the following distributions on X

$$\mathrm{Tr}(\sigma) : h \mapsto \mathrm{Tr}\left(\int_X h(x)\sigma(x)\,\mathrm{d}x\right)$$
$$\mathrm{Tr}(\rho_1 \circ m) : h \mapsto \mathrm{Tr}\left(\int_X h(x)\rho_1(m(x))\,\mathrm{d}x\right)$$

Abusing notations somehow, we may write

$$\mathscr{F}_{1,\dots,n} = \frac{\mathrm{Tr}(\sigma)}{\mathrm{Tr}(\rho_1 \circ m)} \cdot \mathrm{id}.$$

The proof of Theorem 3.5.1 will be concluded by the

Lemma 3.5.6. *Under the assumption that $\bigcap_i \ell_i = \{0\}$,*

1. $\mathrm{Tr}(\rho_1 \circ m) \sim Q \otimes \psi$.
2. $\mathrm{Tr}(\sigma) \sim Q' \otimes \psi$.

Proof of 3.5.1. Granting the lemma, we have $\mathscr{F}_{1,\dots,n} = \gamma' \cdot \mathrm{id}$ for some scalar γ'. Given the preceding remark and lemma, we have $\gamma' \sim \gamma(-q) = \gamma(-\tau(\ell_1, \dots, \ell_n))$; however $|\gamma'| = |\gamma(-q)| = 1$, hence $\mathscr{F}_{1,\dots,n} = \gamma(-\tau(\ell_1, \dots, \ell_n)) \cdot \mathrm{id}$. □

Proof of the lemma. Recall that $\Theta_{\rho_1} \sim \delta \otimes \psi$ after fixing relevant measures, where δ is the Dirac measure at $\{0\} \in W$. Since $m : X \to H$ is submersive, $\mathrm{Tr}(\rho_1 \circ m) \sim m^* \Theta_{\rho_1}$; the latter distribution is

$$\sim (\text{Dirac measure at } \mathrm{Ker}\,\Sigma) \cdot \left((g_1, \dots, g_n) \mapsto \psi\left(\frac{\sum_{i>j} \langle g_i, g_j \rangle}{2} \right) \right) \otimes \psi$$

in which the second first tensor slot is just Q, by formula (2.4). Hence $\mathrm{Tr}(\rho_1 \circ m) \sim Q \otimes \psi$.

As for $\mathrm{Tr}(\sigma)$, it suffices to consider test functions of the form $\phi = (\bigotimes_{i=1}^n h_i) \otimes k$, where $h_i \in \mathscr{S}(\ell_i)$ and $k \in \mathscr{S}(F)$.

Using the formula (3.5) (but restricted to ℓ_i), we see that

$$\forall \theta \in \mathcal{H}_{\ell_i},\ (\rho_i(h_i)\theta)(x) \sim \theta(x)\hat{h}_i(\bar{x}),$$

where $\bar{x}$ is the projection of $x \in W$ to W/ℓ_i. On the other hand the function k acts by

$$\theta(x) \mapsto \int_{t \in F} k(t)\psi(t)\theta(x).$$

The overall conclusion is that

$$(\mathrm{Tr}\sigma)(\phi) \sim \int_{\substack{x \in \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} W/\ell_i \\ x_i - x_{i+1} \in \ell_i + \ell_{i+1}}} \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \psi\left(\frac{1}{2} \epsilon_{i,i+1}(x_i, x_{i+1})(x_i - x_{i+1}) \right) \prod_{i \in \mathbb{Z}/n\mathbb{Z}} \hat{h}_i(x_i)$$
$$\times \int_{t \in F} \psi(t)k(t)$$

The right-hand side is $\langle Q'^{\vee}, (\prod_i h_i)^\wedge \rangle \cdot \langle \psi, k \rangle$ by Proposition 2.5.2, where $\langle -, - \rangle$ now denotes the pairing between distributions and test functions. But this is also

$$\sim \langle (Q'^{\vee})^\wedge, \prod_i h_i \rangle \cdot \langle \psi, k \rangle = \langle Q' \otimes \psi, \phi \rangle.$$

This proved our lemma since h_i and k are arbitrary. □

Chapter 4

The Weil Representation

4.1 Definition of the Weil representation

Fix a symplectic space W. Let (ρ_ψ, S) be a unitary irreducible representation of $H = H(W)$ with central character ψ. Such representations are unique up to unitary equivalence by Stone-von Neumann theorem 3.2.1. The symplectic group $\mathrm{Sp}(W)$ operates on H by

$$(w, t) \mapsto (g(w), t), \qquad (w, t) \in H, g \in \mathrm{Sp}(W).$$

For any $g \in \mathrm{Sp}(W)$. Define a new representation ρ_ψ^g acting on the same space S by

$$\rho_\psi^g(h) = \rho_\psi(g(h)) \quad \text{for all } h \in H$$

the representation still satisfies the requirements of the Stone-von Neumann theorem, hence there exists a bounded operator $M : S \to S$ such that

$$M \circ \rho_\psi = \rho_\psi^g \circ M.$$

From Schur's lemma, M is uniquely determined by g up to a constant. Set $\mathrm{GL}(S) := \mathrm{End}_{\mathrm{conti.}}(S)^\times$ and $\mathrm{PGL}(S) := \mathrm{GL}(S)/(\mathbb{C}^\times \cdot \mathrm{id})$. We obtain a projective representation

$$\bar{\omega}_\psi : \mathrm{Sp}(W) \to \mathrm{PGL}(S).$$

A. Weil showed in [19] that $\bar{\omega}_\psi$ cannot be lifted to an ordinary representation $\mathrm{Sp}(W) \to \mathrm{GL}(S)$ for a local field $F \neq \mathbb{C}$. To study $\bar{\omega}_\psi$, we proceed to construct an universal group that lifts $\bar{\omega}_\psi$ to an ordinary representation.

Definition 4.1.1. Define

$$\begin{aligned} \widetilde{\mathrm{Sp}}_\psi(W) &:= \mathrm{Sp}(W) \times_{\mathrm{PGL}(S)} \mathrm{GL}(S) \\ &= \{(g, M) \in \mathrm{Sp}(W) \times \mathrm{GL}(S) : M \circ \rho_\psi = \rho_\psi^g \circ M\} \end{aligned}$$

Equip $\widetilde{\mathrm{Sp}}_\psi(W)$ with the subspace topology induced from the product $\mathrm{Sp}(W) \times \mathrm{GL}(S)$, where $\mathrm{GL}(S)$ is equipped with the *strong operator topology*, that is, the weakest topology making $M \mapsto M(s)$ continuous for each $s \in S$.

Let $\omega_\psi : \widetilde{\mathrm{Sp}}_\psi(W) \to \mathrm{GL}(S)$ be the projection, we have a commutative diagram of groups with exact rows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \widetilde{\mathrm{Sp}}_\psi(W) & \xrightarrow{\tilde{p}} & \mathrm{Sp}(W) & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \omega_\psi} & \square & \downarrow{\scriptstyle \bar{\omega}_\psi} & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & \mathrm{GL}(S) & \longrightarrow & \mathrm{PGL}(S) & \longrightarrow & 1 \end{array}$$

Therefore, $\omega_\psi : \widetilde{\mathrm{Sp}}_\psi(W) \to \mathrm{GL}(S)$ lifts $\bar{\omega}_\psi$ to an ordinary representation on $\widetilde{\mathrm{Sp}}_\psi(W)$ in a tautological way. This is called the *Weil representation* or *metaplectic representation* [1].

It will be shown in §4.2 that $\widetilde{\mathrm{Sp}}_\psi(W)$ is a locally compact group, as a by-product of the construction of Schrödinger models.

Remark 4.1.2. It follows by abstract nonsense that

$$\begin{array}{c}\text{Data } (G, G \to \mathrm{Sp}(W), G \to \mathrm{GL}(S)) \text{ that lifts } \bar{\omega}_\psi \\ \Big\updownarrow \text{bijection} \\ \text{Data } (G, G \to \widetilde{\mathrm{Sp}}_\psi(W))\end{array}$$

Remark 4.1.3. The Weil representation ω_ψ can also be regarded as a representation of the semi-direct product

$$H \rtimes \widetilde{\mathrm{Sp}}_\psi(W), \qquad \widetilde{\mathrm{Sp}}_\psi(W) \to \mathrm{Aut}(H) \text{ via } (g, M) \cdot h = g(h).$$

acting by $\omega_\psi(h, (g, M)) : s \longmapsto \rho(h)(Ms)$.

Remark 4.1.4. Suppose that W_1, W_2 are two symplectic spaces and $W = W_1 \oplus W_2$. Fix representations $(\rho_{\psi,1}, S_1), (\rho_{\psi,2}, S_2)$ for $H(W_1)$ and $H(W_2)$, then the representation $\rho_\psi := \rho_{\psi,1} \otimes \rho_{\psi,1}$ on $S := S_1 \otimes S_2$ satisfies the Stone-von Neumann theorem, and we have an embedding

$$\mathrm{Sp}(W_1) \times \mathrm{Sp}(W_2) \to \mathrm{Sp}(W)$$

and a homomorphism

$$\mathrm{GL}(S_1) \times \mathrm{GL}(S_2) \to GL(S)$$

They induce a homomorphism

$$j : \widetilde{\mathrm{Sp}}_\psi(W_1) \times \widetilde{\mathrm{Sp}}_\psi(W_2) \to \widetilde{\mathrm{Sp}}_\psi(W),$$

which commutes with projections to symplectic groups in the obvious sense.

The representation $\omega_\psi \circ j$ is equivalent to $\omega_{\psi,1} \boxtimes \omega_{\psi,2}$. Such constructions are used in the setting of Howe correspondence, see [13].

Theorem 4.1.5 (Weil, [19] §43)**.** *There exists a unique subgroup $\widehat{\mathrm{Sp}}_\psi(W)$ of $\widetilde{\mathrm{Sp}}_\psi(W)$ such that $p := \tilde{p}|_{\widehat{\mathrm{Sp}}_\psi(W)} : \widehat{\mathrm{Sp}}_\psi(W) \to \mathrm{Sp}(W)$ is a two-fold covering. Let $\epsilon \in \mathrm{Ker}\,(p)$ be the non-trivial element, there is a short exact sequence*

$$1 \longrightarrow \{1, \epsilon\} \longrightarrow \widehat{\mathrm{Sp}}_\psi(W) \xrightarrow{p} \mathrm{Sp}(W) \longrightarrow 1 \tag{4.1}$$

Therefore, the restriction of ω_ψ to $\widehat{\mathrm{Sp}}_\psi$ lifts the projective representation $\tilde{\omega}_\psi$ of $\mathrm{Sp}(W)$ to an ordinary representation of $\widehat{\mathrm{Sp}}_\psi$.

The uniqueness of $\widehat{\mathrm{Sp}}_\psi(W)$ will be proved shortly after.

Remark 4.1.6. The proof in [19] is actually an existential proof. An explicit construction of $\widehat{\mathrm{Sp}}$ in terms of Maslov index will be given in §4.3.

C. Moore proved in [14] that $\mathrm{Sp}(W)$ has a unique non-trivial two-fold covering $\widehat{\mathrm{Sp}}(W)$. We will henceforth drop the subscript ψ and denote by $\widehat{\mathrm{Sp}}(W)$ the subgroup in $\widetilde{\mathrm{Sp}}_\psi(W)$.

Proposition 4.1.7. *$\widehat{\mathrm{Sp}}(W)$ is equal to its own derived group.*

[1] Or Segal-Shale-Weil representations, oscillator representations...

Proof. Let N be the its derived group. Since $\mathrm{Sp}(W)$ equals its own derived group, $p(N) = \mathrm{Sp}(W)$. Consider the non-trivial element $\epsilon \in \mathrm{Ker}\,(p)$. If $\epsilon \in N$, then $N = \widehat{\mathrm{Sp}}(W)$; otherwise $\widehat{\mathrm{Sp}}(W) = \{1, \epsilon\} \times N$, contradicting the fact that $p : \widehat{\mathrm{Sp}}(W) \to \mathrm{Sp}(W)$ does not split. ☐

Corollary 4.1.8. *$\widehat{\mathrm{Sp}}(W)$ is the derived group of $\widetilde{\mathrm{Sp}}_\psi(W)$. As a consequence, the subgroup $\widehat{\mathrm{Sp}}(W)$ satisfying Theorem 4.1.5 is unique.*

Proof. The derived group of $\widetilde{\mathrm{Sp}}_\psi(W)$ contains $\widehat{\mathrm{Sp}}(W)$ by the previous proposition. On the other hand, $\widetilde{\mathrm{Sp}}_\psi(W) = \widehat{\mathrm{Sp}}(W) \cdot \mathbb{C}^\times$, hence $\widehat{\mathrm{Sp}}(W)$ equals the derived group of $\widetilde{\mathrm{Sp}}_\psi(W)$. ☐

Corollary 4.1.9. *The lifting of $\bar{\omega}_\psi$ on $\widehat{\mathrm{Sp}}(W)$ is unique.*

Proof. Suppose that $\omega_\psi, \omega'_\psi : \widehat{\mathrm{Sp}}(W) \to \mathrm{GL}(S)$ are two liftings of $\bar{\omega}_\psi$, then they differ by a homomorphism $\chi : \widehat{\mathrm{Sp}}(W) \to \mathbb{C}^\times$. However $\mathrm{Ker}\,(\chi)$ contains the derived group of $\widehat{\mathrm{Sp}}(W)$, hence χ is trivial and $\omega_\psi = \omega'_\psi$. ☐

4.2 Models

In order to understand the group $\widetilde{\mathrm{Sp}}_\psi(W)$, the main problem is to find *models* for the Weil representation, that is, a concrete representation (ρ, S) of H that satisfies Theorem 3.2.1, with a mapping $M[-] : \mathrm{Sp}(W) \to \mathrm{GL}(S)$ such that

$$(g, M[g]) \in \widetilde{\mathrm{Sp}}_\psi(W),$$

that is, $g \mapsto (g, M[g])$ gives rise to a section $\mathrm{Sp}(W) \to \widetilde{\mathrm{Sp}}_\psi(W)$.

Corresponding to the examples 3.2.5 and 3.2.7, we have *Schrödinger models* and *lattice models* for the Weil representation.

4.2.1 Schrödinger models

Fix a lagrangian ℓ of W and consider the Schrödinger representation $(\rho_\ell, \mathcal{H}_\ell)$. We proceed to find a model using the canonical intertwiners introduced in §3.4.

For any $g \in \mathrm{Sp}(W)$. Since g acts as an automorphism on W as well as $H(W)$, by transport of structure, we have an isomorphism

$$g_* : \mathcal{H}_\ell \xrightarrow{\sim} \mathcal{H}_{g\ell}$$

which transports the functions via

$$s(-) \mapsto s(g^{-1}(-))$$

It transports densities as well (recall that the elements of $\mathcal{H}_\ell$ are L^2 functions tensored with $\Omega_{1/2}(W/\ell)$). So that the following diagrams commutes for every lagrangians ℓ, ℓ'.

$$\begin{array}{ccc} \mathcal{H}_\ell & \xrightarrow{\mathscr{F}_{\ell,\ell'}} & \mathcal{H}_{\ell'} \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle g_*} \\ \mathcal{H}_{g\ell} & \xrightarrow[\mathscr{F}_{g\ell,g\ell'}]{} & \mathcal{H}_{g\ell'} \end{array} \qquad \begin{array}{ccc} \mathcal{H}_\ell & \xrightarrow{g_*} & \mathcal{H}_{g\ell} \\ {\scriptstyle \rho_\ell}\downarrow & & \downarrow{\scriptstyle \rho^g_{g\ell}} \\ \mathcal{H}_\ell & \xrightarrow[g_*]{} & \mathcal{H}_{g\ell} \end{array}$$

Proposition 4.2.1. *Define*

$$M^{Sch}_\ell[g] := \mathscr{F}_{\ell,g\ell} \circ g_* = g_* \circ \mathscr{F}_{g^{-1}\ell,\ell},$$

where $\mathscr{F}_{g\ell,\ell} : \mathcal{H}_\ell \to \mathcal{H}_{g\ell}$ is the canonical intertwiner introduced in §3.4. Then $g \mapsto M^{Sch}_\ell[g]$ is a model of the Weil representation, called the Schrödinger model.

Proof. This is straightforward. Use the diagrams above to conclude that

$$\rho_\ell^g \circ \mathscr{F}_{\ell,g\ell} \circ g_* = \mathscr{F}_{\ell,g\ell} \circ \rho_{g\ell}^g \circ g_* = \mathscr{F}_{g\ell,\ell} \circ g_* \circ \rho_\ell.$$

□

Remark 4.2.2. The operator $M_\ell^{\mathrm{Sch}}[g]$ is unitary since g_* is.

Explicit formulas. Let $f \in \mathcal{H}_\ell$. Fix another lagrangian ℓ' such that $W = \ell \oplus \ell'$, then $\mathcal{H}_\ell$ can be identified with $L^2(\ell') \otimes \Omega_{1/2}(\ell')$. With respect to this splitting, write $g^{-1} \in \mathrm{Sp}(W)$ as

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$
$$\alpha : \ell \to \ell, \beta : \ell' \to \ell, \gamma : \ell \to \ell', \delta : \ell' \to \ell'$$
$$g^{-1}|_\ell = (\alpha, \gamma), g^{-1}|_{\ell'} = (\beta, \delta)$$

It suffices to specify the effect on smooth vectors in $\mathcal{H}_\ell$; for such a $f \in \mathcal{H}_\ell$, write $f\mu_{\ell,g\ell} = f'\nu$, where $\mu_{\ell,g\ell}$ is the density defined in §3.4, $f' \in \mathscr{S}(\ell') \otimes \Omega_1(g\ell/\ell \cap g\ell)$ and $\nu \in \Omega_{1/2}(g\ell')$. Then, the formula in Theorem 3.4.2 implies that

$$\begin{aligned} g_* \circ \mathscr{F}_{g^{-1}\ell,\ell}(f)(y) &= \int\limits_{x' \in g\ell/\ell \cap g\ell} f'((x',0)(gy,0))g^*\nu \qquad (y \in \ell') \\ &= \int\limits_{x \in \ell/\mathrm{Ker}\,\gamma} f'((gx,0)(gy,0))g^*\nu \qquad (x := g^{-1}x') \\ &= \int\limits_{x \in \ell/\mathrm{Ker}\,\gamma} f'(gx+gy)\psi\left(\frac{\langle gx, gy\rangle}{2}\right) g^*\nu \\ &= \int\limits_{x \in \ell/\mathrm{Ker}\,\gamma} f'(\alpha x + \gamma x + \beta y + \delta y)\psi\left(\frac{\langle \alpha x + \gamma x, \beta y + \delta y\rangle}{2}\right) g^*\nu \\ &= \int\limits_{x \in \ell/\mathrm{Ker}\,\gamma} f'(\gamma x + \delta y)\psi\left(\langle \gamma x, \beta y\rangle + \frac{\langle \gamma x, \alpha x\rangle + \langle \delta y, \beta y\rangle}{2}\right) g^*\nu, \end{aligned}$$

the last term is obtained by separating terms in ℓ and ℓ' and using the invariance property of f'.

Recall that

$$N_{\ell,i} := \{g \in \mathrm{Sp}(W) : \dim g\ell \cap \ell = i\}$$

Things become especially simple when $g \in N_{\ell,i}$: in that case, it is clear that $g \mapsto (g, M_\ell^{\mathrm{Sch}}[g])$ is a continuous section from $N_{\ell,i}$ to $\widetilde{\mathrm{Sp}}_\psi(W)$, hence $\tilde{p}^{-1}(N_{\ell,i}) \simeq N_{\ell,i} \times \mathbb{C}^\times$. In particular, $N_{\ell,0}$ is dense and open in $\mathrm{Sp}(W)$, this implies

Corollary 4.2.3. *$\widetilde{\mathrm{Sp}}_\psi(W)$ is a locally compact group.*

Using the explicit formula, one can obtain some simpler formulas for the Schrödinger model.

Proposition 4.2.4.

$$\begin{aligned} g &= \begin{pmatrix} \alpha & 0 \\ 0 & (\alpha^*)^{-1} \end{pmatrix} &&: (M_\ell^{Sch}[g])f(y) = f'(\alpha^* y) \cdot g^*\nu \\ g &= \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} &&: (M_\ell^{Sch}[g])f(y) = f'(y)\psi\left(\tfrac{\langle \beta y, y\rangle}{2}\right) \cdot g^*\nu \\ g &= \begin{pmatrix} 0 & \gamma \\ (\gamma^*)^{-1} & 0 \end{pmatrix} &&: (M_\ell^{Sch}[g])f(y) = \textstyle\int_{x \in \ell} f'(\gamma^{-1}x)\psi(\langle \gamma^{-1}x, \gamma^* y\rangle) \cdot g^*\nu \\ & &&= \textstyle\int_{x \in \ell} f'(\gamma^{-1}x)\psi(\langle x, y\rangle) \cdot g^*\nu \end{aligned}$$

for all $y \in \ell'$.

Remark 4.2.5. The Schrödinger model is determined by these formulas. Indeed, the first family of g is a Levi component of the standard maximal parabolic subgroup of $\mathrm{Sp}(2n, F)$, and the second is its unipotent radical. The three families of matrices generate $\mathrm{Sp}(2n, F)$ by the Bruhat decomposition.

4.2.2 Lattice models

Another convenient model for the Weil representation, which only exists for non-archimedean F, is the lattice model. It follows the same philosophy of Schrödinger models, namely: transport of structure followed by an intertwiner. However we will state and prove everything by explicit formulas in this section.

Suppose that the residual character of F is not 2. Pick a compact open subgroup $A \subset W$ such that $A^\perp = A$ and take the character $\psi_A := 1 \times \psi$ of A_H. Now consider the representation (ρ_A, S_A) in example 3.2.7. Here we are only concerned with the smooth part instead of its completion to an unitary representation. For any $g \in \mathrm{Sp}(W)$ and $f \in S_A$, put

$$(M_A[g]f)(w) = \sum_{a \in A/gA \cap A} \psi\left(\frac{\langle a, w\rangle}{2}\right) \cdot f(g^{-1}(a+w)). \tag{4.2}$$

If g lies in the stabilizer of A in $\mathrm{Sp}(W)$, which is a maximal compact subgroup, the formula is particularly simple:

$$(M_A[g]f)(w) = f(g^{-1}w), \text{ when } gA = A.$$

Proposition 4.2.6. *The map $g \mapsto M_A[g]$ defines a model of the Weil representation.*

Proof. Fix $g \in \mathrm{Sp}(W)$. Let $f \in S_A$, it is clear that $M_A[g]f \in S_A$. It remains to do straightforward calculations.

$$\begin{aligned}
\rho_A(gv)(M_A[g]f) :w &\mapsto \psi\left(\frac{\langle w, gv\rangle}{2}\right)(M_A[g]f)(w+gv) \\
&= \sum_{a \in A/A\cap gA} \psi\left(\frac{\langle a, w+gv\rangle}{2}\right)\psi\left(\frac{\langle w, gv\rangle}{2}\right) f(g^{-1}(a+w+gv)) \\
&= \sum_{a \in A/A\cap gA} \psi\left(\frac{\langle a, w+gv\rangle}{2}\right)\psi\left(\frac{\langle w, gv\rangle}{2}\right) f(g^{-1}(a+w)+v))
\end{aligned}$$

$$\begin{aligned}
M_A[g](\rho_A(v)f) :w &\mapsto \sum_{a \in A/A\cap gA} \psi\left(\frac{\langle a, w\rangle}{2}\right)(\rho_A(v)f)(g^{-1}(a+w)) \\
&= \sum_{a \in A/A\cap gA} \psi\left(\frac{\langle a, w\rangle}{2}\right)\psi\left(\frac{\langle g^{-1}(a+w), v\rangle}{2}\right) f(g^{-1}(a+w)+v) \\
&= \sum_{a \in A/A\cap gA} \psi\left(\frac{\langle a, w\rangle}{2}\right)\psi\left(\frac{\langle a+w, gv\rangle}{2}\right) f(g^{-1}(a+w)+v)
\end{aligned}$$

The terms inside summation are equal. □

Of course, one can juggle the Haar measures in the formulas to make $M_A[g]$ a unitary operator for all g.

4.3 Construction of $\widehat{\mathrm{Sp}}(W)$

4.3.1 The Maslov cocycle

The Maslov index occurs naturally in the context of Schrödinger models, namely

$$\begin{aligned} M_\ell^{\mathrm{Sch}}[g] \circ M_\ell^{\mathrm{Sch}}[h] &= \mathscr{F}_{\ell,g\ell} \circ g_* \circ \mathscr{F}_{\ell,h\ell} \circ h_* \\ &= \mathscr{F}_{\ell,g\ell} \circ \mathscr{F}_{g\ell,gh\ell} \circ g_* h_* \\ &= \gamma(\tau(\ell, h\ell, gh\ell)) \cdot M_\ell^{\mathrm{Sch}}[gh] \quad \text{by 3.5.1.} \end{aligned}$$

Recall that the central extensions $1 \to Z \to G' \to G \to 1$ can be described by group cohomology $H^2(G; Z)$ by choosing sections $G \to G'$. Using Schrödinger models as sections for $\widetilde{\mathrm{Sp}}_\psi(W) \to \mathrm{Sp}(W)$, this motivates the following definition.

Definition-Proposition 4.3.1 (Maslov cocycles)**.** For a fixed lagrangian ℓ of W, the function on $\mathrm{Sp}(W) \times \mathrm{Sp}(W)$ defined by

$$c_{g,h}(\ell) = \gamma(\tau(\ell, g\ell, gh\ell))$$

is a 2-cocycle on $\mathrm{Sp}(W)$.

Proof. This follows from the formula $M_\ell^{\mathrm{Sch}}[g] \circ M_\ell^{\mathrm{Sch}}[h] = c_{g,h}(\ell) M_\ell^{\mathrm{Sch}}[gh]$. □

This cocycle is not a coboundary, otherwise the extension $\widetilde{\mathrm{Sp}}_\psi(W)$ splits; however, it follows from 2.4.3 that the cocycle can be adjusted by some coboundaries to take value in ± 1. Such a construction first appears in [10]. The detailed construction goes as follows.

Definition 4.3.2 (Metaplectic group via Maslov cocycles)**.** For every $g \in \mathrm{Sp}(W), \ell \in \Lambda(W)$, fix an arbitrary orientation of ℓ and give $g\ell$ the transported orientation. Define

$$m_g(\ell) := m(g\ell, \ell) = \gamma(1)^{\frac{\dim W}{2} - \dim g\ell \cap \ell - 1} \gamma(A_{g\ell,\ell}),$$

this number is independent of the choice of orientation of ℓ (recall Theorem 2.4.3 for the relevant definitions). And define $\widehat{\mathrm{Sp}}(W)$ to be the set of pairs of the form

$$(g, t) \qquad (g \in \mathrm{Sp}(W), t : \Lambda(W) \to \mathbb{C}^\times),$$

such that

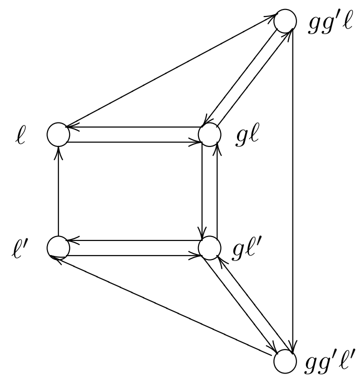
- $t(\ell)^2 = m_g(\ell)^2$.
- For any $\ell, \ell' \in \Lambda(W)$, we have $t(\ell') = \gamma(\tau(\ell, g\ell, g\ell', \ell'))t(\ell)$.
- Multiplication in $\widehat{\mathrm{Sp}}(W)$ is defined by

$$(g, t) \cdot (g', t') = (gg', st \cdot c_{g,g'}).$$

The right-hand side lies in $\widehat{\mathrm{Sp}}(W)$. Indeed, $(stc_{g,g'})(\ell)^2 = m_{gg'}(\ell)^2$ by Theorem 2.4.3 and Proposition 2.4.4; to verify the second defining condition of $\widehat{\mathrm{Sp}}(W)$, it suffices to check that

$$\tau(\ell, g\ell, g\ell', \ell') + \tau(\ell, g'\ell, g'\ell', \ell') + \tau(\ell', g\ell', gg'\ell') - \tau(\ell, g\ell, gg'\ell) = \tau(\ell, gg'\ell, gg'\ell', \ell').$$

The second term is equal to $\tau(g\ell, gg'\ell, gg'\ell', g\ell')$ by symplectic invariance. Then the required condition is explained by the diagram:



Finally, the associativity is equivalent to the cocycle condition.

- The unit element is $(1, \mathbb{1})$, where $\mathbb{1}$ means the constant function 1 on $\Lambda(W)$. Indeed, $m_1(\ell) = m(\ell, \ell) = \gamma(1)^{-1}\gamma(1) = 1$, while $c_{1,g}(\ell) = \gamma(\tau(\ell, \ell, g\ell)) = 1$ for all $g \in \mathrm{Sp}(W)$ since $\tau(\ell, \ell, g\ell)$ is represented by a quadratic space of dimension zero (equation (2.5)).
- The inverse of an element (g, t) is (g^{-1}, t^{-1}) since $c_{g,g^{-1}}(\ell) = 1$ by the same argument.

Proposition 4.3.3. *The projection map* $p : \widehat{\mathrm{Sp}}(W) \to \mathrm{Sp}(W)$ *defined by* $p((g, t)) = g$ *makes* $\widehat{\mathrm{Sp}}(W)$ *a two-fold covering of* $\mathrm{Sp}(W)$*.*

Proof. By the definition of $\widehat{\mathrm{Sp}}(W)$, for any $g \in \mathrm{Sp}(W)$, $p^{-1}(g)$ has either 2 or no elements. Fix $\ell \in \Lambda(W)$. For any $\ell' \in \Lambda(W)$, set

$$t(\ell') := \gamma(\tau(\ell, g\ell, g\ell', \ell')) \cdot m_g(\ell)$$

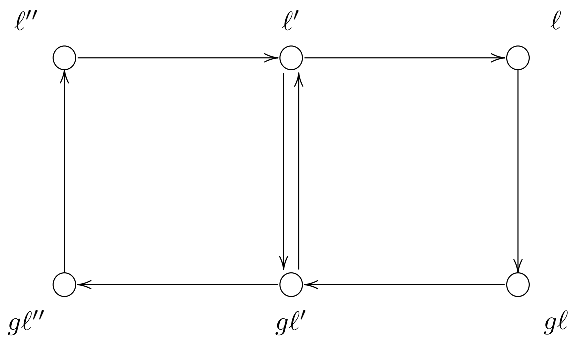
It remains to show that, for any $\ell', \ell'' \in \Lambda(W)$

$$\begin{aligned} t(\ell')^2 &= \gamma(\tau(\ell, g\ell, g\ell', \ell'))^2 m_g(\ell)^2 = m_g(\ell')^2 \\ t(\ell'') &= \gamma(\tau(\ell, g\ell, g\ell'', \ell''))m_g(\ell) = \gamma(\tau(\ell', g\ell', g\ell'', \ell''))t(\ell') \\ &\qquad\qquad = \gamma(\tau(\ell', g\ell', g\ell'', \ell'') + \tau(\ell, g\ell, g\ell', \ell'))m_g(\ell) \end{aligned}$$

The first equality follows from Theorem 2.4.3 and Proposition 2.4.4.

$$\begin{aligned} \gamma(\tau(\ell, g\ell, g\ell', \ell')) &= \pm m(\ell, g\ell)m(g\ell, g\ell')m(g\ell', \ell')m(\ell', \ell) \\ &= \pm m_g(\ell)m(\ell, \ell')m_g(\ell')^{-1}m(\ell, \ell')^{-1} \\ &= \pm m_g(\ell)m_g(\ell')^{-1} \end{aligned}$$

The second equality can be shown by the dihedral symmetry of Maslov indices



$$\tau(\ell', g\ell', g\ell'', \ell'') + \tau(\ell, g\ell, g\ell', \ell') = \tau(\ell, g\ell, g\ell'', \ell'').$$

□

Proposition 4.3.4. *Fix $\ell \in \Lambda(W)$, then the map*

$$\begin{aligned} \widehat{\mathrm{Sp}}(W) &\xrightarrow{\phi_\ell} \widetilde{\mathrm{Sp}}_\psi(W) \\ (g,t) &\longmapsto (g, t(\ell) M_\ell^{Sch}[g]) \end{aligned}$$

gives an embedding of $\widehat{\mathrm{Sp}}(W)$ as a subgroup of $\widetilde{\mathrm{Sp}}_\psi(W)$.

Proof. It is evidently injective. To show that ϕ_ℓ is a homomorphism, note that

$$\begin{aligned} \phi_\ell((g,t)) \cdot \phi_\ell((g',t')) &= (gg', t(\ell)t'(\ell) \cdot M_\ell^{\mathrm{Sch}}[g] M_\ell^{\mathrm{Sch}}[g']) \\ &= (gg', t(\ell)t'(\ell)\gamma(\tau(\ell, g\ell, gg'\ell)) \cdot M_\ell^{\mathrm{Sch}}[gg']) \\ &= \phi_\ell((g,t)\cdot(g',t')) \end{aligned}$$

□

In particular, $\bar{\omega}_\psi$ lifts to an ordinary representation of $\widehat{\mathrm{Sp}}(W)$ via

$$(g,t) \mapsto t(\ell) \cdot M_\ell^{\mathrm{Sch}}[g] = t(\ell) \cdot \mathscr{F}_{\ell, g\ell} \circ g_*. \tag{4.3}$$

Remark 4.3.5. The embedding ϕ_ℓ is independent of ℓ in the following sense: given $\ell, \ell' \in \Lambda(W)$, let $T_{\ell',\ell} : \mathrm{GL}(\mathcal{H}_\ell) \xrightarrow{\sim} \mathrm{GL}(\mathcal{H}_{\ell'})$ be given by

$$M \mapsto \mathscr{F}_{\ell',\ell} \circ M \circ \mathscr{F}_{\ell,\ell'},$$

then the diagram below commutes.

$$\begin{array}{ccccc} \widehat{\mathrm{Sp}}(W) & \xrightarrow{\phi_\ell} & \widetilde{\mathrm{Sp}}_\psi(W) & & \text{(constructed using } \ell) \\ \Big\| & & \big\downarrow{\scriptstyle T_{\ell',\ell}} & & \\ \widehat{\mathrm{Sp}}(W) & \xrightarrow{\phi_{\ell'}} & \widetilde{\mathrm{Sp}}_\psi(W) & & \text{(constructed using } \ell') \end{array}$$

4.3.2 Topological properties

Pick an arbitrary $\ell \in \Lambda(W)$. For any $g \in \mathrm{Sp}(W)$, define a function $\Xi_g^\ell : \Lambda(W) \to \mathbb{C}^\times$ by

$$\forall \ell' \in \Lambda(W),\ \Xi_g^\ell(\ell') := \gamma(\tau(\ell, g\ell, g\ell', \ell')) \cdot m_g(\ell).$$

Also observe that $\Xi_g^\ell(\ell) = m_g(\ell)$.

Give $\widehat{\mathrm{Sp}}(W)$ the subspace topology induced by $\phi_\ell : \widehat{\mathrm{Sp}}(W) \rightarrowtail \widetilde{\mathrm{Sp}}_\psi(W)$, then we have

Lemma 4.3.6. *The map $g \mapsto (g, \Xi_g^\ell)$ defines a continuous section $N_{\ell,i} \to \widehat{\mathrm{Sp}}(W)$ (see* (1.3) *for the definition) for every i.*

Proof. It had been shown in the proof of Proposition 4.3.3 that $(g, \Xi_g^\ell) \in \widehat{\mathrm{Sp}}(W)$. The map $g \mapsto m_g(\ell)$ is locally constant on each $N_{\ell,i}$ by the definition of m_g. On the other hand, $g \mapsto (g, M_\ell^{\mathrm{Sch}}[g]) \in \widetilde{\mathrm{Sp}}_\psi(W)$ is continuous on each $N_{\ell,i}$ by explicit formula. Hence $g \mapsto (g, \Xi_g^\ell)$ defines a continuous section on each $N_{\ell,i}$. □

Define the *evaluation map* $\mathrm{ev} : \Lambda(W) \times \widehat{\mathrm{Sp}}(W) \to \mathbb{C}^\times$ by

$$\mathrm{ev}_\ell(g,t) = t(\ell).$$

Proposition 4.3.7. *The map ev_ℓ is locally constant on each $p^{-1}(N_{\ell,i})$.*

Proof. By the previous lemma, every $p^{-1}(N_{\ell,i})$ is homeomorphic to two copies of $N_{\ell,i}$ via $g \mapsto (g, \pm\Xi_g)$. $\square$

Proposition 4.3.8. *$p : \widehat{\mathrm{Sp}}(W) \to \mathrm{Sp}(W)$ is a local homeomorphism. This makes $\widehat{\mathrm{Sp}}(W)$ into a F-analytic group. In particular, $\widehat{\mathrm{Sp}}(W)$ is a locally compact and totally disconnected group* [2].

Proof. The subset $p^{-1}(N_{\ell,0})$ is non-empty, open and admits a continuous section $N_{\ell,0} \to p^{-1}(N_{\ell,0})$. Hence it is homeomorphic to $N_{\ell,0} \times \{\pm 1\}$ by the lemma.

By transport of structure, this implies that $p : \widehat{\mathrm{Sp}}(W) \to \mathrm{Sp}(W)$ admits local sections, hence p is a local homeomorphism. $\square$

Remark 4.3.9. In fact, the triplet $(\widehat{\mathrm{Sp}}(W), \mathrm{Ker}\,(p), \mathrm{Sp}(W))$ is a semi-simple *almost algebraic group* in the sense of [12] p.257.

Therefore, the machinery of representation theory on locally compact, totally disconnected groups applies to $\widehat{\mathrm{Sp}}(W)$.

4.4 Admissibility

Theorem 4.4.1. *Take the Schrödinger model associated to $\ell \in \Lambda(W)$ for ω_ψ. Let $L^2(W/\ell) = L^2_{even}(W/\ell) \oplus L^2_{odd}(W/\ell)$ be the decomposition into even and odd functions. Then*

1. *ω_ψ decomposes accordingly as $\omega_\psi = \omega_{\psi,even} \oplus \omega_{\psi,odd}$.*
2. *$\omega_{\psi,even}$ and $\omega_{\psi,odd}$ are irreducible.*

Proof. See [5] for a sketch proof when $\dim W = 2$; the general case is similar, namely we use Proposition 4.2.4 to check that the only operators $L^2(W/\ell) \to L^2(W/\ell)$ commuting with ω_ψ are linear combinations of id and $s(x) \mapsto s(-x)$. This can also be regarded as the simplest case of Howe's correspondence. $\square$

Corollary 4.4.2. *The representation ω_ψ is admissible. Hence on can define its character Θ_{ω_ψ} to be the distribution*

$$\Theta_{\omega_\psi}(\phi \ \mathrm{d}x) = \mathrm{Tr}\Big(\int_{\widehat{\mathrm{Sp}}(W)} \phi(x)\omega_\psi(x) \ \mathrm{d}x\Big).$$

for every Schwartz-Bruhat function ϕ on $\widehat{\mathrm{Sp}}(W)$ and every Haar measure $\mathrm{d}x$.

Proof. According to [16] Théorème 1.2.3, every irreducible unitary representation of $\widehat{\mathrm{Sp}}(W)$ is admissible. Thus the admissibility of ω_ψ follows from Theorem 4.4.1; alternatively, one can argue by the lattice model and the proof of Lemma 3.2.2 when the residual characteristic is not 2.

$\square$

It remains to find a good explicit expression of Θ_{ω_ψ}. This is the subject-matter of our next chapter.

[2] This is equivalent to *locally profinite*.

Chapter 5

The character of the Weil representation

5.1 Statement of main results

In this chapter, we will follow [18] rather closely to obtain an explicit formula for the character Θ_{ω_ψ}. The main results are listed below.

Define $\mathrm{Sp}(W)'' := \{g \in \mathrm{Sp}(W) : \det(g-1) \neq 0\}$. It is a dense open subset in $\mathrm{Sp}(W)$.

Theorem 5.1.1. *The distribution Θ_{ω_ψ} is smooth on the dense open subset*

$$\widehat{\mathrm{Sp}}(W)'' := p^{-1}(\mathrm{Sp}(W)'') = \{(g,t) \in \widehat{\mathrm{Sp}}(W) : \det(g-1) \neq 0\} \tag{5.1}$$

and has the explicit form

$$\Theta_{\omega_\psi}(g,t) = \pm \frac{\gamma(1)^{\dim W - 1} \gamma(\det(g-1))}{|\det(g-1)|^{\frac{1}{2}}} \tag{5.2}$$

the sign ambiguity comes from the choice of t such that $p(g,t) = g$.

Recall that an element $x \in \widehat{\mathrm{Sp}}(W)$ is called *semi-simple* [resp. *semi-simple regular*] if $p(x) \in \mathrm{Sp}(W)$ is semi-simple [resp. semi-simple regular]. These semi-simple regular elements are denoted by $\widehat{\mathrm{Sp}}(W)'$, which is an open dense subset. In concordance with Harish-Chandra's regularity theorem, we have the

Proposition 5.1.2.

$$\widehat{\mathrm{Sp}}(W)'' \supset \widehat{\mathrm{Sp}}(W)'$$

Proof. Take any regular semi-simple $x \in \mathrm{Sp}(W)$. It is contained in a maximal F-torus T. By passing to a Galois extension L/F, we may suppose that T splits. Up to a conjugation, we may further suppose that T consists of diagonal matrices $y = (y_1, \ldots, y_n, y_1^{-1}, \ldots, y_n^{-1})$. We have the following roots of $(T, \mathrm{Sp}(W))$.

$$\alpha_i : \alpha_i(y) = y_i^2.$$

As x is regular semi-simple, $x_i^2 \neq 1$ for every i, hence $\det(1 \pm x) \neq 0$. □

Theorem 5.1.1 is actually derived from another character formula. To state it, we need some constructions.

Let $\overline{W}$ be the space W equipped with the symplectic form $-\langle,\rangle$. For any $g \in \mathrm{Sp}(W)$, the graph Γ_g of $g : \overline{W} \to W$ forms a lagrangian of $\overline{W} \oplus W$. On the other hand, given $\ell \in \Lambda(W)$, the space $\ell \oplus \ell$ is also a lagrangian. For every lagrangian ℓ, define a function Θ_ℓ on $\widehat{\mathrm{Sp}}(W)$ by

$$\Theta_\ell(g,t) := t(\ell)\cdot\gamma(\tau(\Gamma_g,\Gamma_1,\ell\oplus\ell)). \tag{5.3}$$

It will be shown later that Θ_ℓ is independent of ℓ and locally constant on $\widehat{\mathrm{Sp}}(W)''$.

Theorem 5.1.3 (K. Maktouf [12], T. Thomas [18])**.** *Fix a $\ell\in\Lambda(W)$. For every $(g,t)\in\widehat{\mathrm{Sp}}(W)''$, we have*

$$\Theta_{\omega_\psi}(g,t) = \frac{\Theta_\ell(g,t)}{|\det(g-1)|^{\frac{1}{2}}}. \tag{5.4}$$

The sign ambiguity thus disappeared at the cost of fixing a lagrangian in the computation.

Furthermore, we will use the machinery developed in §5.2 to express the character as the pull-back of a function on $\widehat{\mathrm{Sp}}(\overline{W}\oplus W)$. See Corollary 5.5.1

5.2 An embedding $\widehat{\mathrm{Sp}}(W)\to\widehat{\mathrm{Sp}}(\overline{W}\oplus W)$

Let $f:\mathrm{Sp}(W)\to\mathrm{Sp}(\overline{W}\oplus W)$ be the embedding

$$g\mapsto(1,g).$$

We are going to define a lifting $\tilde{f}:\widehat{\mathrm{Sp}}(W)\to\widehat{\mathrm{Sp}}(\overline{W}\oplus W)$ of f such that the diagram commutes.

$$\begin{array}{ccc}
\widehat{\mathrm{Sp}}(W) & \xrightarrow{\ \tilde{f}\ } & \widehat{\mathrm{Sp}}(\overline{W}\oplus W)\\
\downarrow & & \downarrow\\
\mathrm{Sp}(W) & \xrightarrow[\ f\ ]{} & \mathrm{Sp}(\overline{W}\oplus W)
\end{array}$$

Fix $\ell\in\Lambda(W)$ and set

$$\tilde{f}(g,t) = ((1,g),f_g(t))\in\widehat{\mathrm{Sp}}(\overline{W}\oplus W)$$

where $f_g(t):\Lambda(\overline{W}\oplus W)\to\mathbb{C}^\times$ is determined by

$$f_g(t)(\ell\oplus\ell) := t(\ell).$$

Proposition 5.2.1. *$\tilde{f}$ is a continuous, injective homomorphism independent of the choice of ℓ.*

Proof. Prove the independence first. For any other $\ell'\in\Lambda(W)$, we have

$$f_g(t)(\ell'\oplus\ell') = \gamma(\tau(\ell\oplus\ell,\ell\oplus g\ell,\ell'\oplus g\ell',\ell'\oplus\ell'))\cdot t(\ell).$$

It remains to show that it equals to

$$t(\ell') = \gamma(\tau(\ell,g\ell,g\ell',\ell'))\cdot t(\ell).$$

Indeed,

$$\tau(\ell\oplus\ell,\ell\oplus g\ell,\ell'\oplus g\ell',\ell'\oplus\ell') = \tau(\ell,g\ell,g\ell',\ell')+\tau(\ell,\ell,\ell',\ell'),$$

and the last term is 0 by (2.5).

A similar argument shows that f is homomorphic, the required condition being that

$$\tau(\ell\oplus\ell,\ell\oplus g\ell,\ell\oplus gh\ell) = \tau(\ell,g\ell,gh\ell).$$

This follows from the symplectic additivity of τ and the fact that $\tau(\ell,\ell,\ell)=0$, which also follows from (2.5).

Finally, f is obviously injective. If $g\in N_{\ell,i}$, then $(1,g)\in N_{\ell\oplus\ell,i+\frac{\dim W}{2}}$ and

$$\tilde{f}(g,\pm\Xi^\ell_g) = ((1,g),(\ell\oplus\ell\mapsto\pm m_g(\ell))).$$

We have $\Xi^{\ell\oplus\ell}_{(1,g)}(\ell\oplus\ell) = m_{(1,g)}(\ell\oplus\ell)$, which is equal to $m(\ell\oplus\ell,\ell\oplus g\ell)=m_g(\ell)$ by Proposition 2.4.4. The function $f_g(\Xi^\ell_g)$ coincides with $\Xi^{\ell\oplus\ell}_{(1,g)}$. Hence $\tilde{f}$ is continuous. □

Now we can enounce some properties of the function Θ_ℓ defined in (5.3).

Proposition 5.2.2.

$$\Theta_\ell = \mathrm{ev}_{\Gamma_1} \circ \tilde{f}$$

In particular, Θ_ℓ is independent of choice of ℓ.

Proof.

$$\mathrm{ev}_{\Gamma_1} \circ \tilde{f} = f_g(t)(\ell \oplus \ell) \cdot \gamma(\tau(\ell \oplus \ell, \ell \oplus g\ell, \Gamma_g, \Gamma_1)).$$

It remains to prove that $\tau(\ell \oplus \ell, \ell \oplus g\ell, \Gamma_g, \Gamma_1) = \tau(\Gamma_g, \Gamma_1, \ell \oplus \ell)$. By the chain condition, their difference is $\tau(\ell \oplus g\ell, \Gamma_g, \ell \oplus \ell)$, which is 0 by (2.5). □

Corollary 5.2.3. *Θ_ℓ is locally constant on $\widehat{\mathrm{Sp}}(W)''$.*

Proof. Given the previous proposition and Proposition 4.3.7, it suffices to observe that $\tilde{f}$ maps $\widehat{\mathrm{Sp}}(W)''$ into the preimage of $N_{\Gamma_1,0}$ in $\widehat{\mathrm{Sp}}(\overline{W} \oplus W)$. Indeed, $\Gamma_1 \cap \Gamma_g \simeq \mathrm{Ker}\,(g-1) = \{0\}$ in this case. □

5.3 Two quadratic spaces

For given $g \in \mathrm{Sp}(W), \ell \in \Lambda(W)$, we are going to associate a quadratic space $(S_{g,\ell}, q_{g,\ell})$ and its dual $(S'_{g,\ell}, q'_{g,\ell})$. They will appear in the integral kernel of the operator $\omega_\psi(g,t)$.

Consider the commutative diagram whose rows are complexes, labelled as $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$.

$$\begin{array}{ccccccccc}
 & & g\ell\cap\ell & \xrightarrow{1-g^{-1}} & \ell & \longrightarrow & W/(g-1)W & & \mathfrak{A}\\
 & & {\scriptstyle g^{-1}}\downarrow & & \downarrow & & \| & & \downarrow\\
 & & \ell & \xrightarrow{g-1} & g\ell+\ell & \longrightarrow & W/(g-1)W & & \mathfrak{B}\\
 & & \downarrow & & \downarrow & & \| & & \downarrow\\
\mathrm{Ker}\,(g-1) & \longrightarrow & W & \xrightarrow{g-1} & W & \longrightarrow & W/(g-1)W & & \mathfrak{C}\\
\| & & \downarrow & & \downarrow & & & & \downarrow\\
\mathrm{Ker}\,(g-1) & \longrightarrow & W/\ell & \xrightarrow{g-1} & W/(g\ell+\ell) & & & & \mathfrak{D}
\end{array}$$

It is easy to verify that

- $\mathfrak{A} \to \mathfrak{B}$ is a quasi-isomorphism.
- $\mathfrak{C}$ is exact.
- $\mathfrak{B} \rightarrowtail \mathfrak{C} \twoheadrightarrow \mathfrak{D}$ is a short exact sequence of complexes of F-vector spaces.
- $\mathfrak{A}, \mathfrak{D}$ are dual via $\langle , \rangle$.

Definition 5.3.1. Define

- $S'_{g,\ell} :=$ the cohomology of $\mathfrak{A}$ at its center term.
- $S_{g,\ell} :=$ the cohomology of $\mathfrak{D}$ at its center term.

Let I be the cohomology of $\mathfrak{B}$ at its center term. Since $\mathfrak{C}$ is exact, $\mathfrak{B} \rightarrowtail \mathfrak{C} \twoheadrightarrow \mathfrak{D}$ is a short exact sequence, the connecting homomorphism gives rise to an isomorphism $\Phi_{g,\ell} : S_{g,\ell} \to S'_{g,\ell}$,

$$\underbrace{S_{g,\ell} \xrightarrow{\sim} I \xleftarrow{\sim} S'_{g,\ell}}_{\Phi_{g,\ell}}\ .$$

Hence we can define the following non-degenerate bilinear forms:

$$q_{g,\ell}(x,y) := \langle \Phi_{g,\ell}(x), y\rangle, \qquad x,y \in S_{g,\ell} \tag{5.5}$$

$$q'_{g,\ell}(a,b) := \langle a, \Phi^{-1}_{g,\ell}(b)\rangle, \qquad a,b \in S'_{g,\ell} \tag{5.6}$$

Description of $q_{g,\ell}, q'_{g,\ell}$. Proceed to describe the forms $q_{g,\ell}, q'_{g,\ell}$ pull-backed to

$$\hat{S}_{g,\ell} := \{x \in W/\ell : (g-1)x \in g\ell + \ell\} \subset W/\ell$$

$$\hat{S}'_{g,\ell} := \ell \cap (g-1)W \subset \ell.$$

Look at our big commutative diagram: given $x \in \hat{S}_{g,\ell}$, write $(g-1)x = ga + b$ with $a, b \in \ell$. However $a + b \equiv ga + b \mod (g-1)\ell$, hence $\Phi_{g,\ell}(x) = a + b \in \ell$ (we have actually followed the usual proof of the snake lemma), and

$$q_{g,\ell}(x,y) = \langle a+b, y\rangle, \quad y \in \hat{S}_{g,\ell} \tag{5.7}$$

On the other hand, given $a, b \in \hat{S}'_{g,\ell}$, then there exists $y \in W$ such that $b = (g-1)y$, thus $\Phi_{g,\ell}(y+\ell) = b$ and

$$q'_{g,\ell}(a,b) = \langle a, y\rangle, \quad b \in \hat{S}'_{g,\ell} \tag{5.8}$$

Proposition 5.3.2. *The forms $q_{g,\ell}, q'_{g,\ell}$ are symmetric.*

Proof. It suffices to prove this for $q'_{g,\ell}$. Given $a, b \in \hat{S}'_{g,\ell}$, suppose that $a = (g-1)x, b = (g-1)y$ as before.

$$q'_{g,\ell}(a,b) = \langle (g-1)x, y\rangle = \langle x, (g^{-1}-1)y\rangle = \langle x, -g^{-1}b\rangle = \langle gx, -b\rangle.$$

Since $gx - x = a \in \ell$, the last term is just $\langle x, -b\rangle = \langle b, x\rangle$, which is $q'_{g,\ell}(b,a)$. ☐

Remark 5.3.3. If $g \in \mathrm{Sp}(W)''$, then $(g-1)$ is invertible and $S_{g,\ell} = \hat{S}_{g,\ell}$, $\hat{S}'_{g,\ell} = \ell$.

If furthermore $g\ell \cap \ell = \{0\}$, then $S_{g,\ell} = \hat{S}_{g,\ell} = W/\ell$ and $S'_{g,\ell} = \hat{S}'_{g,\ell} = \ell$. This is really the case which will be encountered later.

To establish the link between those quadratic spaces and the Maslov index, we begin with a lemma.

Lemma 5.3.4.

$$\begin{aligned}\dim S_{g,\ell} &= \dim S'_{g,\ell} \\ &= \frac{\dim W}{2} - \dim \mathrm{Ker}\,(g-1) - \dim g\ell \cap \ell + 2 \dim \ell \cap \mathrm{Ker}\,(g-1).\end{aligned}$$

Proof. It suffices to compute $\dim S'_{g,\ell}$. The Euler-Poincaré characteristic of $\mathfrak{A}$ in our big diagram is

$$\dim g\ell \cap \ell - \dim \ell + \dim W/(g-1)W,$$

which equals that of its cohomology

$$= \dim \mathrm{Ker}\,(1-g^{-1}) \cap g\ell \cap \ell - \dim S'_{g,\ell} + \dim W/(\ell + (g-1)W).$$

The following simple observations conclude the proof:

$$\begin{aligned}\mathrm{Ker}\,(1-g^{-1}) \cap g\ell \cap \ell &= \ell \cap \mathrm{Ker}\,(g-1) \\ \dim W/(g-1)W &= \dim \mathrm{Ker}\,(g-1) \\ (W/(\ell + (g-1)W))^* &\simeq (\ell + (g-1)W)^\perp = \ell \cap ((g-1)W)^\perp \\ &= \ell \cap \mathrm{Ker}\,(g-1)\end{aligned}$$

☐

Proposition 5.3.5. *The classes of $S_{g,\ell}, S'_{g,\ell}$ in $W(F)$ are equal to the Maslov index*

$$\tau(\Gamma_g, \Gamma_1, \ell \oplus \ell).$$

Proof. Recall that $\tau(\Gamma_g, \Gamma_1, \ell \oplus \ell)$ is represented by the possibly degenerate quadratic space

$$\hat{T} := \{(x, y, z) \in \Gamma_g \times \Gamma_1 \times (\ell \oplus \ell) : x + y + z = 0 \in \overline{W} \oplus W\}.$$

with the quadratic form (using (2.4))

$$q((x, y, z)) = \langle x, z \rangle.$$

Consider the linear map $f : \hat{T} \to \hat{S}'_{g,\ell}$

$$f : s = ((x, gx), (y, y), (a, b)) \longmapsto a - b$$

It indeed maps to $\hat{S}'_{g,\ell}$ by the definition of $\hat{T}$. It is also an isometry, since

$$q(s) = \langle gx, b \rangle - \langle x, a \rangle = \langle x, b \rangle - \langle x, a \rangle = \langle a - b, x \rangle = q'_{g,\ell}(f(s), f(s)).$$

Therefore f gives rise to an injective isometric map $\bar{f}$ between their non-degenerate quotients, namely T and $S'_{g,\ell}$. However, according to the dimension formula (2.5),

$$\begin{aligned}
\dim T = & \frac{3-2}{2} \cdot 2 \dim W - \dim \Gamma_1 \cap \Gamma_g - \dim \Gamma_g \cap (\ell \oplus \ell) - \dim(\ell \oplus \ell) \cap \Gamma_1 + \\
& + 2 \dim \Gamma_1 \cap \Gamma_g \cap (\ell \oplus \ell) \\
& = \dim W - \dim \operatorname{Ker}(g - 1) - \dim g^{-1}\ell \cap \ell - \dim \ell + \\
& + 2 \dim \ell \cap \operatorname{Ker}(g - 1) \\
& = \dim S'_{g,\ell} \qquad \text{, by the previous lemma.}
\end{aligned}$$

This shows that $\bar{f}$ is an isometric isomorphism. The assertion for $\dim S_{g,\ell}$ follows since $S_{g,\ell}$ is dual to $S'_{g,\ell}$. □

5.4 Expression by an integral kernel

A lagrangian $\ell \in \Lambda(W)$ is fixed throughout this section. The trace will be computed on the preimage of the smaller dense open subset

$$\begin{aligned}
\operatorname{Sp}(W)^\ell &:= \operatorname{Sp}(W)'' \cap N_{\ell,0} \\
&= \{g \in \operatorname{Sp}(W) : g\ell \cap \ell = \{0\},\ \det(g - 1) \neq 0\}.
\end{aligned}$$

Recall the Schrödinger model of Weil representation constructed in §4.2

$$\omega_\psi : (g, t) \longmapsto t(\ell) \cdot \mathscr{F}_{\ell, g\ell} \circ g_*,$$

in which the canonical intertwiner $\mathscr{F}_{\ell,g\ell}$ can be represented by an integral kernel (see Lemma 3.5.2 for the relevant definitions)

$$(\mathscr{F}_{\ell,g\ell}\phi)(y) = \int\limits_{\substack{x \in W/g\ell \\ x - y \in \ell + g\ell}} \phi(x) \psi\left(\frac{Q_{g\ell,\ell}((x, y))}{2}\right) \mu_{g\ell,\ell}^{1/2}.$$

Let $K_{g\ell,\ell} := \psi\left(\frac{Q_{g\ell,\ell}((x,y))}{2}\right) \mu_{g\ell,\ell}^{1/2}$. It is a smooth function on $W \times W$ satisfying the transformation rule of $\mathcal{H}^\vee_{g\ell}$ for the first variable, and satisfying that of $\mathcal{H}_\ell$ for the second variable.

Define two homomorphisms

$$\Delta : W/\ell \longrightarrow W/\ell \times W/\ell, \quad x \mapsto (x,x)$$
$$\alpha_g : (W/\ell)^2 \longrightarrow W/g\ell \times W/\ell, \quad (x,y) \mapsto (gx,y)$$

We abuse notation to use α_g to denote the automorphism $(x,y) \mapsto (gx,y)$ of W^2.

Then $\omega_\psi(g,t)$ can be represented by the integral kernel $t(\ell)\cdot\alpha_g^* K_{g\ell,\ell}$ on $(W/\ell)^2$. The integration of $t(\ell)\cdot\Delta^*\alpha_g^* K_{g\ell,\ell}$ will give Θ_{ω_ψ} by the smoothness of $K_{g\ell,\ell}$.

Assume henceforth that $g \in \mathrm{Sp}(W)^\ell$ so that $S_{g,\ell} = W/\ell$ and $S'_{g,\ell} = \ell$. Our strategy is to

- Identify $\Delta^*\alpha_g^* Q_{g\ell,\ell}$. The result will be the quadratic form $q_{g,\ell}$ defined in §5.3.
- Identify $\Delta^*\alpha_g^*\mu_{g\ell,\ell}^{1/2}$. It will yield the factor $|\det(g-1)|^{1/2}$ in Theorem 5.1.3.

First of all, note that $S_{g,\ell} = W/\ell$ and $S'_{g,\ell} = \ell$ since $g \in \mathrm{Sp}(W)^\ell$ by assumption.

Lemma 5.4.1. *If $g \in \mathrm{Sp}(W)^\ell$ then*

$$\Delta^*\alpha_g^* Q_{g\ell,\ell}(x) = q_{g,\ell}(x) \quad ,\textit{for all } x \in W/\ell.$$

Proof. Recall the definition of $Q_{g\ell,\ell}$ in Lemma 3.5.2 and Definition 2.5.1. Given $x, y \in W$ with $x - y = a + b$, where $a \in g\ell, b \in \ell$, then

$$Q_{g\ell,\ell}((x,y)) = \langle a, x\rangle + \langle b, y\rangle.$$

The assertion follows immediately by comparing with (5.7). □

Lemma 5.4.2. *If $g \in \mathrm{Sp}(W)^\ell$ then*

$$\Delta^*\alpha_g^*\mu_{g\ell,\ell}^{1/2} = |\det(g-1)|^{-\frac{1}{2}}\, \mathrm{d}q_{g,\ell},$$

where $\mathrm{d}q_{g,\ell}$ denotes the self-dual measure on W/ℓ with respect to $q_{g,\ell}$.

Proof. Since $g \in \mathrm{Sp}(W)^\ell$, we have $W = \ell \oplus g\ell$ and $\mu_{g\ell,\ell}$ is simply the self-dual measure μ_W on W with respect to the symplectic form. Set $\nu_\ell := \Delta^*\alpha_g^*\mu_{g\ell,\ell}^{1/2}$; it is a Haar measure on W/ℓ. Consider the following pairing

$$\theta : \Omega_1^{\mathbb{R}}(W/\ell) \times \Omega_1^{\mathbb{R}}(\ell) \longrightarrow \Omega_1^{\mathbb{R}}(W) = \mathbb{R}$$
$$(\alpha,\beta) \longmapsto \frac{\alpha\cdot\beta}{\mu_W}$$

Observe that $(\Phi_{g,\ell})_*\,\mathrm{d}q_{g,\ell}$ is the self-dual measure on $S'_{g,\ell} = \ell$. Fix a nonzero $\omega \in \bigwedge^{\max}\ell$, we can regard $g\omega$ as an element of $\bigwedge^{\max}(W/\ell) \simeq \bigwedge^{\max}(g\ell)$ so that $g\omega\wedge\omega \in \bigwedge^{\max} W$. Our assertion is then equivalent to

$$\theta(\nu_\ell, (\Phi_{g,\ell})_*\nu_\ell) = \frac{\nu_\ell(g\omega)\cdot(\Phi_{g,\ell})_*\nu_\ell(\omega)}{\mu_W(g\omega\wedge\omega)} = |\det(g-1)|^{-1}. \tag{5.9}$$

We set out to prove the equality. The first observation is that

$$\nu_\ell(g\omega) = \mu_W(g^2\omega\wedge g\omega)^{1/2} = \mu_W(g\omega\wedge\omega)^{1/2} \tag{5.10}$$

by the definition of ν_ℓ. On the other hand, according to the discussion in §5.3,

$$\Phi_{g,\ell}^{-1} : \ell \xrightarrow{\sim} W/\ell$$
$$x \mapsto (g-1)^{-1}x \mod \ell$$

Composing with the isomorphism $W/\ell \xrightarrow{\sim} g\ell$, we may identify $\Phi_{g,\ell}^{-1}(x)$ as the $g\ell$-component of $(g-1)^{-1}x$ in $W = \ell \oplus g\ell$. Thus

$$\begin{aligned}((\Phi_{g,\ell})_* \nu_\ell)(\omega) &= \nu_\ell(\Phi_{g,\ell}^{-1}\omega) \\ &= \nu_\ell(g\omega) \cdot \frac{\mu_W((g-1)^{-1}\omega \wedge \omega)}{\mu_W(g\omega \wedge \omega)}\end{aligned}$$

in which the last fraction is nothing but the constant c such that $\Phi_{g,\ell}^{-1}\omega = c \cdot g\omega \in \bigwedge^{\max}(W/\ell)$.

However $(g-1)^{-1}\omega \wedge \omega = (g-1)^{-1}(\omega \wedge (g-1)\omega) = (g-1)^{-1}(\omega \wedge g\omega)$, we get

$$(\Phi_{g,\ell})_* \nu_\ell(\omega) = |\det(g-1)|^{-1} \mu_W(g\omega \wedge \omega)^{\frac{1}{2}}. \tag{5.11}$$

Finally, the combination of (5.9), (5.10) and (5.11) proves our assertion. □

The overall conclusion of this section is

$$(\Delta^* \alpha_g^* K_{g\ell,\ell})(x) = |\det(g-1)|^{-\frac{1}{2}} \psi\left(\frac{q_{g,\ell}(x)}{2}\right) \,\mathrm{d}q_{g,\ell} \tag{5.12}$$

5.5 The character formula

The goal of this section is to prove Theorem 5.1.1 and 5.1.3. Fix a Haar measure on $\widehat{\mathrm{Sp}}(W)$, the choice doesn't affect the character and we will simply omit the measure in the integrals.

Proof of Theorem 5.1.3. Consider any Schwartz-Bruhat function ϕ on $\widehat{\mathrm{Sp}}(W)$ supported on $\widehat{\mathrm{Sp}}(W)''$. Let $\widehat{\mathrm{Sp}}(W)^\ell := p^{-1}(\mathrm{Sp}(W)^\ell)$. Use the result of §5.4 to write the operator $\omega_\psi(\phi)$ as

$$\begin{aligned}(\omega_\psi(\phi)s)(y) &= \int_{(g,t)\in\widehat{\mathrm{Sp}}(W)} \phi(g,t)\cdot t(\ell) \int_{x\in W/\ell} s(x)(\alpha_g^* K_{g\ell,\ell})(x,y) \\ &= \int_{(g,t)\in\widehat{\mathrm{Sp}}(W)^\ell} \phi(g,t)\cdot t(\ell) \int_{x\in W/\ell} s(x)(\alpha_g^* K_{g\ell,\ell})(x,y)\end{aligned}$$

This operator has finite rank, hence is a trace class operator. Since the kernel is smooth, Theorem A.0.6 implies

$$\mathrm{Tr}(\omega_\psi(\phi)) = \int_{x\in W/\ell} \int_{(g,t)\in\widehat{\mathrm{Sp}}(W)^\ell} \phi(g,t)t(\ell)\cdot(\Delta^*\alpha_g^* K_{g\ell,\ell})(x) \tag{5.13}$$

This only makes sense as iterated integral. Take a smooth function h of compact support on W/ℓ such that $h(0) = 1$ and that $h^\wedge$ is a positive measure. Set $h_s(x) := h(sx)$ for every $s \in F$.

Claim: $\mathrm{Tr}(\omega_\psi(\phi)) = \lim_{s\to 0} \mathrm{Tr}(h_s \cdot \omega_\psi(\phi))$.

Indeed, we have

$$\mathrm{Tr}(h_s\cdot\omega_\psi(\phi)) := \int_{x\in W/\ell} h_s(x) \int_{(g,t)\in\widehat{\mathrm{Sp}}(W)^\ell} \phi(g,t)t(\ell)\cdot(\Delta^*\alpha_g^* K_{g\ell,\ell})(x). \tag{5.14}$$

We have $|h_s| \le 1$ and $h_s \to 1$ pointwise. The dominated convergence theorem can be applied to the outer integral. This proves the claim.

Fubini's theorem can now be applied to the truncated integral (5.14),

$$\mathrm{Tr}(h_s\cdot\omega_\psi(\phi)) = \int_{(g,t)\in\widehat{\mathrm{Sp}}(W)^\ell} \phi(g,t)t(\ell) \int_{x\in W/\ell} h_s(x)\cdot(\Delta^*\alpha_g^* K_{g\ell,\ell})(x).$$

The function $h(x)$ satisfies the requirements of Proposition 1.2.13. Therefore (5.12) and Proposition 1.2.13 shows that the inner integral is bounded by $|\det(g-1)|^{-1/2}$ and approaches $\gamma(q_{g,\ell}) \cdot |\det(g-1)|^{-1/2}$ as $s \to 0$. The dominated convergence theorem implies

$$\lim_{s \to 0} \operatorname{Tr}(h_s \cdot \omega_\psi(\phi)) = \int\limits_{(g,t) \in \widehat{\operatorname{Sp}}(W)^\ell} \phi(g,t) \cdot t(\ell) \cdot |\det(g-1)|^{-\frac{1}{2}} \cdot \gamma(q_{g,\ell}).$$

Recalling Proposition 5.3.5, we have actually established Theorem 5.1.3. □

As an easy corollary, we can now express Θ_{ω_ψ} as the pull-back of a function on $\widehat{\operatorname{Sp}}(\overline{W} \oplus W)$:

Corollary 5.5.1.

$$\Theta_{\omega_\psi}(g,t) = \frac{\operatorname{ev}_{\Gamma_1}(\tilde{f}(g,t))}{|\det(g-1)|^{\frac{1}{2}}}.$$

Proof. This follows from Theorem 5.1.1 and Proposition 5.2.2. □

Proof of Theorem 5.1.1. In view of 5.1.3, the main issue is to show

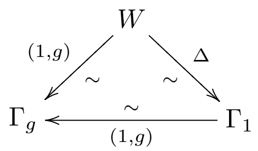
$$\Theta_\ell(g,t) = \pm\gamma(1)^{\dim W - 1}\gamma(\det(g-1)) \text{ for all } g \in \operatorname{Sp}(W)''.$$

By Proposition 5.2.2, we have

$$\Theta_\ell(g,t) = f_g(t)(\Gamma_1) = \pm m_{(1,g)}(\Gamma_1) = \pm m(\Gamma_g, \Gamma_1).$$

According to the definition of $m_{(1,g)}$, the orientations of Γ_1 and Γ_g are related by $(1,g) \in \widehat{\operatorname{Sp}}(\overline{W} \oplus W)$. Recalling the definition (2.8) for $m(-,-)$, it suffices to show that $A_{\Gamma_g,\Gamma_1} = \det(g-1) \mod F^{\times 2}$.

Consider the commutative diagram



Note that the horizontal arrow respects orientations. Since $\Gamma_1 \cap \Gamma_g = \{0\}$, the aim is simply to compute the discriminant of the symplectic pairing between Γ_g and Γ_1 with respect to the prescribed orientations. In virtue of the diagram above, we may identify both spaces with W, and the symplectic pairing becomes

$$(x,y) \mapsto \langle (x,gx), (y,y) \rangle_{\overline{W} \oplus W} = \langle (g-1)x, y \rangle.$$

Therefore $A_{\Gamma_g,\Gamma_1} = \det(g-1) \mod F^{\times 2}$, as asserted. □

Appendix A

Trace class operators

This appendix collects some standard results on trace class operators.

Definition A.0.2. Let H be a Hilbert space. Let A be a bounded operator from H to itself, and let A^* be its hermitian adjoint. Define $|A| := (A^*A)^{1/2}$. Take any orthonormal basis $\{e_i : i \in I\}$ indexed by a set I, then A is called a *trace class operator* if its *trace norm*

$$\|A\|_1 := \sum_{i \in I} (e_i, |A|e_i)$$

is finite. This condition is independent of the chosen orthonormal basis.

For a trace class operator A, we define its *trace* by

$$\mathrm{Tr}(A) := \sum_{i \in I} (e_i, Ae_i).$$

This is again independent of the chosen orthonormal basis.

Remark A.0.3. If A is normal, then $\mathrm{Tr}(A)$ is just the sum of its eigenvalues.

Proposition A.0.4. *The trace class operators form a two-sided ideal in the algebra of bounded operators on H. If A is trace class, so is its adjoint A^*.*

Proposition A.0.5. *We have the following hierarchy of operators from H to H:*

Bounded of finite rank ⇒ trace class ⇒ Hilbert-Schmidt ⇒ compact.

One can certainly introduce much more families of operators, but we will content ourselves with this. Only the first implication is used in this thesis.

Let X be a locally compact, σ-compact Hausdorff topological space. Fix a Radon measure on X. The keystone of this appendix is to compute the trace of an operator on $L^2(X)$ by integration along diagonal. There is a variant of the classical *Mercer's theorem.*

Theorem A.0.6. *With the above hypotheses on X, let $K(x, y)$ be a square-integrable function on $X \times X$. Consider the bounded operator $A_K : L^2(X) \to L^2(X)$ defined by $(A_K f)(y) = \int_X K(x, y) f(x)\, \mathrm{d}x$. Suppose that*

1. *A_K is a trace class operator.*
2. *$K(x, y)$ is continuous on $X \times X$.*

Then $K(x, x)$ is integrable over X and we have

$$\mathrm{Tr}(A_K) = \int_X K(x, x)\, \mathrm{d}x.$$

Proof. See [4], Théorème V.3.3.1 . □

The topological conditions on X are met for all cases encountered in this thesis.

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