On the inverse problem for deformation rings of representations

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Chapter 1

Introduction

Let G be a finite group, let k be a finite field of characteristic $p > 0$. An n-dimensional *representation* of G over k is a group homomorphism

$$\rho\colon G \to \mathrm{GL}_n(k)$$

In the same way, if A is a complete Noetherian local ring with residue field k, an n-dimensional representation of G over A is a group homomorphism $G \to \mathrm{GL}_n(A)$. We say that $(A, \tilde{\rho})$ is a *lift* of ρ if A is a complete Noetherian local ring with residue field k and $\tilde{\rho}$ is a group homomorphism for which the diagram

$$\begin{tikzcd} & \mathrm{GL}_n(A) \arrow[d] \\ G \arrow[ur, "\tilde{\rho}"] \arrow[r, "\rho"] & \mathrm{GL}_n(k) \end{tikzcd}$$

commutes. Two lifts $\tilde{\rho}_1, \tilde{\rho}_2\colon G \to \mathrm{GL}_n(A)$ of ρ over A are said to be *equivalent* if there exists a matrix K in $\ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$ for which $K^{-1}\tilde{\rho}_1(g)K = \tilde{\rho}_2(g)$ for every g in G. An equivalence class of lifts is called *deformation* of ρ. Given a representation of G over k and a complete Noetherian local ring A, we define the set $\mathrm{Def}(\rho, A)$ to be the set of all deformations of ρ to A.

For a representation $\rho\colon G \to \mathrm{GL}_n(k)$ the *universal deformation ring* R_ρ is a lift (R_ρ, ρ^u) for which the following universal property holds: for any lift $(A, \tilde{\rho})$ of ρ there exists a unique homomorphism $\varphi\colon R^u \to A$ such that the following diagram:

$$\begin{tikzcd} & \mathrm{GL}_n(R) \arrow[d, "\hat{\varphi}"] \\ G \arrow[ur, "\rho^u"] \arrow[r, "\tilde{\rho}"] & \mathrm{GL}_n(A) \end{tikzcd}$$

commutes, where the vertical arrow $\hat{\varphi}$ is the map induced by $\varphi\colon R^u \to A$.

Deformation theory was developed by B. Mazur, in particular to study Galois representations, see [11] for a detailed description of Mazur's work. It has been a powerful tool in Wiles's proof of Fermat's last Theorem.
Mazur found that for an absolutely irreducible representation $\bar{\rho}$, there is a universal solution to the problem of classifying deformations of $\bar{\rho}$.
He works in the category $\mathcal{C}$ of complete Noetherian local $W(k)$-algebras with residue field k, where $W(k)$ is the ring of Witt vectors of k. The objects of $\mathcal{C}$, also called *coefficient rings*, are endowed with the natural profinite topology, a base of open ideals being given by the powers of its maximal ideal $\mathfrak{m}_A$:

$$A = \varprojlim_{\nu} A/\mathfrak{m}_A^{\nu}$$

whose morphisms are continuous maps $A' \to A$ such that the inverse image of the maximal ideal $\mathfrak{m}_A$ is the maximal ideal $\mathfrak{m}_{A'} \subset A'$ and the induced homomorphism on residue fields is the identity.

Proposition 1. *If N is a positive integer, G a finite group and*

$$\bar{\rho}\colon G \to \mathrm{GL}_N(k)$$

is absolutely irreducible, there is a "universal coefficient ring" $R = R(\bar{\rho})$ with residue field k, and a "universal" deformation,

$$\rho^u\colon G \to \mathrm{GL}_N(R)$$

of $\bar{\rho}$ to R; it is universal in the sense that given any coefficient-ring A with residue field k, and any deformation

$$\rho\colon G \to \mathrm{GL}_N(A)$$

of $\bar{\rho}$ to A, there is one and only one homomorphism $h\colon R \to A$ inducing the identity isomorphism on residue fields for which the composition of the universal deformation ρ^u with the homomorphism $\mathrm{GL}_N(R) \to \mathrm{GL}_N(A)$ coming from h is equal the deformation ρ. In other terms the functor

$$D_{\bar{\rho}}\colon \mathcal{C} \to \mathbf{Set}$$

for which $D_{\bar{\rho}}(A) = \mathrm{Def}(\bar{\rho}, A)$, is representable by R, i.e.

$$D_{\bar{\rho}} \cong \mathrm{Hom}_{W(k)\text{-alg}}(R, A)$$

where $W(k)$ is the ring of Witt vectors of k.

However Lenstra and de Smit proved in [15], following an argument due to Faltings, that we can skip the hypothesis of absolute irreducibility if we require the weaker condition $\mathrm{End}_{k[G]}(k^n) = k$ for the representation

$\rho\colon G \to \mathrm{GL}_n(k)$, we will state this result later as Theorem 2.1.5.

The aim of this thesis is to study an inverse problem. One can ask what kind of ring can occur as universal deformation ring. Namely, given a ring R is it possible to find a group G and a representation whose universal deformation ring is R? We will answer this question for a particular class of rings: $\mathbb{Z}/p^n\mathbb{Z}$ with $p > 3, n \geq 1$, $\mathbb{Z}_p$ and $\mathbb{Z}_p[[t]]/(p^n, p^m t)$ with $p > 3, n, m \geq 1$.
One of the most powerful tools to compute the deformation ring of a representation

$$\bar{\rho}\colon G \to \mathrm{GL}_n(k)$$

is to compute $H^1(G, M_{n\times n}(k))$, where G acts on $M_{n\times n}(k)$ by conjugation, because an important result of Mazur states:

Theorem 1. *The universal deformation ring $R_{\bar{\rho}}$ of the representation $\bar{\rho}$ is a quotient of the formal power series ring $W(k)[[t_1, \ldots, t_d]]$ whose minimal number of variables is $d = \dim_k H^1(G, M_{n\times n}(k))$, where $W(k)$ is the ring of the Witt vectors of k.*

In the thesis we will not use cohomological methods but we will only study the set $\mathrm{Def}(\bar{\rho}, A)$, in particular we will put more attention for $A = k[\epsilon]$ with $\epsilon^2 = 0$. Moreover we will only study representations over the finite field $\mathbb{F}_p$, for which it is known that $W(\mathbb{F}_p) = \mathbb{Z}_p$.

The thesis is organized as follows. In the first chapter we briefly recall the basic notions and results pertaining to *Deformation Theory*, namely we give the definition of the *Zariski tangent space* and we state some important properties that show how it is linked to the set $\mathrm{Def}(\rho, k[\epsilon])$. The most important result, due to Mazur (see [11]), is:

$$\mathrm{Def}(\rho, k[\epsilon]) \cong \mathrm{Hom}_{\mathcal{C}}(R_\rho, k[\epsilon]) \cong t_\rho \cong H^1(G, M_{n\times n}(k))$$

where k is a finite field of characteristic $p > 0$, $\rho\colon G \to \mathrm{GL}_n(k)$ is the residual representation, R_ρ is the universal deformation ring, t_ρ is the Zariski tangent space of R_ρ and $\mathcal{C}$ is the category of the Noetherian complete local $W(k)$-algebras with residue field k, and G acts on $M_{n\times n}(k)$ by conjugation. Then we study one-dimensional representations of finite groups and we compute the universal deformation ring for such representations. It turns out that the universal deformation ring of a representation $\rho\colon G \to \mathbb{F}_p^\times$ is $R_\rho = \mathbb{Z}_p[G^{(p)}]$, where $G^{(p)}$ is the biggest abelian quotient of G that is a p-group.

In the second chapter we give an example of a group and a representation whose universal deformation ring is $\mathbb{F}_p$ (for $p > 3$), $G = \mathrm{GL}_2(\mathbb{F}_p)$ and ρ the identity:

$$\rho\colon G \to \mathrm{GL}_2(\mathbb{F}_p)$$

Instead for $p = 2, 3$ we find that the universal deformation ring is $\mathbb{Z}_p$. In this chapter we give also an important property, that is:

Proposition 2 (3.1.5)**.** *Let p be a prime number,*

$$T = \begin{pmatrix} \mathbb{F}_p^\times & 0 \\ 0 & \mathbb{F}_p^\times \end{pmatrix}$$

and $\rho \colon T \to \mathrm{GL}_2(\mathbb{F}_p)$ be the inclusion, then the universal deformation ring is $\mathbb{Z}_p$.

It is clear that ρ is not irreducible and that $\mathrm{End}_{\mathbb{F}_p[T]}(\mathbb{F}_p^2) \neq \mathbb{F}_p$, but we get the existence of the universal deformation ring.

In the third chapter we study the natural 2-dimensional representation over $\mathbb{F}_p$ of the group:

$$G = \begin{pmatrix} \mu_{p-1} & \mathbb{Z}/p^n\mathbb{Z} \\ 0 & \mu_{p-1} \end{pmatrix}$$

The group G can be viewed as a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$, since $\mu_{p-1} = \{x \in \mathbb{Z}_p : x^{p-1} = 1\} \subseteq \mathbb{Z}_p^\times$ and so there exists an injective homomorphism $\mu_{p-1} \to (\mathbb{Z}/p^n\mathbb{Z})^\times$. We show that the universal deformation ring is $\mathbb{Z}/p^n\mathbb{Z}$.

In the fourth chapter we show that for the group

$$G = \begin{pmatrix} \mu_{p-1} & \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z} \\ 0 & \mu_{p-1} \end{pmatrix} = \big(\mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z}\big) \rtimes (\mu_{p-1} \times \mu_{p-1})$$

with $1 \leq m \leq n$ and a 2-dimensional representation that we will define in the following the universal deformation ring is:

$$\big(\mathbb{Z}/p^n\mathbb{Z}\big)[[t]]/(p^m t) \cong \mathbb{Z}_p[[t]]/(p^n, p^m t)$$

This ring is not a complete intersection, hence we have an answer to the question of M. Flach (for a more general example see [3]).
Moreover this is an example that answers in negative way two questions by T. Chinburg and F. Bleher, see [4] and [5].

Chapter 2

Deformations

2.1 Preliminaries and basic definition

2.1.1 Representation theory

Let k be a field and let G be a finite group. An n-dimensional *representation* of G is a $k[G]$-module V for wich $\dim_k V = n$. Similarly, an n-dimensional *representation* of G over a ring R is an $R[G]$-module free of rank n as R-module.

An *homomorphism* of representations over the field k (resp. over the ring R) is a $k[G]$-linear map (resp. an $R[G]$-linear map), i.e. a morphism $\varphi\colon V \to W$ of k-vector space (resp. R-module) such that

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "g"'] & W \arrow[d, "g"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

commutes for every g in G.

A *subrepresentation* of a representation V is a k-linear subspace W of V which is invariant under the action of G. A nonzero representation V of a group G is called *irreducible* if there is no proper nonzero invariant subspace W of V. A representation over the field k is called *absolutely irreducible* if it remains irreducible even after any finite extension of the field.

2.1.2 Deformation theory

Let k be a field of characteristic $p > 0$ and $\mathcal{O}$ be a Noetherian local ring with residue field k and fix the surjective map $\mathcal{O} \twoheadrightarrow k$. We define the category $\mathcal{C}$ to be the full subcategory of $\mathcal{O}$-**alg** whose object are topological local $\mathcal{O}$-algebras A that are complete Noetherian local rings with residue field k such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O} \arrow[r] \arrow[d] & R \arrow[d] \\
k \arrow[r] & R/\mathfrak{m}_R
\end{tikzcd}$$

and the arrow $k \to R/\mathfrak{m}_R$ is an isomorphism.
The objects of $\mathcal{C}$ are endowed with the natural profinite topology, a base of open ideals being given by the powers of its maximal ideal $\mathfrak{m}_A$ and since they are complete it holds the following property:

$$A = \varprojlim_{\nu} \frac{A}{\mathfrak{m}_A^{\nu}}$$

Let G be a finite group and V a $k[G]$-module, of dimension n as vector space over k. We give the following:

Definition 2.1.1. A *lift* of V to a ring A in $\mathcal{C}$ is a pair (M, φ), where M is an $A[G]$-module, free of rank n as A-module and $\varphi \colon M \otimes_A k \to V$ is an isomorphism of $k[G]$-modules.

Given G and a $k[G]$-module with an abuse of notation we will refer to a lift as the first element of the couple. In particular we always get the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Aut}_A(M) \arrow[d] \\
G \arrow[ur, "\tilde{\rho}"] \arrow[dr, "\bar{\rho}"'] & \mathrm{Aut}_k(M \otimes_A k) \arrow[d, "\hat{\varphi}"] \\
 & \mathrm{Aut}_k(V)
\end{tikzcd}$$

where, given $f \colon M \otimes_A k \to M \otimes_A k$, we define $\bar{\varphi}(f)$ to be the map

$$\varphi^{-1} \circ f \circ \varphi \colon V \to V$$

Let (M, φ) and (M', φ') be two lifts of the $k[G]$-module V to the ring A , we say that they are *equivalent* if there exists an $A[G]$-module isomorphism $f \colon M \to M'$ such that the diagram:

$$\begin{tikzcd}
M \otimes_A k \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & M' \otimes_A k \arrow[dl, "\varphi'"] \\
 & V &
\end{tikzcd}$$

commutes.

Definition 2.1.2. A *deformation* of the representation V to the ring A is an equivalence class of lifts of V to A. By $\mathrm{Def}_V(A)$ we will denote the set of the deformations of V over A or by $\mathrm{Def}(\bar{\rho}, A)$ if we have the group homomorphism $\bar{\rho}\colon G \to \mathrm{Aut}_k(V)$.

For a $k[G]$-module V, the definition 2.1.2 allows us to define a functor:

$$D_V : \mathcal{C} \to \mathbf{Set}$$

that sends a ring A to $\mathrm{Def}_V(A)$, and if $f\colon A \to B$ is a morphism in $\mathcal{C}$ we define

$$D_V(f)\colon \mathrm{Def}_V(A) \to \mathrm{Def}_V(B)$$

to be the map that sends the deformation $[M, \varphi]$ of V over A to the deformation $[B \otimes_{A,f} M, \varphi_f]$ of V over B, where φ_f is the composition:

$$k \otimes_B (B \otimes_{A,f} M) \cong k \otimes_A M \xrightarrow{\varphi} V$$

Let V be an n-dimensional $k[G]$-module, we have the homomorphism of groups $\bar{\rho}\colon G \to \mathrm{Aut}_k(V)$. We give the following two definitions.

Definition 2.1.3. A ring $R = R^v$ in $\mathcal{C}$ is said to be the *versal deformation ring* for $\bar{\rho}$ if there exists a lift (M, φ) to R^v such that:

- for all rings A in $\mathcal{C}$ the map

$$\begin{array}{rccc} f_A\colon & \mathrm{Hom}_{\mathcal{C}}(R^v, A) & \to & \mathrm{Def}_V(A) \\ & \alpha & \mapsto & D_V(\alpha)[M, \varphi] \end{array}$$

  is surjective;

- if $A = k[\epsilon], \epsilon^2 = 0$, f_A is bijective.

In [10], Proposition 1, Mazur proved that the versal deformation ring always exists and it is unique, for another proof of the uniqueness see Proposition 2.9 in [13].

Definition 2.1.4. A ring $R = R_{\bar{\rho}} = R^u$ is said to be the *universal deformation ring* of the representation V if it is the versal deformation ring and the map f_A is bijective for every ring A in $\mathcal{C}$.

In other words R^u is the universal deformation ring of the representation V if the functor D_V is *represented by* R^u, i.e. $D_V(-)$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{C}}(R^u, -)$.

Existence of the universal deformation ring

An important result states that the universal deformation ring exists if the residual representation is absolutely irreducible. In [15] H. Lenstra and B. de Smit proved the following theorem for a representation $\bar{\rho}\colon G \to \mathrm{GL}_k(V)$, the category $\mathcal{C}$ is the one defined above.

Theorem 2.1.5. *If* $\mathrm{End}_{k[G]}(V) = k$ *then*

1. *there are a ring R in $\mathcal{C}$ and a deformation $\rho^u \in \mathrm{Def}(\bar{\rho}, R)$ such that for all rings A in $\mathcal{C}$ we have a bijection $f_A\colon \mathrm{Hom}_{\mathcal{C}}(R, A) \to \mathrm{Def}(\bar{\rho}, A)$;*

2. *the pair (R, ρ^u) is determined up to unique $\mathcal{C}$-isomorphism by the property in* (1).

Let us fix a basis of V so we have an isomorphism $V \to k^n$, hence we can identify $\mathrm{GL}_k(V)$ with $\mathrm{GL}_n(k)$, and we can speak in terms of matrices. The existence of the universal deformation ring R^u for a representation $\bar{\rho}\colon G \to \mathrm{GL}_n(k)$ implies also the existence of *the universal deformation* ρ^u, we get the following commutative diagram:

$$\begin{tikzcd} & \mathrm{GL}_n(R^u) \arrow[d] \\ G \arrow[ur, "\rho^u"] \arrow[r, "\bar{\rho}"] & \mathrm{GL}_n(k) \end{tikzcd}$$

The universal representation ρ^u satisfies the following universal property: for every deformation $\rho\colon G \to \mathrm{GL}_n(A)$ of $\bar{\rho}$ to the ring A there is a unique homomorphism $\varphi\colon R^u \to A$ such that the diagram:

$$\begin{tikzcd} & \mathrm{GL}_n(R^u) \arrow[d, "\varphi"] \\ G \arrow[ur, "\rho^u"] \arrow[r, "\tilde{\rho}"] & \mathrm{GL}_n(A) \end{tikzcd}$$

commutes, up to conjugation by an element in $\ker\big(\mathrm{GL}_n(R^u) \to \mathrm{GL}_n(A)\big)$. The universal deformation ring is uniquely identified by this universal property.

Zariski tangent space

In §15 of [11] Mazur describes the set $\mathrm{Def}(\bar{\rho}, A)$ from a cohomological point of view. First let us recall the definition of the *Zariski (co)tangent space.*

Definition 2.1.6. Let R be an element of $\mathcal{C}$. The *Zariski cotangent space* of the $\mathcal{O}$-algebra R is

$$t_R^* := \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}}R)$$

Observe that t_R^* is naturally endowed with the structure of a vector space over $k = R/\mathfrak{m}_R$. Since R is Noetherian t_R^* is finite-dimensional and so is its dual $t_R = \mathrm{Hom}_{k\text{-v.sp.}}(t_R^*, k)$.

Proposition 2.1.7. *There is a natural isomorphism of k-vector spaces*

$$t_R \cong \mathrm{Hom}_{\mathcal{O}\text{-alg}}(R, k[\epsilon])$$

Consider a representation $\bar{\rho}\colon G \to \mathrm{Aut}_k(V)$ and let $\tilde{\rho}\colon G \to \mathrm{Aut}_R(M)$ be a lift to the ring R.

Proposition 2.1.8. *There is a natural isomorphism of R-modules*

$$t_R \cong H^1(G, \mathrm{End}_R(M))$$

In particular if $R_{\bar{\rho}}$ is the universal deformation ring, we get

$$\mathrm{Hom}_{\mathcal{C}}(R_{\bar{\rho}}, k[\epsilon]) \cong \mathrm{Def}(\bar{\rho}, k[\epsilon]) \cong t_{R_{\bar{\rho}}} \cong H^1(G, \mathrm{End}_k(V)) \tag{2.1}$$

where G acts on $\mathrm{End}_k(V)$ by conjugation. These isomorphisms are the reason why we will often consider lifts to $k[\epsilon]$ to compute the deformation ring.
For a complete proof of the isomorphisms in (2.1) we refer to [11]. We will just remark how the maps between them work.

Remark 2.1.9. Let $\bar{\rho}\colon G \to \mathrm{GL}_n(k)$ be an n-dimensional representation of a finite group G over a field k of positive characteristic p and let R be the universal deformation ring:

$$\begin{tikzcd} & \mathrm{GL}_n(R) \arrow[d] \\ G \arrow[ur, "\rho^u"] \arrow[r, "\bar{\rho}"] & \mathrm{GL}_n(k) \end{tikzcd}$$

Let $t_R = \mathrm{Hom}(\mathfrak{m}/(\mathfrak{m}^2, p), k)$ be the Zariski tangent space of R and take $\psi \in t_R$. We get the following diagram:

$$\begin{tikzcd} \mathfrak{m}/(\mathfrak{m}^2, p) \arrow[r] \arrow[d, "\psi"'] & R/(\mathfrak{m}^2, p) \arrow[d] \\ k \cdot \epsilon \arrow[r] & k \oplus k \cdot \epsilon \end{tikzcd}$$

Thus we can construct the homomorphism $\psi^\sharp\colon R \to k[\epsilon]$.
The map $t_R \to \mathrm{Def}(\bar{\rho}, k[\epsilon])$ associates to ψ the deformation given by the class of $\psi^\sharp \circ \rho^u\colon G \to \mathrm{GL}_n(k[\epsilon])$.
Since $\mathrm{GL}_n(k[\epsilon]) \cong (1 + \epsilon \cdot M_{n\times n}(k)) \rtimes \mathrm{GL}_n(k)$, see Proposition 1 in §21 of [11], we get:

$$(\psi^\sharp \circ \rho^u)(\cdot) = (1 + \epsilon\sigma(\cdot))\bar{\rho}(\cdot)$$

where $\sigma\colon G \to M_{n\times n}(k)$ is a 1-cocycle.
The map $\mathrm{Def}(\bar\rho, k[\epsilon]) \to H^1(G, M_{n\times n}(k))$ is given by the one that associate to $\psi^\sharp \circ \rho^u$ the class $[\sigma]$.

From now on we will consider the field $k = \mathbb{F}_p$ and $\mathcal{O} = \mathbb{Z}_p$. Hence the category $\mathcal{C}$ is the category whose objects are complete Noetherian local rings R with $R/\mathfrak{m}_R = \mathbb{F}_p$, and the morphisms are $\mathbb{Z}_p$-algebra homomorphisms.

2.2 The 1-dimensional case

In this section we study 1-dimensional representations, i.e. a group homomorphism $\rho\colon G \to \mathbb{F}_p^\times$, and we will compute the universal deformation ring. Let start from this:

Example 2.2.1. Let G be a finite group. Let $\rho\colon G \to \mathbb{F}_p^\times$ be the trivial representation, i.e. $\rho(G) = \{1\}$. Let φ be a lift to A, ring of $\mathcal{C}$

$$\begin{tikzcd} & A^\times \arrow[d] \\ G \arrow[ur, "\varphi"] \arrow[r, "\rho"] & \mathbb{F}_p^\times \end{tikzcd}$$

We know that $A^\times = (1+\mathfrak{m}_A)\times\mu_{p-1}$, where $\mu_{p-1} := \{a \in A : x^{p-1} = 1\}$ that by Hensel's Lemma is isomorphic to $\mathbb{F}_p^\times$. Let $K := \ker(\varphi(G) \to \mathbb{F}_p^\times)$. Since $K \subseteq 1+\mathfrak{m}_A$ we get that K is a p-group. Moreover $\varphi(G) = (p\text{-group}) \times \mu_{p-1}$. Now, consider the biggest abelian quotient of G that is a p-group,

$$G^{(p)} = \frac{G}{[G,G]} \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

We claim that $R_\rho = \mathbb{Z}_p[G^{(p)}]$.
Take the projection $\tilde\rho\colon G \to G^{(p)}$, by the homomorphism theorem $\frac{G}{\ker\tilde\rho} \cong G^{(p)}$. Being K is a p-group we get $G_0 := \ker\tilde\rho \subseteq \ker\rho$ (for every lift φ), so we get this commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\pi"'] & A^\times \\ G^{(p)} \arrow[ur, "\tilde\varphi"] & \end{tikzcd}$$

where $\tilde\varphi(g \ker\tilde\rho) = \varphi(g)$.
Therefore there exists only one homomorphism $\mathbb{Z}_p[G^{(p)}] \to A$, defined on the generators $\bar g \mapsto \varphi(g)$ such that the following diagram:

$$\begin{tikzcd} & \mathbb{Z}_p[G^{(p)}]^\times \arrow[d] \\ G \arrow[ur] \arrow[r] & A^\times \end{tikzcd}$$

commutes. Hence the ring $\mathbb{Z}_p[G^{(p)}]$ satisfies the universal property and so it is the *universal deformation ring* of the trivial representation.

Remark 2.2.2. If G is an abelian p-group we claim that $\mathbb{Z}_p[G]$ is an object of $\mathcal{C}$.
Since G is abelian it has the decomposition $G = \mathbb{Z}/p^{n_1}\mathbb{Z} \times \ldots \times \mathbb{Z}/p^{n_l}\mathbb{Z}$ with $n_i \leq n_{i+1}$.
We have the identification $\mathbb{Z}_p[G] = \mathbb{Z}_p[x_1, \ldots, x_l]/(x_i^{p^{n_i}} - 1, i = 1, \ldots, l)$. Since $\mathbb{Z}_p[G]$ has characteristic p we have $x_i^{p^{n_i}} - 1 = (x_i - 1)^{p^{n_i}}$ and we can do the change of variables $x_i - 1 = t_i$. Hence we get the equality

$$\mathbb{Z}_p[G] = \mathbb{Z}_p[t_1, \ldots, t_l]/(t_i^{p^{n_i}}, i = 1, \ldots, l)$$

This is a quotient of the formal power series ring $\mathbb{Z}_p[[t_1, \ldots, , t_l]]$. It is local and it is Noetherian by the *Hilbert's basis theorem.*
If G is not a p-group we may have $\mathbb{Z}_p[G]$ not local. For instance take $\mathbb{Z}_p[C_2] = \mathbb{Z}_p[x]/(x^2 - 1)$; this is isomorphic to $\mathbb{Z}_p[t]/(t^2)$ if $p = 2$, hence it is local, and to $\mathbb{Z}_p \times \mathbb{Z}_p$ if $p \neq 2$ hence it is not local.

If G a finite group we will write $G^{(\mathrm{ab})}$ for the biggest quotient of G that is abelian, i.e. $G^{(ab)} = G/[G, G]$.

If we have a representation $\rho\colon G \to \mathbb{F}_p^\times$ we can construct $\rho^{(\mathrm{ab})}\colon G^{(\mathrm{ab})} \to \mathbb{F}_p^\times$, defining $\rho^{(\mathrm{ab})}(\bar{g}) = \rho(g)$, such that $\rho^{(\mathrm{ab})} \circ \pi = \rho$, where $\pi\colon G \to G^{(\mathrm{ab})}$, $\rho^{(\mathrm{ab})}$ is well defined because if $\bar{g} = \bar{h}$ there exist $a, b \in G$ such that $h^{-1}g = [a, b]$, hence $\rho(g) = \rho(h[a, b]) = \rho(h)\rho([a, b]) = \rho(h)$.

Proposition 2.2.3. *Let $\rho\colon G \to \mathbb{F}_p^\times$ be a representation of the finite group G then ρ and $\rho^{(\mathrm{ab})}$ have the same deformation ring, i.e. $R_\rho = R_{\rho^{(\mathrm{ab})}}$.*

Proof. The universal property of the deformation ring tells us that we have $\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{alg}}(R_\rho, A) \cong \mathrm{Def}(\rho, A)$ and $\mathrm{Hom}_{\mathbb{Z}_p-\mathrm{alg}}(R_{\rho^{(\mathrm{ab})}}, A) \cong \mathrm{Def}(\rho^{(\mathrm{ab})}, A)$.
Indeed we have $\mathrm{Def}(\rho, A) \cong \mathrm{Def}(\rho^{(\mathrm{ab})}, A)$.
In fact let A be a lift of $\rho' = \rho^{(\mathrm{ab})}$, we have the commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "\rho"] \arrow[d, "\pi"'] & \mathbb{F}_p^\times \\
G^{(\mathrm{ab})} \arrow[r, "\varphi'"] \arrow[ur, "\rho'"] & A^\times \arrow[u]
\end{tikzcd}$$

from which we understand that A is also a lift for ρ.
Now, we start from a lift of ρ and we define a map φ' such that this diagram commute:

$$\begin{tikzcd}
G^{(ab)} \arrow[r, "\varphi'"] & A^\times \arrow[d] \\
G \arrow[u, "\pi"] \arrow[r, "\rho"] \arrow[ur, "\varphi"] & \mathbb{F}_p^\times
\end{tikzcd}$$

where $\varphi'(\bar{g}) = \varphi(g)$ and it is well defined.
Therefore $\mathrm{Def}(\rho', A) = \mathrm{Def}(\rho, A)$, and $R_\rho = R_{\rho^{(\mathrm{ab})}}$. *q.e.d.*

Thanks to this proposition we can restrict ourselves to studying only representations of abelian groups. Every finite abelian group can be expressed as a direct product of its Sylow subgroups. Hence we have two different cases: G is a p-group and p does not divide the order of G. Let us start with the first.

Proposition 2.2.4. *Let G be an abelian p-group. Then the universal deformation ring of a linear representation of G over $\mathbb{F}_p$ is $\mathbb{Z}_p[G]$.*

Proof. First of all we observe that if $\rho\colon G \to \mathbb{F}_p^\times$ is a representation then ρ is trivial because for every $g \in G$, $\rho(g)^{p^n} = 1$ and since $\mathbb{F}_p$ does not contain element of order a power of p we must have $\rho(g) = 1$.
Thus we obtain the equality $R_\rho = \mathbb{Z}_p[G]$ thanks to Example 2.2.1. *q.e.d.*

Proposition 2.2.5. *Let G be a group whose order is not divisible by p. For every 1-dimensional representation of G over $\mathbb{F}_p$ the set $\mathrm{Def}(\rho, A)$ is of order one.*

Proof. As in the Example 2.2.1 if A is a lift, the multiplicative group is $A^\times = (1 + \mathfrak{m}_A) \times \mu_{p-1}$ and the unique φ is $\varphi(g) = (1, \rho(g))$, since $1 + \mathfrak{m}_A$ is a p-group. *q.e.d.*

Corollary 2.2.6. *Let G be a group whose order is not divisible by p. For every 1-dimensional representation $R_\rho = \mathbb{Z}_p$.*

Proof. We know that $\mathrm{Hom}_{\mathbb{Z}_p-alg}(R_\rho, A) = \mathrm{Def}(\rho, A)$, since it has only one element for every ring A in $\mathcal{C}$, R_ρ is an initial object in the category of $\mathbb{Z}_p$-algebras so it is $\mathbb{Z}_p$. *q.e.d.*

Now we know how to compute the deformation ring for finite abelian p-group and for finite abelian group with no p torsion part. For every group G we know that $G^{(ab)} = G_p \times G_{non-p}$. At this point we would like to have a property that links the deformation ring of a product with the two deformation rings of each representations. If we have the representations of two groups $\rho_1\colon G_1 \to \mathbb{F}_p^\times$ and $\rho_2\colon G_2 \to \mathbb{F}_p^\times$ we can define

$$\rho = \rho_1 \times \rho_2\colon G_1 \times G_2 \to \mathbb{F}_p^\times$$

$\rho(g_1, g_2) = \rho_1(g_1)\rho_2(g_2)$, and it is clear that ρ is an homomorphism of groups.

Proposition 2.2.7. *Let G_1 be an abelian p-group and G_2 be an abelian group whose order is not divisible by p. Let $\rho_1\colon G_1 \to \mathbb{F}_p^\times$ and $\rho_2\colon G_2 \to \mathbb{F}_p^\times$ be group homomorphisms. Consider the representation $\rho_1 \times \rho_2\colon G_1 \times G_2 \to \mathbb{F}_p^\times$. Then $R_{\rho_1\times\rho_2} = R_{\rho_1} \otimes_{\mathbb{Z}_p} R_{\rho_2}$.*

Proof. We have the following functorial isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}_p-alg}(R_1 \otimes R_2, A) &\cong \mathrm{Hom}_{\mathbb{Z}_p-alg}(R_1, A) \times \mathrm{Hom}_{\mathbb{Z}_p-alg}(R_2, A)\\ &\cong \mathrm{Def}(\rho_1, A) \times \mathrm{Def}(\rho_2, A)\\ &\cong \mathrm{Def}(\rho_1 \times \rho_2, A)\\ &\cong \mathrm{Hom}_{\mathbb{Z}_p-alg}(R_{\rho_1\times\rho_2}, A)\end{aligned}$$

The first isomorphism is a classical property, as the third one. The second one and the last one are the definition of the universal deformation ring. *q.e.d.*

Let G be a finite group and ρ a 1-dimensional representation. Then ρ factors through $\rho^{(\mathrm{ab})}$:

$$\begin{tikzcd} G \arrow[r, two heads] \arrow[dr, "\rho"'] & G^{(\mathrm{ab})} \arrow[r, equal] \arrow[d, "\rho^{(\mathrm{ab})}"] & G_p \times G_{\text{non-}p} \arrow[dl, "\rho_p\times\rho_{\text{non-}p}"] \\ & \mathbb{F}_p^\times & \end{tikzcd}$$

and $G^{(\mathrm{ab})}$ is the biggest abelian quotien of G, so it is product of its p-Sylow subgroup. In particular we can write $G^{(\mathrm{ab})} = G_p \times G_{\text{non-}p}$ and we get the equality $\rho^{(\mathrm{ab})} = \rho_p \times \rho_{\text{non-}p}$. We have proved that $R_\rho = R_{\rho^{(\mathrm{ab})}}$, hence thanks to the last proposition $R_{\rho_p\times\rho_{\text{non-}p}} = R_{\rho_p} \otimes_{\mathbb{Z}_p} R_{\rho_{\text{non-}p}}$. Thanks to Proposition 2.2.4 and Corollary 2.2.6 we get $R_{\rho_{\text{non-}p}} = \mathbb{Z}_p$ and $R_{\rho_p} = \mathbb{Z}_p[G_p]$.
Therefore $R_\rho = \mathbb{Z}_p[G_p] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p = \mathbb{Z}_p[G_p]$.

Chapter 3

The deformation ring of $\mathrm{GL}_2(\mathbb{F}_p)$

In this chapter we will compute the deformation ring of the identity representation of $\mathrm{GL}_2(\mathbb{F}_p)$. It turns out that if $p > 3$, the deformation ring is $\mathbb{F}_p$ while for $p = 2, 3$ the deformation ring is $\mathbb{Z}_p$.

3.1 Case $p > 3$

Theorem 3.1.1. *Let p be a prime number greater than 3. The universal deformation ring of the identity representation $\mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$ is $\mathbb{F}_p$.*

Let us start from this.

Proposition 3.1.2. *Let p be a prime greater than 3 and $\rho\colon \mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$ be the identity representation . There is no lift of ρ to $\mathbb{Z}/p^2\mathbb{Z}$.*

Proof. Suppose that there exists such a lift:

$$\begin{tikzcd} & \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) \arrow[d] \\ \mathrm{GL}_2(\mathbb{F}_p) \arrow[ur, "\varphi"] \arrow[r, "\rho"] & \mathrm{GL}_2(\mathbb{F}_p) \end{tikzcd}$$

Then the following short exact sequence splits:

$$1 \longrightarrow (\mathbb{Z}/p\mathbb{Z})^4 \longrightarrow \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z}) \longrightarrow 1$$

where $\ker\big(\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{F}_p)\big) = 1 + M_{2\times 2}(p\mathbb{Z}/p^2\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^4$.
Let $\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, $\varphi(\sigma)$ is a lift of σ and it has to be of the form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} ap & bp \\ cp & dp \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + M \in \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$$

with $0 \leq a, b, c, d \leq p-1$ and clearly $M^2 = 0$ in $M_{2\times 2}(\mathbb{Z}/p^2\mathbb{Z})$.
Since $\sigma^p = 1$ we should have the same for its lift.
Let us consider $M_1 = M + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\begin{aligned} \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + M\right]^p &= (1+M_1)^p \\ &= 1 + pM_1 + \frac{p(p-1)}{2}M_1^2 + \frac{p(p-1)(p-2)}{6}M_1^3 \end{aligned}$$

Since p annihilates M

$$pM_1 = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix}$$

thus $pM_1^2 = 0$, moreover $M_1^2 \equiv 0 \mod p$ that implies $M_1^4 = 0$, finally we get:

$$(\sigma + M)^p = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$$

Therefore the sequence does not split. *q.e.d.*

Now we introduce the following two lemmas in order to complete the computation of the universal deformation ring:

Lemma 3.1.3. *A morphism $B \to A$ in $\mathcal{C}$ is surjective if and only if the induced map from t_B^* to t_A^* is surjective.*

Proof. see [13] Lemma 1.1. *q.e.d.*

Lemma 3.1.4. *Let $\rho\colon G \to \mathrm{GL}_n(k)$ be a representation of G, where k is a field of characteristic $p > 0$, and let H be a subgroup of G. If the index of H in G and p are relatively prime then $R_{\rho_{|_H}} \to R_\rho$ is surjective.*

Proof. Let H be a subgroup of G and let A be the ring $M_{n\times n}(k)$. The groups G and H act on A by conjugation. It is known that the composition:

$$H^1(G, A) \xrightarrow{res} H^1(H, A) \xrightarrow{cor} H^1(G, A)$$

is the multiplication by the index $[G : H]$ (see chapter VII Prop. 6 in [14]). By the hypothesis $\gcd([G : H], \mathrm{char} k) = 1$. So the above map is injective, in particular also the map *res* is injective, thus by the isomorphism (2.1) the associated map between the cotangent space is onto:

$$\mathfrak{m}_{\rho_{|_H}}/(\mathfrak{m}^2_{\rho_{|_H}}, p) \twoheadrightarrow \mathfrak{m}_\rho/(\mathfrak{m}^2_\rho, p)$$

where $\mathfrak{m}_\rho$ and $\mathfrak{m}_{\rho_{|H}}$ are the maximal ideals of R_ρ and $R_{\rho_{|H}}$, respectively.
By the universal property of the universal deformation ring of $\rho_{|H}$ there exists a unique map $\varphi\colon R_{\rho_{|H}} \to R_\rho$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathrm{GL}_n(R_{\rho_{|H}}) \arrow[d, "\varphi"] \\ H \arrow[ur, "(\rho_{|H})^u"] \arrow[r, "(\rho^u)_{|H}"'] & \mathrm{GL}_n(R_\rho) \end{tikzcd}$$

The map φ induces a morphism φ^* between the Zariski tangent spaces and we claim that the following diagram commutes:

$$\begin{tikzcd} t^*_{R_\rho} \arrow[r, "\varphi^*"] \arrow[d] & t^*_{R_{\rho_{|H}}} \arrow[d] \\ H^1(G,A) \arrow[r, "res"] & H^1(H,A) \end{tikzcd}$$

where the vertical arrow are the isomorphisms give by (2.1). Therefore by Lemma 3.1.3, φ is surjective.
It remains to prove the claim. The map φ gives rise to maps $\tilde{\varphi}$ and $\tilde{\varphi}_t$ for which the following diagram:

$$\begin{tikzcd} \mathfrak{m}_H/(\mathfrak{m}_H^2, p) \arrow[r] \arrow[d, "\tilde{\varphi}_t"'] & R_H/(\mathfrak{m}_h^2, p) \arrow[d, "\tilde{\varphi}"] \\ \mathfrak{m}/(\mathfrak{m}^2, p) \arrow[r] & R/(\mathfrak{m}^2, p) \end{tikzcd}$$

commutes, where we denote by R and R_H the universal deformation ring of ρ and $\rho_{|H}$, respectively. By definition of $\tilde{\varphi}$ also the following diagram commutes:

$$\begin{tikzcd} R_H \arrow[r] \arrow[d, "\varphi"] & R_H/(\mathfrak{m}_H^2, p) \arrow[d, "\tilde{\varphi}"] \\ R \arrow[r] & R/(\mathfrak{m}^2, p) \end{tikzcd}$$

and now thanks to Remark 2.1.9 it is easy to see that the claim is true. In fact, using the notation of the remark, if ψ is an element of the Zariski tangent space of R, i.e. $\psi\colon \mathfrak{m}/(\mathfrak{m}^2, p) \to k$ then $\varphi^*(\psi) = \tilde{\varphi}_t \circ \psi$ and the image of this map in $\mathrm{Def}(\rho_{|H}, k[\epsilon])$ is

$$\begin{aligned} (\psi \circ \varphi)^\sharp \circ (\rho_{|H})^u &= \psi^\sharp \circ \varphi \circ (\rho_{|H})^u \\ &= \psi^\sharp \circ (\rho^u)_{|H} \end{aligned}$$

and this deformation is sent to $[\sigma_{|H}] \in H^1(H, M_{2\times 2}(k))$, that is the image of $[\sigma] \in H^1(G, M_{2\times 2}(k))$ via *res*. *q.e.d.*

Consider these subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$:

$$T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \leq B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \leq \mathrm{GL}_2(\mathbb{F}_p)$$

we will show that $R_{\rho_{|_T}} = \mathbb{Z}_p, R_{\rho_{|_B}} = \mathbb{F}_p$.

Proposition 3.1.5. *For every prime number p, the universal deformation ring of the the natural representation $T \to \mathrm{GL}_2(\mathbb{F}_p)$ is $\mathbb{Z}_p$.*

Proof. The group T acts diagonally on $\mathbb{F}_p^2$ so we have the decomposition in eigenspaces: $\mathbb{F}_p^2 = V(\chi_1) \oplus V(\chi_2)$, each of dimension 1, where $T \underset{\chi_2}{\overset{\chi_1}{\rightrightarrows}} \mu_{p-1}$ are two characters.
Let A be a ring of $\mathcal{C}$ and W be a free A-module of rank 2 with a group homomorphism $T \to \mathrm{GL}_A(W)$ that lifts the representation:

$$\begin{tikzcd} & \mathrm{GL}_A(W) \arrow[d] \\ T \arrow[ur] \arrow[r] & \mathrm{GL}_2(\mathbb{F}_p) \end{tikzcd}$$

by definition of lift, $W \otimes_A \mathbb{F}_p \cong_{\mathbb{F}_p[G]} \mathbb{F}_p^2$.
Now, take M a $\mathbb{Z}_p[T]$-module. Since

$$\mathbb{Z}_p[T] = \prod_{\chi \in T^\vee} \mathbb{Z}_p$$

where $T^\vee = \mathrm{Hom}_{\mathbf{Grp}}(T, \mu_{p-1})$ is the set of all characters of T, we get the decomposition of the module in direct sum:

$$M = \oplus_{\chi \in T^\vee} M(\chi)$$

Also W decomposes in the same way:

$$W = \oplus_{\chi \in T^\vee} W(\chi) = W(\chi_1) \oplus W(\chi_2)$$

because

$$\dim W(\chi) = \begin{cases} 1 & \text{if } \chi = \chi_1 \text{ or } \chi_2 \\ 0 & \text{otherwise} \end{cases}$$

Thus there is only one deformation of the representation $T \to \mathrm{GL}_2(\mathbb{F}_p)$ to the ring A, for every ring A in $\mathcal{C}$. Therefore the universal deformation ring is $\mathbb{Z}_p$. *q.e.d.*

Although neither the representation $T \to \mathrm{GL}_2(\mathbb{F}_p)$ is absolutely irreducible nor it holds that $\mathrm{End}_{\mathbb{F}_p[T]}(V) = \mathbb{F}_p$, we have found that it has the universal deformation ring $\mathbb{Z}_p$.

Remark 3.1.6. The previous proposition implies that $\mathrm{Hom}_{\mathbb{Z}_p-alg}(R_{\rho_{|T}}, A)$ is trivial for every lift A of $\rho_{|T}$. In particular from the isomorphisms (2.1) $\mathrm{Def}(\rho_{|T}, \mathbb{F}_p[\epsilon])$ and $H^1(T, M_{2\times 2}(\mathbb{F}_p))$ are sets of one element.

Since $[\mathrm{GL}_2(\mathbb{F}_p) : B] = p+1$ we can apply Lemma 3.1.4. Let us study the deformation ring of the subgroup $B \leq \mathrm{GL}_2(\mathbb{F}_p)$.

Proposition 3.1.7. *If p is a prime greater than 3, the universal deformation ring of the natural representation $\rho\colon B \to \mathrm{GL}_2(\mathbb{F}_p)$ is $\mathbb{F}_p$.*

Proof. Take $\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let C be the group generated by σ. We have that C is a normal subgroup in B. The group T acts on it by conjugation, let $t \in T, \sigma^a \in C$:

$$tat^{-1} = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & at_1t_2^{-1} \\ 0 & 1 \end{pmatrix}$$

For any $h \in \mathbb{Z}$ such that $h \equiv \chi_1\chi_2^{-1}(t) \mod p$ we get $tat^{-1} = \sigma^h$. Thus $C = C(\chi_1\chi_2^{-1})$ and C is a $\mathbb{F}_p[T]$-module of dimension 1 over $\mathbb{F}_p$.
Let A be a ring in the category $\mathcal{C}$ and take an homomorphism $B \to \mathrm{GL}_2(A)$ that lifts the representation ρ. We can consider the representation $T \to \mathrm{GL}_A(\mathrm{End}(A^2))$, this is semisimple and

$$M_{2\times 2}(A) = \mathrm{End}(A^2) = A \oplus A \oplus A \oplus A$$

where the first two characters are 1 and last two are $\chi_1\chi_2^{-1}, \chi_1^{-1}\chi_2$, i.e.:

$$t\begin{pmatrix} a & b \\ c & d \end{pmatrix}t^{-1} = \begin{pmatrix} a & \tilde{\varphi}(\chi_1\chi_2^{-1}(t))b \\ \tilde{\varphi}(\chi_1^{-1}\chi_2(t))c & d \end{pmatrix}$$

where $\tilde{\varphi}\colon \mathbb{F}_p^\times \to A^\times$ is the composition of the Teichmuller lift $\mathbb{F}_p^\times \to \mathbb{Z}_p^\times$ and the natural map $\mathbb{Z}_p \to A$. Let W a free A-module of rank 2 together with a group homomorphism $\tilde{\rho}\colon B \to \mathrm{GL}_A(W)$ that is a lift of ρ, $\tilde{\rho}(\sigma) = \tilde{\sigma}$. For any $h \in \mathbb{Z}$ congruent to $\chi_1\chi_2^{-1}(t)$ modulo p we have $t\sigma t^{-1} = \sigma^{\chi_1\chi_2^{-1}(t)}$, for every t. Thus for all $n \in \mathbb{Z}$ congruent to $\chi_1\chi_2^{-1}(t)$ modulo p we have $\tilde{\rho}(t)\tilde{\sigma}\tilde{\rho}(t)^{-1} = \tilde{\sigma}^n$.
If $p > 3$ we can take $t, t' \in T$ such that $\chi_1\chi_2^{-1}(t) = -1$ and $\chi_1\chi_2^{-1}(t') \equiv 2 \mod p$. For t we get:

$$\begin{aligned} \tilde{\rho}(t)\tilde{\sigma}\tilde{\rho}(t)^{-1} = \tilde{\rho}(t)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\tilde{\rho}(t)^{-1} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \\ \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \\ \begin{pmatrix} a^2-bc & (a-d)b \\ (d-a)c & d^2-bc \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Since $b \in 1 + \mathfrak{m}_A$ we get the following system:

$$\begin{cases} a^2 - bc &= 1 \\ a &= d \end{cases}$$

For t':

$$\begin{aligned} \tilde{\rho}(t') \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tilde{\rho}(t')^{-1} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \\ \begin{pmatrix} a & \tilde{2}b \\ \tilde{2}^{-1}c & d \end{pmatrix} &= \begin{pmatrix} a^2 + bc & (a+d)b \\ (a+d)c & d^2 + bc \end{pmatrix} \end{aligned}$$

where $\tilde{2}$ is the image via $\tilde{\varphi}$ of $\chi_1 \chi_2^{-1}(t') = 2$.
Hence since $b \in 1 + \mathfrak{m}_A$ we get the following:

$$\begin{cases} a^2 + bc &= a \\ a + d &= \tilde{2} \\ (a+d)c &= \tilde{2}^{-1}c \end{cases}$$

From the last two equations we get $(\tilde{2} - \tilde{2}^{-1})c = 0$ in A, and since $p > 3$ we get $c = 0$, finally we have:

$$\begin{cases} a &= 1 \\ d &= 1 \\ c &= 0 \end{cases}$$

Therefore $\tilde{\sigma}$ is:

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

Moreover $\tilde{\sigma}$ raised to the power p has to be 1 so $pb = 0$, that means $p = 0$ in A.
All choices of b give equivalent lifts in the sense that we can find a matrix K in $\ker(\mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{F}_p))$ for which:

$$K \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} K^{-1} = \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix}$$

where $K = \begin{pmatrix} 1+x & y \\ 0 & 1+z \end{pmatrix}$, with $(x+1)b = b'(1+z)$.
Thus the set $\mathrm{Def}(\rho, A)$ has at most 1 element. Hence the universal deformation ring of ρ is a quotient of $\mathbb{Z}_p$. In the proof of the Proposition 3.1.2 we saw that there is no lift of σ to $\mathbb{Z}/p^2\mathbb{Z}$, therefore the universal deformation ring of ρ is $R_\rho = \mathbb{F}_p$. *q.e.d.*

Proof of Theorem 3.1.1. Thanks to Lemma 3.1.4 there exists a surjective homomorphism

$$R_{\rho_{|B}} \twoheadrightarrow R_\rho$$

therefore $R_\rho = \mathbb{F}_p$. *q.e.d.*

3.2 Other cases

We want to show that the universal deformation ring of the identity representation $\rho\colon \mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{F}_p)$ is $R_\rho = \mathbb{Z}_p$ when $p = 2$ or $p = 3$.

3.2.1 $p = 2$

Let A be an object of the category $\mathcal{C}$. We know that $\mathrm{GL}_2(\mathbb{F}_2) \cong S_3$. Let us consider the group ring $A[S_3]$; this is isomorphic to $A[S_3/A_3] \times M_{2\times 2}(A)$. Let M be an $A[S_3]$-module free of rank 2 over A, then $M = M_1 \oplus M_2$, where M_1 is an $A[S_3/A_3]$-module and M_2 is an $M_{2\times 2}(A)$-module. Since M is an A-module free of rank 2 we have that M_1, M_2 are free modules of rank:

- $\operatorname{rank} M_1 = 2$, $\operatorname{rank} M_2 = 0$;
- $\operatorname{rank} M_1 = 1$, $\operatorname{rank} M_2 = 1$;
- $\operatorname{rank} M_1 = 0$, $\operatorname{rank} M_2 = 2$;

In the first two cases the commutator subgroup $A_3 = [S_3, S_3]$, which is a cyclic group of order three, acts trivially on M_1 so we can only have the third one. Hence if M is an $A[S_3]$-module then M is a $M_{2\times 2}(A)$-module free of rank 2 over A.
It is known that given a ring A we construct the ring of $n \times n$ matrices on A and they are Morita equivalent, in the sense of Definition 2.2.2 of [2], so by Proposition 2.2.5 of [2] we have an equivalence of abelian categories between A-**Mod** and **Mod**-$M_{n\times n}(A)$.
Hence in our case we get that the $M_{2\times 2}(A)$-module M is the standard module A^2. Therefore for every A there is only 1 deformation, i.e. $\mathrm{Def}(\rho, A)$ has one element, and so the universal deformation ring is $\mathbb{Z}_p$.

3.2.2 $p = 3$

Proposition 3.2.1. *For the representation $\rho\colon \mathrm{GL}_2(\mathbb{F}_3) \to \mathrm{GL}_2(\mathbb{F}_3)$, the set $\mathrm{Def}(\rho, \mathbb{F}_3[\epsilon])$ has one element.*

Proof. Let φ be a lift of ρ to $\mathbb{F}_3[\epsilon]$:

$$\begin{tikzcd} & \mathrm{GL}_2(\mathbb{F}_3[\epsilon]) \arrow[d] \\ \mathrm{GL}_2(\mathbb{F}_3) \arrow[ur, "\varphi"] \arrow[r, "\rho"] & \mathrm{GL}_2(\mathbb{F}_3) \end{tikzcd}$$

Let us call $\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then

$$\varphi(\sigma) = 1 + M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a\epsilon & 1 + b\epsilon \\ c\epsilon & d\epsilon \end{pmatrix}$$

Since $\sigma^3 = 1$ we get $(1+M)^3 = 1 + M^3 = 1$ which implies $c = 0$, because

$$M^3 = \begin{pmatrix} 0 & c\epsilon \\ 0 & 0 \end{pmatrix}$$

Now, let $t \in T$, with T the subgroup of $\mathrm{GL}_2(\mathbb{F}_3)$ consisting of diagonal matrices. Since $t\sigma t^{-1} = \sigma^l$ with $l \in \mathbb{Z}$ congruent to $\chi_1\chi_2^{-1}(t) \mod p$, we have that for every $h \in \mathbb{Z}$ that are congruent to $\chi_1\chi_2^{-1}(t)$ modulo p:

$$t(1+M)t^{-1} = (1+M)^h$$

We can take $t \in T$ for which $\chi_1\chi_2^{-1}(t) \equiv 2 \mod p$, then $t(1+M)t^{-1} = (1+M)^2$ that is:

$$\begin{aligned} \begin{pmatrix} 1+a\epsilon & 2(1+b\epsilon) \\ 0 & 1+d\epsilon \end{pmatrix} &= \begin{pmatrix} 1+a\epsilon & 1+b\epsilon \\ 0 & 1+d\epsilon \end{pmatrix}^2 \\ &= \begin{pmatrix} 1+2a\epsilon & 2+(2b+a+d)\epsilon \\ 0 & 1+2d\epsilon \end{pmatrix} \end{aligned}$$

hence $a = d = 0$ and:

$$\varphi(\sigma) = \begin{pmatrix} 1 & 1+b\epsilon \\ 0 & 1 \end{pmatrix}$$

If $b \neq 0$ then the lift φ is equivalent to the lift $\tilde{\rho}\colon \mathrm{GL}(\mathbb{F}_3) \to \mathrm{GL}(\mathbb{F}_3[\epsilon])$ for which $b = 0$:

$$\tilde{\rho}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in fact we can consider the matrix:

$$K = \begin{pmatrix} 1+x\epsilon & 0 \\ 0 & 1+y\epsilon \end{pmatrix} \in 1 + M_{2\times 2}(\mathbb{F}_p[\epsilon]) = \ker\big(\mathrm{GL}_2(\mathbb{F}_3[\epsilon]) \to \mathrm{GL}_2(\mathbb{F}_3)\big)$$

such that $y - x = b$ and it holds:

$$\begin{aligned} K\begin{pmatrix} 1 & 1+b\epsilon \\ 0 & 1 \end{pmatrix}K^{-1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ K\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}K^{-1} &= \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \end{aligned}$$

Therefore there is only one deformation to $\mathbb{F}_3[\epsilon]$. *q.e.d.*

Thanks the isomorphisms (2.1) the universal deformation ring is a quotient of $\mathbb{Z}_3$. In order to show that it is in fact $\mathbb{Z}_3$ we will construct a lift to $\mathbb{Z}_3$.

Explicit construction of a lift to $\mathbb{Z}_3$ of ρ

First let observe that the "dihedral group" $D_8 = \langle\zeta\rangle \rtimes \langle\sigma\rangle$, with $\zeta^8 = 1$, $\sigma^2 = 1$, $\sigma\zeta\sigma = \zeta^{-1}$, contains the quaternion group Q. In fact Q is isomorphic to

$$\{1, -1, \zeta^2, \zeta^6, \zeta\sigma, \zeta^3\sigma, \zeta^5\sigma, \zeta^7\sigma\}$$

under the identification $i \mapsto \zeta^2, j \mapsto \zeta^3\sigma, k \mapsto \zeta^5\sigma$.

Let ζ be an 8-th root of unity and consider the quadratic extension $\mathbb{Z}_9 = \mathbb{Z}_3[\zeta]$ with $\zeta = (1-i)r$, where $r \in \mathbb{Z}_3$, $r^2 = -\frac{1}{2}$ and $r \equiv 1 \mod 3$. The Frobenius automorphism σ acts on $\mathbb{Z}_9$ and it sends $\zeta \mapsto \zeta^3$, so let us take the twisted group algebra $\mathbb{Z}_9[\sigma]$. In fact we have that $\mathbb{Z}_9[\sigma] = \mathrm{End}_{\mathbb{Z}_3}(\mathbb{Z}_9)$, because $\mathbb{Z}_9 \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \cdot \zeta$. Thus we can view $\langle\zeta, \sigma\rangle$ as a subgroup of $\mathbb{Z}_9[\sigma]^\times$ and we get a lift of some representation of the dihedral group D_8 over $\mathbb{F}_3$ to $\mathbb{Z}_3$:

$$\begin{tikzcd} \langle\zeta,\sigma\rangle = D_8 \arrow[r] \arrow[dr] & \mathrm{GL}_2(\mathbb{Z}_3) \arrow[d] \\ & \mathrm{GL}_2(\mathbb{F}_3) \end{tikzcd}$$

We claim that there is an element $\rho = a + b\sigma \in \mathbb{Z}_9[\sigma]^\times$ of order 3, such that $S_3 = \langle\sigma, \rho\rangle$ and such that ρ acts by conjugation on the quaternion group:

$$\rho\colon i \mapsto j \mapsto k \mapsto i \tag{3.1}$$

for which $\langle\sigma, \rho, i, j\rangle = Q \rtimes S_3 \subseteq \mathbb{Z}_9[\sigma]^\times$. In order to construct ρ it is enough to consider the following system:

$$\begin{cases} \rho^2 + \rho + 1 &= 0 \\ \sigma\rho\sigma &= \rho^2 \\ \rho i \rho^2 &= \zeta^3\sigma \end{cases}$$

We find $\rho = -\zeta^7 r - ri\sigma$. Hence we get the following diagram, where we write $\mathrm{GL}_2(\mathbb{Z}_3) = \mathbb{Z}_9[\sigma]^\times, \mathrm{GL}_2(\mathbb{F}_3) = \mathbb{F}_3[\sigma]^\times$:

$$\begin{tikzcd} & \mathrm{GL}_2(\mathbb{Z}_3) \arrow[d] \\ Q \rtimes S_3 = \langle\sigma,\rho,i,j\rangle \arrow[ur] \arrow[r] & \mathrm{GL}_2(\mathbb{F}_3) \end{tikzcd}$$

where in $Q \rtimes S_3$ the action of ρ on Q is given by (3.1), while σ acts by conjugation as:

$$\begin{array}{rccc} \sigma\colon & i & \mapsto & i^{-1} \\ & j & \mapsto & k^{-1} \\ & k & \mapsto & j^{-1} \end{array}$$

The images of the generators in $\mathrm{GL}_2(\mathbb{Z}_3)$ are:

$$\begin{aligned} \sigma &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \rho \mapsto \begin{pmatrix} -\frac{1}{2} & -r+\frac{1}{2} \\ -r-\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \\ i &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ j \mapsto \begin{pmatrix} r & r \\ r & -r \end{pmatrix} \end{aligned}$$

and in $\mathrm{GL}_2(\mathbb{Z}_3)$:

$$\begin{aligned} \sigma &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \rho \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ i &\mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ j \mapsto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

The images of Q and S_3 in $\mathrm{GL}_2(\mathbb{F}_3)$ are:

$$\begin{aligned} \tilde{Q} &= \left\{ \pm 1, \pm \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \pm \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\} \\ \tilde{S_3} &= \left\{ 1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \right\} \end{aligned}$$

The fact that $\tilde{Q} \cap \tilde{S_3} = \{1\}$ implies $|\tilde{Q}\tilde{S_3}| = |\tilde{Q}||\tilde{S_3}| = |\mathrm{GL}_2(\mathbb{F}_3)|$ thus $\mathrm{GL}_2(\mathbb{F}_3) = \tilde{Q}\tilde{S_3}$ and:

$$\mathrm{GL}_2(\mathbb{F}_3) = \tilde{Q} \rtimes \tilde{S_3} \cong Q \rtimes S_3$$

Thus we get a lift of $\mathrm{GL}_2(\mathbb{F}_3) \to \mathrm{GL}_2(\mathbb{F}_3)$ to $\mathbb{Z}_3$:

$$\begin{tikzcd} & \mathrm{GL}_2(\mathbb{Z}_3) \arrow[d] \\ \mathrm{GL}_2(\mathbb{F}_3) \arrow[ur] \arrow[r, "\rho"] & \mathrm{GL}_2(\mathbb{F}_3) \end{tikzcd}$$

hence the universal deformation ring of ρ is $\mathbb{Z}_3$.

Chapter 4

Example of groups whose deformation rings are $\mathbb{Z}/p^n\mathbb{Z}$

4.1 Introduction

Let define $\mu_{p-1} = \{x \in \mathbb{Z}_p : x^{p-1} = 1\}$, the set of the $(p-1)$-th roots of unity of $\mathbb{Z}_p$, that is isomorphic to $\mathbb{F}_p^\times$. Let $n \geq 1$. Since $\mu_{p-1} \subseteq \mathbb{Z}_p^\times$ we have the following maps:

$$\begin{tikzcd} \mu_{p-1} \arrow[r] \arrow[dr] & \mathbb{Z}_p^\times \arrow[d] \\ & (\mathbb{Z}/p^n\mathbb{Z})^\times \end{tikzcd}$$

where the map $\mu_{p-1} \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ is injective, hence the group

$$G = \begin{pmatrix} \mu_{p-1} & \mathbb{Z}/p^n\mathbb{Z} \\ 0 & \mu_{p-1} \end{pmatrix}$$

can be viewed as a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$.

We have a natural 2-dimensional represention over $\mathbb{F}_p$:

$$\bar{\rho} \colon G \to \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{F}_p)$$

where the first arrow is the inclusio and the second one is the natural projection of all the entries of the matrix.
For this representation we have the following property:

Proposition 4.1.1. *Let p be prime greater than 3. For the representation $\bar{\rho}$ defined above we have* $\mathrm{End}_G(\bar{\rho}) = \mathrm{End}_{\mathbb{F}_p[G]}(\mathbb{F}_p^2) = \mathbb{F}_p$.

Proof. Let $\varphi\colon \mathbb{F}_p^2 \to \mathbb{F}_p^2$ be an $\mathbb{F}_p[G]$-linear map. The image of the basis vectors of $\mathbb{F}_p^2$ via φ are:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} c \\ d \end{pmatrix}$$

Since φ is G-linear we have:

$$\varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$
$$\varphi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

from which we deduce $a = d$ and $b = 0$. Acting with a diagonal matrix we get:

$$\varphi\left(\begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}$$

and $c = 0$. Thus we conclude that φ is the multiplication by a nonzero element of $\mathbb{F}_p$. *q.e.d.*

Thanks to Theorem 2.1.5, we get that the universal deformation ring exists.

Theorem 4.1.2. *Let p be a prime greater than 3. Then the universal deformation ring of the representation $\bar{\rho}$ is $\mathbb{Z}/p^n\mathbb{Z}$.*

4.2 Proof of Theorem 4.1.2

Proposition 4.2.1. *Let p be a prime greater than 3. Then for the group G and the representation $\bar{\rho}$ defined above $\mathrm{Def}(\bar{\rho}, \mathbb{F}_p[\epsilon])$ has one element.*

Proof. Let φ be a lift of ρ to $A = \mathbb{F}_p[\epsilon]$:

$$\begin{tikzcd} & \mathrm{GL}_2(A) \arrow[d] \\ G \arrow[ur, "\varphi"] \arrow[r] & \mathrm{GL}_2(\mathbb{F}_p) \end{tikzcd}$$

Let T be the torus, subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$. The group T, acts on A^2 by the characters χ_1, χ_2 and under the action of T we have the decomposition $A = A_1 \oplus A_2$, with both A_1, A_2 free of rank 1 over A.
Let us call $L \in \mathrm{GL}_2(A)$ the image via φ of:

$$\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$$

First we should show that in fact there exists a lift of σ, i.e. that $L^p = 1$. Since $L \equiv \sigma \mod \epsilon$ we can write:

$$L = \begin{pmatrix} 1 + a\epsilon & 1 + b\epsilon \\ c\epsilon & 1 + d\epsilon \end{pmatrix}$$

Thus:

$$L^p = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a\epsilon & 1 + b\epsilon \\ c\epsilon & d\epsilon \end{pmatrix} \right]^p = (1 + M)^p = 1 + M^p$$

Since $M^2 \in \epsilon \cdot M_{2\times 2}(\mathbb{F}_p)$ we get $M^4 = 0$, and since $p > 3$ we have $L^p = 1$. Now, consider $t \in T$. If $h \in \mathbb{Z}$ such that $h \equiv \chi_1 \chi_2^{-1}(t) \mod p$ we get:

$$t \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^h$$

so the same we must have for L in $\mathrm{GL}_2(A)$

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} L \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}^{-1} = L^h$$

We can take $t \in T$ for which $\chi_1 \chi_2^{-1}(t) \equiv 2 \mod p$, so $tLt^{-1} = L^2$ is:

$$\begin{pmatrix} 1 + a\epsilon & 2(1 + b\epsilon) \\ 2^{-1}c\epsilon & 1 + d\epsilon \end{pmatrix} = \begin{pmatrix} 1 + (2a + c)\epsilon & 2 + (a + 2b + d)\epsilon \\ 2c\epsilon & 1 + (c + 2d)\epsilon \end{pmatrix}$$

so we get the following system:

$$\begin{cases} a + c &= 0 \\ a + d &= 0 \\ d + c &= 0 \\ 2^{-1}c &= 2c \end{cases}$$

thus $a = c = d = 0$ and:

$$L = \begin{pmatrix} 1 & 1 + b\epsilon \\ 0 & 1 \end{pmatrix}$$

Observe that if $b \neq 0$ the lift $\varphi\colon \mathrm{GL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{F}_p[\epsilon])$ is equivalent to the lift $\tilde{\rho}$:

$$\tilde{\rho} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in fact $\tilde{\rho}(g) = K\varphi(g)K^{-1}$ for every $g \in \mathrm{GL}_2(\mathbb{F}_p)$ for

$$K = \begin{pmatrix} 1 + x\epsilon & 0 \\ 0 & 1 + y\epsilon \end{pmatrix} \in \ker\bigl(\mathrm{GL}_2(\mathbb{F}_p[\epsilon]) \to \mathrm{GL}_2(\mathbb{F}_p)\bigr)$$

with $y - x = b$.
Therefore there is only one deformation to the ring $\mathbb{F}_p[\epsilon]$. *q.e.d.*

In particular thanks to the isomorphisms (2.1) $H^1(G, M_2(\mathbb{F}_p))$ is trivial and $\mathrm{Hom}_{\mathbb{Z}_p-alg}(R_\rho, \mathbb{F}_p[\epsilon])$ is a set of one element. Therefore the universal deformation ring is a quotient of $\mathbb{Z}_p$.

Proposition 4.2.2. *There is no lift to $\mathbb{Z}/p^{n+1}\mathbb{Z}$.*

Proof. Suppose that we can lift ρ to $\mathbb{Z}/p^{n+1}\mathbb{Z}$. Let us call $\tilde{\sigma}$ the lift of σ. Since we can consider the standard lift to $\mathbb{Z}/p^n\mathbb{Z}$, and this is the only deformation over $\mathbb{Z}/p^n\mathbb{Z}$ we can write $\tilde{\sigma}$ in the following form:

$$\tilde{\sigma} = \begin{pmatrix} 1+a & 1+b \\ c & 1+d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z})$$

where a, b, c, d are in $p^n\mathbb{Z}/p^{n+1}\mathbb{Z}$. Let us compute the p^n-th power of $\tilde{\sigma}$:

$$\begin{aligned} \tilde{\sigma}^{p^n} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & 1+b \\ c & d \end{pmatrix}\right]^{p^n} &= (1+M)^{p^n} \\ &= 1 + p^n M + \frac{p^n(p^n-1)}{2} M^2 \\ &+ \frac{p^n(p^n-1)(p^n-2)}{6} M^3 \end{aligned}$$

since all the entries of M^2 are divisible by p^n we get $M^4 = 0$ and $pM^2 = 0$. Finally we have $\tilde{\sigma}^{p^n} = 1 + p^n M \neq 1$, hence there is no lift to $\mathbb{Z}/p^{n+1}\mathbb{Z}$. *q.e.d.*

Proof of Theorem 4.1.2. By Proposition 4.2.1 the universal deformation ring is a quotient of $\mathbb{Z}_p$, by Proposition 4.2.2 we cannot lift ρ further than $\mathbb{Z}/p^n\mathbb{Z}$. Therefore the universal deformation ring is $\mathbb{Z}/p^n\mathbb{Z}$ and the universal deformation is the natural homomorphism:

$$\rho^u \colon G = \begin{pmatrix} \mu_{p-1} & \mathbb{Z}/p^n\mathbb{Z} \\ 0 & \mu_{p-1} \end{pmatrix} \to \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$$

q.e.d.

Chapter 5

Main result

5.1 Introduction

Let p be a prime number. Take the group $G = \begin{pmatrix} \mu_{p-1} & \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z} \\ 0 & \mu_{p-1} \end{pmatrix}$, $1 \leq m \leq n$ and the representation:

$$\begin{array}{cccc} \bar{\rho}\colon & G & \to & \mathrm{GL}_2(\mathbb{F}_p) \\ & \begin{pmatrix} x & (a,b) \\ 0 & y \end{pmatrix} & \mapsto & \begin{pmatrix} x & (a \mod p) \\ 0 & y \end{pmatrix} \end{array}$$

Remark 5.1.1. It is known that $\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + \mathfrak{m}_{\mathbb{Z}_p})$, so we have a natural projection $\pi\colon \mathbb{Z}_p^\times \to \mu_{p-1}$, let $\chi_1, \chi_2\colon \mathbb{Z}_p^\times \times \mathbb{Z}_p^\times \to \mu_{p-1}$ defined as $\chi_i = \pi \circ \pi_i$ and π_i is the projection on the i-th component.
Let M be a $\mathbb{Z}_p$-module. Let us consider the group $G_M = M \rtimes (\mu_{p-1} \times \mu_{p-1})$, where the action is defined as $\chi_1\chi_2^{-1}$, i.e. if $t = (t_1, t_2) \in \mu_{p-1} \times \mu_{p-1}$ and $m \in M$ then $tmt^{-1} = \chi_1\chi_2^{-1}(t)m$. It is easy to see that the group G_M can be viewed as the matrix group:

$$G_M = \begin{pmatrix} \mu_{p-1} & M \\ 0 & \mu_{p-1} \end{pmatrix}$$

Thus in our case $M = \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z}$ and the group G_M is

$$(\mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z}) \rtimes (\mu_{p-1} \times \mu_{p-1})$$

Also in this case Proposition 4.1.1 holds and the universal deformation ring exists.

Theorem 5.1.2. *Let p be a prime greater than 3. Then the universal deformation ring of the representation $\bar{\rho}$ is $\mathbb{Z}_p[[t]]/(p^n, p^m t)$.*

Thanks to the homomorphism $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^m\mathbb{Z}$ for which $a \mapsto (a, 0)$, the group studied in the previous chapter is a subgroup of G that we will denote as G_0, thus $\bar{\rho}_{|G_0}$ is the representations studied before.

5.2 Proof of Theorem 5.1.2

First, let us show that there is more than one deformation to the ring $k[\epsilon]$.

Proposition 5.2.1. *The set* $\mathrm{Def}(\bar{\rho}, \mathbb{F}_p[\epsilon])$ *has more than one element.*

Proof. It is enough to show that we can find at least two different deformation: we can take ρ_1, ρ_2 defined as follows:

$$\rho_1 \begin{pmatrix} \zeta_1 & (a,b) \\ 0 & \zeta_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 & a \mod p \\ 0 & \zeta_2 \end{pmatrix}$$
$$\rho_2 \begin{pmatrix} \zeta_1 & (a,b) \\ 0 & \zeta_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 & a \mod p + b\epsilon \\ 0 & \zeta_2 \end{pmatrix}$$

of course $\pi \circ \rho_1 = \pi \circ \rho_2 = \bar{\rho}$, where $\pi \colon \mathbb{F}_p[\epsilon] \to \mathbb{F}_p$ is the natural projection. Hence they are elements of $\mathrm{Def}(\bar{\rho}, \mathbb{F}_p[\epsilon])$ and of course they are not the same because the images in $\mathrm{GL}_2(\mathbb{F}_p[\epsilon])$ of

$$\sigma' = \begin{pmatrix} 1 & (0,1) \\ 0 & 1 \end{pmatrix}$$

are not conjugate, in fact σ' acts trivially via ρ_1 while it does not via ρ_2.
q.e.d.

Theorem 5.2.2. *The ring* $\mathbb{Z}_p[[t]]/(p^n, p^m t)$ *is the versal deformation ring of the representation* $\bar{\rho}$ *defined above.*

Proof. Let us call

$$\sigma = \begin{pmatrix} 1 & (1,0) \\ 0 & 1 \end{pmatrix}, \quad \sigma' = \begin{pmatrix} 1 & (0,1) \\ 0 & 1 \end{pmatrix}$$

Let ρ be a lift of $\bar{\rho}$ to a ring A in the category $\mathcal{C}$:

$$\begin{tikzcd} & \mathrm{GL}_2(A) \arrow[d] \\ G \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}"] & \mathrm{GL}_2(\mathbb{F}_p) \end{tikzcd}$$

Since we know that the universal deformation ring of G_0 is $\mathbb{Z}/p^n\mathbb{Z}$ we get that p^n has to annihilate A, and that the image of σ via ρ, after suitable choice of basis, is:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

In G, the elements σ and σ' commute so the same must hold for their images via ρ. The computation given in the proof of Proposition 4.1.1 shows the image of σ' to be of the form:

$$\rho(\sigma') = \begin{pmatrix} a & t \\ 0 & a \end{pmatrix}$$

with $a \in 1 + \mathfrak{m}_A$ and $t \in \mathfrak{m}_A$. We also know that $[\rho(\sigma')]^{p^m} = 1$.
It is easy to show that a must be 1. In fact, let us consider

$$s \in T = \begin{pmatrix} \mu_{p-1} & 0 \\ 0 & \mu_{p-1} \end{pmatrix} \leq \mathrm{GL}_2(\mathbb{F}_p)$$

and let us call

$$\rho(s) = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \in \mathrm{GL}_2(A)$$

As usual the torus acts by conjugation and ρ respects its action, thus:

$$\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} a & t \\ 0 & a \end{pmatrix} \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}^{-1} = \begin{pmatrix} a & \chi_1\chi_2^{-1}(s)t \\ 0 & a \end{pmatrix}$$

For every $h \in \mathbb{Z}$ such that $h \equiv \chi_1\chi_2^{-1}(s) \mod p^m$ we have $\rho(s\sigma' s^{-1}) = \rho(\sigma'^h) = \rho(\sigma)^h$ that is:

$$\begin{aligned} \begin{pmatrix} a & \chi_1\chi_2^{-1}(s)t \\ 0 & a \end{pmatrix} &= \begin{pmatrix} a & t \\ 0 & a \end{pmatrix}^h \\ &= \begin{pmatrix} a^h & * \\ 0 & a^h \end{pmatrix} \end{aligned}$$

We find $a^h = a$ and taking $s \in T$ for which $h = -1$ we get $a^2 - 1 = 0$, since $a + 1$ is invertible we get $a = 1$.
Therefore the only possibilities for the image of σ' is

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

where t lies in the maximal ideal $\mathfrak{m}_A$ and $p^m t = 0$ since $\rho(\sigma')^{p^m} = 1$.
We define the lift $\tilde{\rho}\colon G \to \mathrm{GL}_2(\mathbb{Z}_p[[T]]/(p^n, p^m T))$ as:

$$\begin{aligned} \tilde{\rho}(\sigma) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ \tilde{\rho}(\sigma') &= \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Therefore the map:

$$\begin{aligned} \tilde{\rho}_A^*\colon \mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(R, A) &\to \mathrm{Def}(\bar{\rho}, A) \\ f &\mapsto [f \circ \tilde{\rho}] \end{aligned}$$

is surjective, where $R = \mathbb{Z}_p[[T]]/(p^n, p^m T)$. If $A = \mathbb{F}_p[\epsilon]$, then $\mathrm{Hom}_{\mathbb{Z}_p\text{-alg}}(R, A)$ is one-dimensional and $\tilde{\rho}_A^*$ is surjective. Thus, thanks to the previous proposition, $\mathrm{Def}(\bar{\rho}, R)$ is one-dimensional and $\tilde{\rho}_A^*$ is an isomorphism.
Thus R is *the versal deformation ring* of the representation $\bar{\rho}$ of G. *q.e.d.*

For the representation $\bar{\rho}$ we have that $\mathrm{End}_G(\bar{\rho}) = \mathrm{End}_{\mathbb{F}_p[G]}(\mathbb{F}_p^2) = \mathbb{F}_p$, hence by Theorem 2.1.5 the universal deformation ring exists and it is the versal deformation ring $R^u = \mathbb{Z}_p[[t]]/(p^n, p^m t)$.

5.3 Observations

Question by Matthias Flach

First of all let us recall the following:

Definition 5.3.1. Let $\mathcal{O}$ be a complete Noetherian local ring and let A be a complete Noetherian local $\mathcal{O}$-algebra. Let m defined by the formula $\dim A = \dim \mathcal{O} + m$. The $\mathcal{O}$-algebra A is said to be a *complete intersection* over $\mathcal{O}$ if it is of the form

$$\mathcal{O}[[t_1, \ldots, t_n]]/(f_1, \ldots, f_{n-m})$$

for some integer n and $f_1, \ldots, f_{n-m} \in \mathcal{O}[[t_1, \ldots, t_n]$.

The following question is due to Matthias Flach.

Question 1. *Are there G and ρ for which R_ρ is Noetherian but not a local complete intersection?*

In this chapter we have shown that the representation $\bar{\rho}$ of the group G has $R = \mathbb{Z}_p[[t]]/(p^n, p^m t)$ as universal deformation ring. It holds that R is not complete intersection, to check this we can apply the same argument used in [3] in the proof of Corollary 2.3.
In [3] Frauke M. Bleher, Ted Chinburg, Bart de Smit give an example of representations and groups whose universal deformation rings are not complete intersection for every characteristic of the field k.

Question by F. M. Bleher, T. Chinburg 1

In [4] the authors pose the following question:

Question 2. *Suppose k is a field of characteristic $p > 0$, G is a finite group and V is a $k[G]$-module of finite dimension over k which belongs to a block B of $k[G]$ having defect group D which has nilpotency r. Suppose further that the stable endomorphism ring $\mathrm{End}_{k[G]}(V)$ of V is one-dimensional over k, so that $R(G, V)$ is well defined. Is it the case that $\dim(R(G, V)) - \mathrm{depth}(R(G, V)) \leq r - 1$?*

For a short introduction to the concepts of "Block Theory" we refer to [2] chapter 6, for our pourpose we just need to know that the defect group is a subgroup of the p-Sylow subgroup of G, see Theorem 6.1.1 in [2]. In our case for the representation defined at the beginning of the chapter $\bar{\rho}\colon G \to \mathrm{GL}_2(\mathbb{F}_p)$ we have that the defect group is

$$D \leq \begin{pmatrix} 1 & \mathbb{Z}/p^n\mathbb{Z} \otimes \mathbb{Z}/p^m\mathbb{Z} \\ 0 & 1 \end{pmatrix} \leq G$$

in particular D is abelian and so it has nilpotency class $r = 1$.
For the definition of the depth of a ring see [9], chapter 6. A commutative Noetherian local ring R has depth zero if and only if there is a nonzero element $x \in R$ such that x annihilates the maximal ideal $\mathfrak{m}_R$ of R.
Let $R = \mathbb{Z}_p[[t]]/(p^n, p^m t)$ as in our example. If $n = m$ then $R = \mathbb{Z}/p^n\mathbb{Z}[[t]]$ and it has non zero depth, while if $n > m$ we find that it has zero depth, in fact for every element r of R that is not a unit it holds $rp^{n-1} = 0$. The ring R has dimension 1 and we get:

$$\dim(R) - \operatorname{depth}(R) = 1 - 0 = 1 > 0 = r - 1$$

Question by F. M. Bleher, T. Chinburg 2

In [5] the authors pose the following question:

Question 3. *Let k be an algebraically closed field of positive characteristic p, define $W = W(k)$ to be the ring of Witt vectors over k. Suppose G is a finite group and that V is a finitely generated $k[G]$-module such that* $\dim_k \mathrm{End}_{k[G]}(V) \leq 1$. *Is $R(G, V)$ a subquotient ring of the group ring $W[D]$ over W of a defect group D of the block B of $k[G]$ associated to V?*

Our example is a negative answer to this question, as well. In fact the group ring $W[D]$ is finitely generated as $\mathbb{Z}_p$-module since D is a finite group, while $R = \mathbb{Z}_p[[t]]/(p^n, p^m t)$ is not finitely generated, so R can not be a subquotient of $W[D]$.

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