Tensor products in Riesz space theory

Jan van Waaij

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Thesis advisors
dr. O.W. van Gaans
dr. M.F.E. de Jeu

Mathematical Institute, University of Leiden

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Preface

The aim of my master project is to study several tensor products in Riesz space theory and in particular to give new constructions of tensor products of integrally closed directed partially ordered vector spaces, also known as integrally closed pre-Riesz spaces, and of Banach lattices without making use of the constructions by D.H. Fremlin [3, 4]. To do this, we made use of so-called free Riesz spaces and free Banach lattices.
Our second aim was to find out if positive linear maps as universal mappings are really so natural as they seems to be. The so-called Riesz* homomorphism and bimorphisms seems to be more natural as universal mappings for pre-Riesz spaces, since Riesz* homomorphisms extends to Riesz homomorphism between the Riesz completions. Therefore we define the integrally closed Riesz* tensor product. Under some conditions, Riesz* bimorphisms extends to Riesz bimorphisms between the Riesz completions and we can prove that the integrally closed Riesz* tensor product actually exists. In the cases where it exists, it is equal to the usual tensor product of integrally closed pre-Riesz spaces; the positive tensor product.
It turns out that the positive tensor product $E \otimes F$ of integrally closed pre-Riesz spaces E and F has the nice property that the Riesz completion of $E \otimes F$ is the Archimedean Riesz tensor product of the Riesz completions E^r and F^r. Moreover, $\otimes : E \times F \to E \otimes F$ is a Riesz* bimorphism and the restriction of the Riesz bimorphism $\otimes : E^r \times F^r \to E^r \bar{\otimes} F^r$ (where we view E as a subspace of E^r and F as a subspace of F^r). We give a positive answer to the open problem in [7]; the Archimedean tensor cone is equal to the Fremlin tensor cone. After all we consider the tensor product of partially ordered vector spaces with Fremlin norm and norm closed cone and we calculate the positive tensor product of polyhedral cones.

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List of symbols

A^l	set of lower bounds of a set A
A^u	set of upper bounds of a set A
$[x, y]$	order interval
$x \leq A$	for non-empty set A we have $x \leq a$ for all $a \in A$
$A \leq y$	for non-empty set A we have $a \leq y$ for all $a \in A$
$x \leq A \leq y$	$x \leq A$ and $A \leq y$
$x \perp y$	x is orthogonal to y
$x \vee y$	supremum of x and y
$x \wedge y$	infimum of x and y
$\lvert x \rvert$	absolute value of x
E^+	positive elements of E
$E^\sim$	the space of all order bounded linear functionals on E
$\mathrm{FBL}(A)$	free Banach lattice over non-empty set A
$\mathrm{FNRS}(A)$	free normed Riesz space over non-empty set A
$\mathrm{FVS}(A)$	free vector space over set A
$\mathrm{FRS}(A)$	free Riesz space over set A
K_F	Fremlin tensor cone
K_I	integrally closed tensor cone
K_T	projective tensor cone
$E \otimes F$	positive tensor product or vector space tensor product of E and F
$L \tilde{\otimes} M$	Riesz space tensor product of L and M
$L \bar{\otimes} M$	Archimedean Riesz space tensor product of L and M
$L \widehat{\otimes} M$	Banach lattice tensor product of L and M
$r_A : \mathbb{R}^B \to \mathbb{R}^A$	restriction map for subset $A \subset B$
$j_A : \mathbb{R}^{\mathbb{R}^A} \to \mathbb{R}^{\mathbb{R}^B}$	$j_A(f)(\xi) = f(\xi\vert_A), f \in \mathbb{R}^{\mathbb{R}^A}, \xi \in \mathbb{R}^B$
$\omega_\xi : \mathrm{FRS}(A) \to \mathbb{R}, \xi \in \mathbb{R}^A$	$\omega_\xi(f) = f(\xi)$

1 Riesz space theory

Many vector spaces have a natural ordering. For example $\mathbb{R}$ or $\mathbb{R}^n$ but also the continuous functions on $\mathbb{R}, C(\mathbb{R})$, with the point-wise ordering. In this section we treat the most basic theory about Riesz spaces and partially ordered vector spaces. For the standard theory and notation we refer to [1, 2].

1.1 Partially ordered vector spaces

Definition 1.1. A partially ordered vector space is a real vector space E with a partially ordering $\leq$ defined on it, such that for all $x, y, z \in E$ and $\alpha \in \mathbb{R}^+$

(i) $x \leq y$ implies that $x + z \leq y + z$,

(ii) $x \leq y$ impies that $\alpha x \leq \alpha y$.

The ordering on a partially ordered vector space is called a *vector space ordering*.

An alternative notation for $y \leq x$ is $x \geq y$. If $x \leq y$ and $x \neq y$ we also write $x < y$ or $y > x$. We call an element $x \in E$ positive if $x \geq 0$ and negative if $x \leq 0$. Two elements x and y are comparable if $x \leq y$ or $y \leq x$ and we denote this by $x \sim y$. We denote by $\mathrm{Fin}(E)$ the set of all *finite* subsets of E. The *trivial ordering* $\leq$ on a vector space E, is defined by $x \leq y$ if and only if $x = y$ for all $x, y \in E$. The set of all positive elements of a partially ordered vector space E is denoted by E^+. It is called the (positive) cone.

Definition 1.2. Let E be a partially ordered vector space. Let $A \subset E$ be a set. We call A

(i) *bounded from below* if there is an $x \in E$ such that $x \leq a$ for all $a \in A$, and we write $x \leq A$ or $A \geq x$.

(ii) *bounded from above* if there is an $x \in E$ such that for all $a \in A$ we have that $a \leq x$. In this case we write $A \leq x$ or $x \geq A$.

(iii) *order bounded* if it is both bounded from below and from above.

(iv) *solid* if $A = \bigcup_{x \in A} [-x, x]$, or equivalently, A is solid, if $[-x, x] \subset A$ for all $x \in A$ [5, Definition 351I]. A linear subspace of E that is solid is called an *(order) ideal* or *solid subspace*.

We define

(i) the set of all lower bounds of A to be $A^l = \{x \in E : \text{ for all } a \in A, x \leq a\}$,

(ii) the set of all upper bounds of A to be $A^u = \{x \in E : \text{ for all } a \in A, a \leq x\}$.

We write A^{ul} for $(A^u)^l$, etc.

Remark 1.3. (i) $[x, y] \neq \emptyset$ if and only if $x \leq y$.

(ii) If E is a Riesz space and $I \subset E$ a subspace, then is I an order ideal if and only if for all $x \in E$ and $y \in I, |x| \leq |y|$ implies that $y \in I$.

(iii) Every ideal of a Riesz space is a Riesz subspace.

(iv) $\emptyset^l = \emptyset^u = E$.

Proposition 1.4. *Let E be a partially ordered vector space.*

(i) If $A \subset E$ then $A^{ulu} = A^u$,

(ii) If $E \neq 0$ then $E^l = E^u = \emptyset$.

Proof. Let E be a partially ordered vector space.

(i) Let $A \subset E$ be a subset. Let $x \in A^u$. Then $x \geq A^{ul}$, so $x \in A^{ulu}$, and hence $A^u \subset A^{ulu}$. Let $x \in A^{ulu}$. Let $a \in A$, then is $a \leq A^u$, so $a \in A^{ul}$, and therefore $a \leq x$. Hence $x \in A^u$, so $A^{ulu} \subset A^u$. We conclude that $A^u = A^{ulu}$.

(ii) Suppose $E \neq 0$. If E has the trivial ordering, then the statements are clear. Suppose E has a non-trivial ordering. Then there is an $x \in E$ with $x < 0$. Suppose that $E^l \neq \emptyset$. Let $y \in E^l$. Then $y \leq x < 0$. But then $2y < y$ and $2y \in E$, so $y \notin E^l$. Contradiction. So $E^l = \emptyset$.
There is an $x \in E, x > 0$. Suppose that $E^u \neq \emptyset$. Let $y \in E^u$. Then $y \geq x > 0$. But then $2y > y$ and $2y \in E$, thus $y \notin E^u$. Contradiction. So $E^u = \emptyset$.

□

Definition 1.5. Let E be a partially ordered vector space. Let $A \subset E$ be a subset. An element $x \in E$ is called the *infimum* of A provided x is a lower bound of A and $y \leq x$, for all $y \in A^l$. Likewise $x \in E$ is called the *supremum* of A if it is an upper bound of A and $x \leq y$, for all $y \in A^u$.

Definition 1.6. A partially ordered vector space E is called

1. *directed* or *generating* if for all $x, y \in E$ there is a $z \in E$ such that $x \leq z$ and $y \leq z$,
2. *Archimedean* if for all $x, y \in E$ with the property that $nx \leq y$ for all $n \in \mathbb{Z}$, we have that $x = 0$,
3. *integrally closed* if for all $x, y \in E$ with the property that $nx \leq y$ for all $n \in \mathbb{N}_0$, one has $x \leq 0$,
4. *a pre-Riesz space* if E is directed and for all $X \in \text{Fin}(E)\backslash\{\emptyset\}$ and for all $x \in E$ with $(x + X)^u \subset X^u$ we have that $x \geq 0$ [9, Definition 1.1(viii)],
5. a *Riesz space* if for all $x, y \in E$ the supremum $x \vee y$ and infimum $x \wedge y$ of x and y exists,
6. *a (σ)-Dedekind complete Riesz space* if for every (countable) bounded set X in E the supremum $\sup X$ and infimum $\inf X$ of X exists.

Proposition 1.7. *For a partially ordered vector space E the following statements hold*

(i) E is directed if and only if $E = E^+ - E^+$,

(ii) if E is integrally closed, then E is Archimedean. The converse is not true in general.

(iii) Every Riesz space is pre-Riesz and every directed integrally closed partially ordered vector space is pre-Riesz[9, Theorem 1.7].

(iv) E is a Riesz space if and and only if one of the following is satisfied

(a) $x \vee y$ exists, for all $x, y \in E$, or,

(b) $x \wedge y$ exists, for all $x, y \in E$, or,

(c) for all $x \in E$, the absolute value $|x| = x \vee (-x)$ of x exists, or,

(d) for all $x \in E$, the positive part of $x, x^+ = x \vee 0$ exists, or,

(e) for all $x \in E$, the negative part of $x, x^- = (-x) \vee 0$ exists.

(Follows directly from [1, Theorem 1.7].)

(v) Let $\mathbb{I}$ be an index set and let for every $i \in \mathbb{I}, E_i$ be a partially ordered vector space. The point wise ordering on $E := \prod_{i\in\mathbb{I}} E_i$ is given by $(x_i)_{i\in\mathbb{I}} \leq (y_i)_{i\in\mathbb{I}}$ if and only if $x_i \leq y_i$, for all $i \in \mathbb{I}$. E is directed, pre-Riesz, Riesz or (σ-)Dedekind complete, respectively if and only if for each $i \in \mathbb{I}, E_i$ is directed, pre-Riesz, Riesz or (σ-)Dedekind complete, respectively.

(vi) For a Riesz space the notions 'Archimedean' and 'integrally closed' are equivalent.

(vii) A σ-Dedekind complete Riesz space is Archimedean.

Proof. Let E be a partially ordered vector space. We will only prove (i), (ii), (vi) and (vii). The other statements are trivial or we refer to the literature.

(i) Suppose E is directed. Let $x \in E$. Then there is a $y \in E$ with $0 \leq y$ and $x \leq y$. Thus $y - x \geq 0$ and $x = y - (y - x)$, so we have $E = E^+ - E^+$. On the other hand, suppose $E = E^+ - E^+$. Let $x, y \in E$. Then there are $x_1, x_2, y_1, y_2 \in E^+$ such that $x = x_1 - x_2$ and $y = y_1 - y_2$. Hence $x \leq x_1 + y_1$ and $y \leq x_1 + y_1$. So E is directed.

(ii) Suppose E is integrally closed. Let $x, y \in E$ with $nx \leq y$ for all $n \in \mathbb{Z}$. Then $nx \leq y$ and $n(-x) \leq y$ for all $n \in \mathbb{N}_0$. Thus $x \leq 0$ and $-x \leq 0$. So $x = 0$. It follows that E is Archimedean.
For a counter-example consider $\mathbb{R}^2$ with $(x_1, x_2) \leq (y_1, y_2)$ if $x_1 < y_1$ and $x_2 < y_2$, or $(x_1, x_2) = (y_1, y_2)$. Suppose $n(x_1, x_2) \leq (y_1, y_2)$ for all $n \in \mathbb{Z}$. Then $nx_1 \leq y_1$ and $nx_2 \leq y_2$, for all $n \in \mathbb{Z}$. Thus $(x_1, x_2) = 0$. So $(\mathbb{R}^2, \leq)$ is Archimedean. Note that $n(-1, 0) \leq (1, 1)$, for all $n \in \mathbb{N}_0$ and $(-1, 0) \not\leq 0$. Therefore $(\mathbb{R}^2, \leq)$ is not integrally closed.

(vi) By statement (ii) we only need to prove that an Archimedean Riesz space E is integrally closed. Let $x, y \in E$ and suppose that $nx \leq y$ for all $n \in \mathbb{N}_0$. Then $nx^+ \leq y$ for all $n \in \mathbb{Z}$. Thus $x^+ = 0$ and $x \leq 0$. It follows that E is integrally closed.

(vii) Suppose E is a σ-Dedekind complete Riesz space. Let $x, y \in E$ and suppose that $nx \leq y$, for all $n \in \mathbb{N}_0$. Then $x \leq \left\{ \frac{y}{n} : n \in \mathbb{N} \right\} =: A$. Since A is bounded and E is a σ-Dedekind complete Riesz space $i = \inf A$ exists. We will prove that $i = 0$. Suppose $i \neq 0$. Since $0 \leq A$, we have that $i > 0$. We have $ni \leq y$ for all $n \in \mathbb{N}_0$. Thus $B := \{ni : n \in \mathbb{N}_0\}$ is a bounded set. Let $s = \sup B$. Then $(n+1)i \leq s$, for all $n \in \mathbb{N}_0$. So $ni \leq s - i$ for all $n \in \mathbb{N}_0$. Since s is the supremum of B we have that $s \leq s - i$ so $i \leq 0$. Therefore $i > 0$ and $i \leq 0$ and that is a contradiction. We conclude that $i = 0$, thus $x \leq 0$. It follows that E is Archimedean.

□

Proposition 1.8. *Let E be a partially ordered vector space. Then the following statements are equivalent.*

(i) E is integrally closed,

(ii) $0 = \inf \left\{ \frac{x}{n} : n \in \mathbb{N} \right\}$, for all $x \in E^+$.

Proof. Let E be a partially ordered vector space. Assume (i). Let $x \in E^+$. Clearly, 0 is a lower bound of $\left\{ \frac{1}{n}x : n \in \mathbb{N} \right\}$. Let $y \in \left\{ \frac{1}{n}x : n \in \mathbb{N} \right\}^l$. Then, $ny \leq x$, for all $n \in \mathbb{N}_0$. Since E is integrally closed, we have that $y \leq 0$. So the infimum of $\left\{ \frac{x}{n} : n \in \mathbb{N} \right\}$ exists and is equal to 0.
Assume (ii). Let $x, y \in E$, and assume that $nx \leq y$ for all $n \in \mathbb{N}_0$. So $y \geq 0$ and x is a lower bound of $\left\{ \frac{y}{n} : n \in \mathbb{N} \right\}$. Hence $x \leq \inf \left\{ \frac{y}{n} : n \in \mathbb{N} \right\} = 0$. We conclude that E is integrally closed. □

Cones

Cones in a vector space are in a one-to-one correspondence to the possible vector space orderings.

Definition 1.9. Let E be vector space. A subset $K \subset E$ is called a cone provided

1. $\alpha x, x + y \in K$ for all $x, y \in K$ and $\alpha \in \mathbb{R}^+$,
2. $K \cap (-K) = 0$.

If K satisfies (i), then it is called a wedge.

Proposition 1.10. *Let E be a vector space. Let K be a cone in E. Define $\leq$ on E by $x \leq y$ if and only if $y - x \in K$. Then $\leq$ is a partially ordered vector space. On the other hand if $\leq$ is a vector space ordering, then E^+ is a cone. The vector space ordering and the ordering induced from E^+ coincide.*

Proof. Let E be a vector space and K a cone in E. Let $\leq$ be defined through $x \leq y$ if and only if $y - x \in K$. Let $x, y, z \in E$ and suppose that $x \leq y$ and $y \leq z$. Then $z - y, y - x \in K$. Thus $z - y + y - x = z - x \in K$, so $x \leq z$. If $x \leq y$ and $y \leq x$, then $x - y, -(x - y) \in K$, so $x - y \in K \cap (-K) = 0$, thus $x = y$. For all $x \in E$ we have $x \leq x$. Thus $\leq$ is a partially ordering. It is straightforward that $\leq$ is a vector space ordering.
On the other hand, suppose $\leq$ is a vector space ordering on E. Note that $x + y, \alpha x \in E^+$, for all $x, y \in E^+$ and $\alpha \in \mathbb{R}^+$. Suppose $x \in E^+ \cap (-E^+)$, then $x \geq 0$ and $-x \geq 0$. Thus $x = 0$. Therefore E^+ is a cone. The last statement is trivial. □

Remark 1.11. If $\leq$ is a vector space ordering induced from a cone K, we denote $(E, \leq)$ sometimes by (E, K).

1.2 Morphisms

Definition 1.12. Let E and F be partially ordered vector spaces and let $\phi : E \to F$ be linear, then we call ϕ

1. *order bounded* if for every order bounded set $A \subset E$, the set $\phi(A)$ is order bounded,
2. *positive* or *increasing* if for all $x, y \in E$ with $x \leq y$ one has $\phi(x) \leq \phi(y)$ or equivalently, $\phi(x) \geq 0$ as soon as $x \geq 0$,
3. *bipositive* if $x \geq 0$, for all $x \in E$, is equivalent to $\phi(x) \geq 0$,
4. an *order isomorphism* if it is bipositive and surjective,
5. *regular* or *a difference of positive linear maps* if there are positive linear maps $\psi, \omega : E \to F$ such that $\phi = \psi - \omega$.

If E and F are directed, then we call ϕ

1. a Riesz* homomorphism if $\phi(\{x, y\}^{ul}) \subset \phi(\{x, y\})^{ul}$ for all $x, y \in E$ [9, Definition 5.1],
2. a Riesz homomorphism if $\phi(\{x, y\}^u)^l = \phi(\{x, y\})^{ul}$ for all $x, y \in E$ [9, Definition 2.1(i)],
3. a complete Riesz* homomorphism if for every set X in E bounded from above, we have that $\phi(X^{ul}) \subset \phi(X)^{ul}$[9, Definition 5.11],
4. a complete Riesz homomorphism if for every set X in E with $\inf X = 0$, also $\inf \phi(X) = 0$[9, Definition 2.1(ii)].

Remark 1.13. (i) Every positive operator is regular and every regular operator is order bounded. Not every order bounded linear map is regular, for a counterexample, see the Example of Lotz in [1, Example 1.16].

(ii) Every bipositive linear map is injective.

(iii) Every Riesz homomorphism is a Riesz* homomorphism[9, Remark 5.2(i)].

(iv) If E and F are Riesz spaces, then every Riesz* homomorphism $\phi : E \to F$ is also a Riesz homomorphism[9, Remark 5.2(ii)].

(v) The composition of two Riesz homomorphism is not necessarily a Riesz homomorphism[9, Remark 2.3].

(vi) There exists an example of a Riesz homomorphism $\phi : E \to F$ and a directed subspace $D \subset E$, such that $\phi|_D : D \to F$ is not a Riesz homomorphism[9, Remark 2.10].

(vii) The composition of two Riesz* homomorphisms is a Riesz* homomorphism[9, Remark 5.2(iii)].

(viii) If E and F are Riesz spaces, then our definition of a Riesz homomorphism coincide with the usual definition of a Riesz homomorphism[9, Remark 2.2].

(ix) A complete Riesz homomorphism $\phi : E \to F$ is not necessarily a Riesz homomorphism [9, Remark 2.2], but if E is pre-Riesz, then ϕ is also a Riesz homomorphism [9, Corollary 2.7].

Theorem 1.14. *Let E be a directed partially ordered vector space. Let $\mathbb{I}$ be an index set. Let F_i be a directed partially ordered vector space for every $i \in I$. Let, for every $i \in \mathbb{I}, \phi_i : E \to F_i$ be a linear map. Let $F = \prod_{i\in\mathbb{I}} F_i$ with the pointwise ordering. Define $\phi : E \to F$ by $\phi(x)_i = \phi_i(x)$. Then*

(i) ϕ is a Riesz homomorphism if and only if for each $i \in \mathbb{I}, \phi_i$ is a Riesz* homomorphism.*

(ii) ϕ is a Riesz homomorphism if and only if for each $i \in \mathbb{I}, \phi_i$ is a Riesz homomorphism.

Proof. (For the proof of (ii) see also [9, Theorem 2.14].) Note that for $x, y \in E$

$$\phi(\{x,y\}^{ul}) = \prod_{i\in\mathbb{I}} \phi_i(\{x,y\}^{ul}),$$

$$\phi(\{x,y\}^{u})^{l} = \prod_{i\in\mathbb{I}} \phi_i(\{x,y\}^{u})^{l}$$

and

$$\phi(\{x,y\})^{ul} = \prod_{i\in\mathbb{I}} \phi_i(\{x,y\})^{ul}.$$

Hence the Theorem follows. □

Proposition 1.15. *Let L and M be Riesz spaces. If $\phi : L \to M$ is an injective Riesz homomorphism, then ϕ is bipositive.*

Proof. Let L and M be Riesz spaces. Let $\phi : L \to M$ be an injective Riesz homomorphism. Clearly, ϕ is positive. Let $x \in L$ and $\phi(x) \geq 0$. Then $\phi(x) = \phi(x) \vee 0 = \phi(x) \vee \phi(0) = \phi(x \vee 0)$. Since ϕ is injective, we have that $x = x \vee 0 \geq 0$. Thus ϕ is bipositive. □

Not every injective positive linear map is bipositive, for example $f : (\mathbb{R}, \{0\}) \to (\mathbb{R}, \mathbb{R}^+), x \mapsto x$ is an injective positive linear map that is not bipositive.

1.3 Order ideals

Recall that a subspace I of a partially ordered vector space E is called an ideal, if for all $x \in I$ the order interval $[-x, x]$ is contained in I. We will give some results here about ideals and what they are useful for.

Proposition 1.16. *Let E and F be partially ordered vector spaces. Let $\phi : E \to F$ be a positive linear map. Then $\ker \phi$ is an order ideal.*

Proof. Let E and F be partially ordered vector spaces. Let $\phi : E \to F$ be a positive linear map. Let $x \in \ker \phi$. For all $y \in [-x, x]$, we have $0 = \phi(-x) \leq \phi(y) \leq \phi(x) = 0$. Hence $\phi(y) = 0$, therefore $y \in \ker \phi$. We conclude that $\ker \phi$ is an order ideal. □

Theorem 1.17. *Let E be a partially ordered vector space and let $J \subset E$ be an order ideal. Denote an equivalence class $x + J$ of $x \in E$ by $[x]$. Define a partially ordering $\leq$ on E/J by, $[x] \leq [y]$ if there is an $z \in J$ such that $x \leq y + z$. This turns E/J into a partially ordered vector space. We have for all $X, Y \in E/J, X \leq Y$ if and only if there are $x, y \in E$ such that $x \leq y, [x] = X$ and $[y] = Y$. Moreover $(E/J)^+ = \{[x] : x \in E^+\}$. Define $q : E \to E/J$ by $x \mapsto [x]$. If E is directed, then q is a Riesz homomorphism.*

Proof. For the first statements, see [5, Proposition 351J]. Suppose E is directed. Let $x, y \in E$. Then $q(\{x, y\}^u)^l = \{[z] : z \in \{x, y\}^u\}^l \subset \{[x], [y]\}^{ul} = q(\{x, y\})^{ul}$. Since q is in particular positive, the converse inclusion holds by [9, Lemma 2.4]. Thus q is a Riesz homomorphism. □

We call q the quotient Riesz homomorphism and E/J the quotient space.

Proposition 1.18. *(See [10, Corollary 1.3.14] and [11, Theorem 62.3]). Let L be a normed Riesz space and I a norm closed order ideal of L. Then L/I is a normed Riesz space with quotient norm*

$$||[x]|| = \inf\{||y|| : y \in L, [y] = [x]\}.$$

If L is a Banach lattice, then L/I is also a Banach lattice.

1.4 Relatively uniform topology

Suppose I is an ideal of a Riesz space L, then L/I is also a Riesz space. Unfortunately, even in the case that L is Archimedean, L/I need not be Archimedean. A sufficient and necessary condition for L/I to be Archimedean is that I is relatively uniformly closed, which will be defined next.

Definition 1.19. Let E be a partially ordered vector space. A sequence $\{x_n\}_{n\geq 1}$ converges relatively uniformly (ru-converges) to $x \in E$, if there exist a $u \in E^+$ and a sequence of positive real numbers $\{\varepsilon_n\}_{n\geq 1}$ with $\varepsilon_n \downarrow 0$ such that $-\varepsilon_n u \leq x_n - x \leq \varepsilon_n u$, for all $n \in \mathbb{N}$. A set $A \subset E$ is relatively uniformly closed (ru-closed) if for every sequence $\{x_n\}_{n\geq 1} \subset A$ that is relatively uniformly convergent to an $x \in L$, we have that $x \in A$. Clearly, the intersection of an arbitrary collection of relatively uniformly closed sets if relatively uniformly closed. We define the relatively uniformly closure (ru-closure) of a set A to be the intersection of all relatively uniformly closed sets B that contain A. This collection is not empty since it contains E.

Remark 1.20. (i) If E is a Riesz space, then x_n converges relatively uniformly to x, if there is a $u \in E^+$ and a real valued sequence $\{\varepsilon_n\}_{n\geq 0}, \varepsilon_n \downarrow 0$ such that $|x_n - x| \leq \varepsilon_n u$, for all $n \in \mathbb{N}$.

(ii) Suppose L is a non-Archimedean Riesz space. So there are $x, y \in L$ such that $nx \leq y$, for all $n \in \mathbb{Z}$, but $x \neq 0$. Note that $y = |y| \geq 0$. For all $n \in \mathbb{N}$ we have that $nx \leq y$ and $-nx \leq y$, thus $n|x| \leq y$, for all $n \in \mathbb{N}_0$. We conclude that $|0 - x| \leq \frac{1}{n}y$, for all $n \in \mathbb{N}$, and that the constant sequence $\{0\}_{n\geq 1}$ converges to $x \neq 0$. We see that $\{0\}$ is not ru-closed and that that the constant sequence $\{0\}_{n=1}^\infty$ converges to at least two different elements, namely 0 and x.

Theorem 1.21. *Let E be a partially ordered vector space. If E^+ is relatively uniformly closed, then E is integrally closed.*

Proof. Let E be a partially ordered vector space with relatively uniformly closed positive cone E^+. Let $x, y \in E$ be such that $nx \leq y$, for all $n \in \mathbb{N}_0$. Then $y \geq 0$ and $0 \leq \frac{1}{n}y - x$, for all $n \in \mathbb{N}$. Hence

$$-\tfrac{1}{n}y \leq \tfrac{1}{n}y - x - (-x) = \tfrac{1}{n}y,$$

for all $n \in \mathbb{N}$. We see that the sequence $\{\frac{1}{n}y - x\}_{n\geq 1} \subset E^+$ converges relatively uniformly to $-x$. Since E^+ is relatively uniformly closed we have that $-x \in E^+$, thus $x \leq 0$ and E is integrally closed. □

Lemma 1.22. *Let E be a partially ordered vector space. Let $A \subset E$ be a non-empty relatively uniformly closed set. Then for all $x \in E$ we have that $x + A = \{x + y : y \in A\}$ is relatively uniformly closed.*

Proof. Let E be a partially ordered vector space. Let $A \subset E$ be a non-empty relatively uniformly closed set. Let $x \in E$. Suppose $\{x+x_n\}_{n=1}^{\infty}$ is a sequence in $x+A$ that converges relative uniformly to y. Then, there exists a real sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ and $u \in E^+$ such that

$$-\varepsilon_n u \leq x + x_n - y \leq \varepsilon_n u,$$

for all $n \in \mathbb{N}$. Thus for all $n \in \mathbb{N}$ we have that

$$-\varepsilon_n u \leq x_n - (y - x) \leq \varepsilon_n u.$$

It follows that $\{x_n\}_{n=1}^{\infty}$ is a sequence in A that ru-converges to $y-x$. Since A is ru-closed, we have that $y - x \in A$, thus $y \in x + A$. Hence $x + A$ is ru-closed. $\square$

Theorem 1.23. *Let E be a partially ordered vector space. Let I be an order ideal of E. Then E/I is Archimedean if and only if I is relatively uniformly closed.*

Proof. Let E be a partially ordered vector space and let I be an order ideal of E. Let $q : E \to E/I$ be the canonical quotient map. Suppose E/I is Archimedean. Let $\{x_n\}_{n\geq 1}$ be a sequence in I that relative uniform converges to $x \in E$. By definition, there is a sequence of positive real numbers $\{\varepsilon_n\}_{n\geq 1}$ and a $u \in E^+$ such that $\varepsilon_n \downarrow 0$ and $-\varepsilon_n u \leq x_n - x \leq \varepsilon_n u$, for all $n \in \mathbb{N}$. Thus $-\varepsilon_n q(u) \leq -q(x) \leq \varepsilon_n q(u)$, for all $n \in \mathbb{N}$. For every $m \in \mathbb{N}$ there is an $n_m \in \mathbb{N}$ such that $\frac{1}{m} \geq \varepsilon_{n_m}$, so $\frac{1}{m} q(u) \leq -q(x) \leq \frac{1}{m} q(u)$, for all $m \in \mathbb{N}$. It follows that $mq(x) \leq q(u)$, for all $m \in \mathbb{N}$. Since E/I is Archimedean we have that $q(x) = 0$. Thus $x \in I$ and I is relatively uniformly closed.
On the other hand, suppose that I is relatively uniformly closed. Let $x, y \in E$ be such that $n[x] \leq [y]$, for all $n \in \mathbb{Z}$, then $[y] \geq 0$. Therefore we may assume that $y \geq 0$. We have for all $n \in \mathbb{N}$,

$$-\tfrac{1}{n}[y] \leq [x] \leq \tfrac{1}{n}[y].$$

Thus there is an $x_n \in I$ such that

$$-\tfrac{1}{n} y \leq x + x_n \leq \tfrac{1}{n} y,$$

for all $n \in \mathbb{N}$. Hence $\{x+x_n\}_{n=1}^{\infty}$ is a sequence in $x+I$ that converges relatively uniformly to 0. By Lemma 1.22 on the preceding page $x + I$ is ru-closed and hence $0 \in x + I$, so $[x] = 0$. We conclude that E/I is Archimedean. $\square$

Theorem 1.24. *Let E be a partially ordered vector space. Then E is Archimedean if and only if every relatively uniformly convergent sequence in E has a unique limit.*

Proof. First assume that $E = E/\{0\}$ is Archimedean. By Theorem 1.23, $\{0\}$ is ru-closed. Let $\{x_n\}_{n\geq 0}$ be a sequence in E that converges relatively uniformly to x and y in E. By definition there are sequences of positive real numbers $\{\varepsilon_n\}_{n\geq 1}$ and $\{\delta_n\}_{n\geq 1}$ and $u, v \in E^+$ with $\varepsilon_n \downarrow 0$ and $\delta_n \downarrow 0$ and

$$-\varepsilon_n u \leq x_n - x \leq \varepsilon_n u \text{ and } -\delta_n v \leq x_n - y \leq \delta_n v,$$

for all $n \in \mathbb{N}$. Thus $-\delta_n v \leq y - x_n \leq \delta_n v$, for all $n \in \mathbb{N}$. It follows that

$$-\varepsilon_n u - \delta_n v \leq y - x \leq \varepsilon_n u + \delta_n v,$$

for all $n \in \mathbb{N}$. We conclude that

$$-(\varepsilon_n \vee \delta_n)(u + v) \leq 0 - (x - y) \leq (\varepsilon_n \vee \delta_n)(u + v),$$

for all $n \in \mathbb{N}$. Note that $\varepsilon_n \vee \delta_n \downarrow 0$ and that $u + v \geq 0$ and hence $\{0\}_{n\geq 0}$ ru-converges to $x - y$. Since $\{0\}$ is ru-closed, we have that $x - y = 0$, that is $x = y$. We conclude that $\{x_n\}_{n=1}^{\infty}$ has a unique limit.
On the other hand, suppose that E is not Archimedean. Then there are $x, y \in E$ such that $nx \leq y$, for all $n \in \mathbb{Z}$, but $x \neq 0$. We have $-\frac{1}{n} y \leq x - 0 \leq \frac{1}{n} y$ for all $n \in \mathbb{N}$. Thus the constant sequence $\{x\}_{n=1}^{\infty}$ converges to 0, but clearly it also converges to $x \neq 0$. So $\{x\}_{n=1}^{\infty}$ has at least two different limits. $\square$

Theorem 1.25. *Let E and F be partially ordered vector spaces with F Archimedean. Let $\phi : E \to F$ be positive. Then* $\ker\phi$ *is ru-closed.*

Proof. Let E and F be partially ordered vector spaces with F Archimedean. Let $\phi : E \to F$ be positive. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\ker\phi$ that is ru-convergent to $x \in E$. So there exists a real sequence $\{\varepsilon_n\}_{n=1}^{\infty}, \varepsilon_n \downarrow 0$ and a $u \in E^+$, such that

$$-\varepsilon_n u \leq x_n - x \leq \varepsilon_n u,$$

for all $n \in \mathbb{N}$. Thus

$$-\varepsilon_n\phi(u) \leq \phi(x) \leq \varepsilon_n\phi(u),$$

for all $n \in \mathbb{N}$. For every $m \in \mathbb{N}$, there is an $n_m \in \mathbb{N}$, such that $\varepsilon_{n_m} \leq \frac{1}{n}$. Thus

$$-\tfrac{1}{n}\phi(u) \leq \phi(x) \leq \tfrac{1}{n}\phi(u),$$

for all $n \in \mathbb{N}$. Furthermore $\phi(u) \geq 0$, so $n\phi(x) \leq \phi(u)$, for all $n \in \mathbb{Z}$. Since F is Archimedean, we have that $\phi(x) = 0$, that is $x \in \ker\phi$. We conclude that $\ker\phi$ is ru-closed. ☐

The relatively uniform topology

Clearly, if Λ is an index set and, for all $\lambda \in \Lambda$, A_λ is a ru-closed subset of a partially ordered vector space E, then $\bigcap_{\lambda\in\Lambda} A_\lambda$ is ru-closed too. Trivially, $\emptyset$ and E are ru-closed subsets of E.

Lemma 1.26. *Let E be a partially ordered vector space and A and B ru-closed subsets of E, then $A \cup B$ is ru-closed.*

Proof. Let E be a partially ordered vector space and let A and B be ru-closed subsets of E. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $A\cup B$ that ru-converges to $x \in E$. Then there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k} \in C$, for all $k \in \mathbb{N}$, where C is either A or B. It follows directly from the definition of a ru-convergent sequence that $\{x_{n_k}\}_{k=1}^{\infty}$ ru-converges to x. Since C is ru-closed, we have that $x \in C \subset A \cup B$. So $A \cup B$ is ru-closed. ☐

Now, the following theorem follows immediately.

Theorem 1.27. *Let E be a partially ordered vector space. The complements of ru-closed sets of E form a topology; the relative uniform topology or ru-topology $\mathcal{T}_{ru}$. A set $\mathcal{O} \subset E$ is open if and only if $E\backslash\mathcal{O}$ is ru-closed.*

1.5 Norms on partially ordered vector spaces

In this subsection we consider norms on partially ordered vector spaces that respect the order structure.

Definition 1.28. Let E be a partially ordered vector space. A norm ρ on E is called a Fremlin norm provided that for all $x, y \in E$ with $-y \leq x \leq y$ we have that $\rho(x) \leq \rho(y)$.

Definition 1.29. Let L be a Riesz space. A Riesz norm or lattice norm $||\cdot||$ on L is a norm on L such that if $|x| \leq |y|$ then $||x|| \leq ||y||$, for all $x, y \in L$. The pair $(L, ||\cdot||)$ is called a normed Riesz space. If the induced metric by $||\cdot||$ on L is complete, then we call $(L, ||\cdot||)$ a Banach lattice.

Proposition 1.30. *Let L be a Riesz space and let ρ be a norm on L. Then ρ is a Riesz norm if and only if ρ is a Fremlin norm and $\rho(x) = \rho(|x|)$, for all $x \in L$.*

Proof. Let L be a Riesz space and ρ a norm on L. Suppose ρ is a Riesz norm on L. Suppose $x, y \in L$ and $-y \leq x \leq y$. Then $-|y| \leq -y \leq \pm x \leq y \leq |y|$ thus $|x| \leq |y|$. Hence $\rho(x) \leq \rho(y)$. So ρ is a Fremlin norm. Clearly, $\rho(x) = \rho(|x|)$, for all $x \in L$. On the other hand suppose that ρ is a Fremlin norm and $\rho(x) = \rho(|x|)$, for all $x \in L$. Let $x, y \in L$ and suppose $|x| \leq |y|$. Then $x \leq |x| \leq |y|$, and $-x \leq |x| \leq |y|$. Thus $x \geq -|y|$. Hence $-|y| \leq x \leq |y|$. Since ρ is a Fremlin norm, we have that $\rho(x) \leq \rho(|y|)$. Note that $\rho(|y|) = \rho(y)$ thus $\rho(x) \leq \rho(y)$. We conclude that ρ is a Riesz norm. ☐

It follows that every Riesz norm is a Fremlin norm. A Fremlin norm on a Riesz space is not necessarily a Riesz norm as the following example shows. The example is from [6, Example 1.8(i)]. The condition that $\rho(x) = \rho(|x|)$ for all $x \in L$ is really necessary.

Example 1.31. Consider $\mathbb{R}^2$ with the standard ordering. Then $\mathbb{R}^2$ is a Riesz space. Define $\rho(x_1, x_2) = |x_1| + |x_2| + |x_1 + x_2|$. Let $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$ and suppose that $-y \leq x \leq y$. Then $-y_1 \leq x_1 \leq y_1$ and $-y_2 \leq x_2 \leq y_2$ and $-y_1 - y_2 \leq x_1 + x_2 \leq y_1 + y_2$. Thus $|x_1| \leq |y_1|, |x_2| \leq |y_2|$ and $|x_1 + x_2| \leq |y_1 + y_2|$. It follows that $\rho(x) \leq \rho(y)$. Hence ρ is a Fremlin norm. Note that $|(1,1)| = (1,1) = |(1,-1)|$, but $\rho((1,1)) = 4 \neq 2 = \rho((1,-1))$. Thus by Proposition 1.30 on the preceding page ρ is not a Riesz norm.

Proposition 1.32. *Let E be a partially ordered vector space with Fremlin norm ρ. Then E is Archimedean. If E^+ is closed under ρ, then E is integrally closed.*

Proof. Let E be a partially ordered vector space with Fremlin norm ρ. Let $x, y \in E$ be such that $nx \leq y$, for all $n \in \mathbb{Z}$. Thus $-\frac{1}{n}y \leq x \leq \frac{1}{n}y$, for all $n \in \mathbb{N}$. So $\rho(x) \leq \rho\left(\frac{1}{n}y\right) = \frac{1}{n}\rho(y)$, for all $n \in \mathbb{N}$. Hence $\rho(x) = 0$ and therefore $x = 0$. It follows that E is Archimedean.
Suppose further that E^+ is ρ-norm closed. Let $x, y \in E$ be such that $nx \leq y$, for all $n \in \mathbb{N}_0$. We have $0 \leq -x + \frac{1}{n}y$, for all $n \in \mathbb{N}$, which ρ-converges to $-x$, as $n \to \infty$. Since E^+ is ρ-closed, we have that $-x \in E^+$, that is $x \leq 0$. We conclude that E is integrally closed. □

Definition 1.33. A Fremlin space is a partially ordered vector space E with Fremlin norm ρ such that E^+ is closed under ρ.

Remark 1.34. E need not be directed. An example is $\mathbb{R}$ with the trivial ordering and Fremlin norm $|\cdot|$.

Every Fremlin space can be considered as a subspace of a Banach lattice.

Theorem 1.35 (Van Gaans)**.** *([6, Corollary 4.8]). The following statements are equivalent.*

(i) F is a Fremlin space.

(ii) There exists a Banach lattice $\mathbb{F}$ and an isometric bipositive linear map $\iota : F \to \mathbb{F}$.

Remark 1.36. Example [6, Example 4.9] shows that the embedding of F in $\mathbb{F}$ is not necessarily a Riesz homomorphism.

1.6 Order denseness

Order denseness is a very important notion in the theory about general partially ordered vector spaces, in particular in the theory about Riesz and Dedekind completions. We give also a proof that order denseness is transitive, because there is no proof in the literature, as far as we know.

Definition 1.37. Let E be a partially ordered vector space. A subspace $D \subset E$ is called order dense in E if for all $x \in E$ we have that $x = \inf_E\{d \in D : x \leq d\}$ and $x = \sup_E\{d \in D : d \leq x\}$.

The following example is illustrative.

Example 1.38. Let $D \subset L^1(\mathbb{R})$ be the space of all integrable simple functions. Then D is order dense in $L^1(\mathbb{R})$.

The following characterization of order denseness is useful.

Proposition 1.39. *Let E be a partially ordered vector space and let $D \subset E$ be a subspace. Then the following statements are equivalent.*

1. *$D \subset E$ is order dense,*
2. *$x = \inf_E\{d : x \leq d, d \in D\}$, for all $x \in E$,*

3. $x = \sup_E\{d : d \leq x, d \in D\}$, *for all* $x \in E$.

Proof. Clearly, (1)⇒(2) and (1)⇒(3).
(2)⇒(1): Suppose that (2) holds. Let $x \in E$ and note that $\{d \in D : d \leq x\} = \{d \in D : -x \leq -d\} = \{-d : -x \leq d, d \in D\} = -\{d \in D : -x \leq d\}$. Since (2) holds, we have $\sup_E\{d \in D : d \leq x\} = -\inf_E\{d \in D : -x \leq d\} = -(-x) = x$.
(3)⇒(1): Similar to the proof of (2)⇒(1). □

Proposition 1.40. *Let G be a partially ordered vector space, let $E \subset G$ be an order dense subspace and let $F \subset G$ be a subspace such that $E \subset F \subset G$. Then $F \subset G$ is order dense.*

Proof. Let E, F and G be as in the proposition. Let $x \in G$. Note that $x = \inf_G\{e \in E : x \leq e\}$. Trivially, $x \leq \{f \in F : x \leq f\}$. Let $x' \in G$ with $x' \leq \{f \in F : x \leq f\}$. Then $x' \leq \{e \in E : x \leq e\}$, since $\{e \in E : x \leq e\} \subset \{f \in F : x \leq f\}$. Thus $x' \leq x = \inf_G\{e \in E : x \leq e\}$. It follows that $x = \inf_G\{f \in F : x \leq f\}$. □

In my bachelor thesis [14, Lemmas 1.2.4-1.2.6 and Theorem 1.27], I proved that order denseness is transitive. That is, if $E \subset F$ is order dense and $F \subset G$ is order dense, then $E \subset G$ is order dense. To do this we need three lemmas.

Lemma 1.41. *Let E be a partially ordered vector space and let D be an order dense subspace of E. Suppose that $S \subset D$ is a subset such that the infimum of S in the space D exists and is equal to s. Then the infimum of S exists in the space E and is equal to s.*

Proof. Let E be a partially ordered vector space and let $D \subset E$ be an order dense subspace of E. Let $S \subset D$ be a set, such that the infimum s of S in D exists. Note that $s \leq S$ in E. Suppose that $t \in E$ and $t \leq S$. Suppose that $d \in D$ with $d \leq S$, then $d \leq s = \inf_D S$. If $d \in D$ with $d \leq t$, then $d \leq S$, so $d \leq s = \inf_D S$. Thus $\{d \in D : d \leq t\} \subset \{d \in D : d \leq s\}$. Since $D \subset E$ is order dense, we have that $t = \sup_E\{d \in D : d \leq t\} \leq \sup_E\{d \in D : d \leq s\} = s$. It follows that s is the infimum of S in E. □

Lemma 1.42. *Let E be a partially ordered vector space and let D be an order dense subspace of E. Suppose that $x, x' \in E$ are such that $x' \not\leq x$. Then there is a $d \in D$ such that $x \leq d$ and $x' \not\leq d$.*

Proof. Let E be a partially ordered vector space and let $D \subset E$ be an order dense subspace. Let $x, x' \in E$ be such that $x' \not\leq x$. We argue by contradiction. Suppose that there is no $d \in D$ with $x \leq d$ and $x' \not\leq d$. Then for any $d \in D$ with $x \leq d$ one has $x' \leq d$. So $\{d \in D : x \leq d\} \subset \{d \in D : x' \leq d\}$. Therefore, since $D \subset E$ is order dense, $x = \inf_E\{d \in D : x \leq d\} \geq \inf_E\{d \in D : x' \leq d\} = x'$. That contradicts our assumption that $x \not\geq x'$. □

Lemma 1.43. *Let G be a partially ordered vector space and let E and F be subspaces of G such that $E \subset F \subset G, E \subset F$ is order dense and $F \subset G$ is order dense. Then for every $z \in G$ there exist an $x \in E$ such that $z \leq x$.*

Proof. Let E, F and G be as in the lemma. Let $z \in G$. Since $z = \inf_G\{y \in F : z \leq y\}$ we have $\{y \in F : z \leq y\} \neq \emptyset$. Let $y_0 \in \{y \in F : z \leq y\}$. Since $y_0 = \inf_F\{x \in E : y_0 \leq x\}$ we have $\{x \in E : y_0 \leq x\} \neq \emptyset$. Let $x_0 \in \{x \in E : y_0 \leq x\}$. Then $x_0 \geq y_0 \geq z$. □

Theorem 1.44. *Let G be a partially ordered vector space. Let E and F be subspaces of G such that $E \subset F \subset G, E \subset F$ is order dense and $F \subset G$ is order dense. Then $E \subset G$ is order dense.*

Proof. Let E, F and G be as in the theorem. We argue by contradiction. So suppose that $E \subset G$ is not order dense. Then there is a $z \in G$ such that z is not the infimum of $S = \{x \in E : z \leq x\}$ in G. By Lemma 1.43 we have $S \neq \emptyset$. Note that $z \leq S$. Thus there is a $z' \in G$ with

$$z' \leq S \tag{1}$$

and $z' \not\leq z$. Since $F \subset G$ is order dense, we have by Lemma 1.42 on the preceding page a $y \in F$ such that

$$z \leq y \tag{2}$$

and

$$z' \not\leq y. \tag{3}$$

Since $E \subset F$ is order dense and $F \subset G$ is order dense we have by Lemma 1.41 on the previous page that $y = \inf_F\{x \in E : y \leq x\} = \inf_G\{x \in E : y \leq x\}$. Let $x \in E, x \geq y$ then $x \geq z$, by (2), thus $x \in S$. From (1) follows that $x \geq z'$. Since $z' \leq S \supset \{x \in E : y \leq x\}$ and $y = \inf_G\{x \in E : y \leq x\}$ we have that $z' \leq y$. This contradicts (3). We are forced to conclude that $E \subset G$ is order dense. $\square$

Theorem 1.44 on the preceding page implies that for every *finite* sequence of order dense subspaces $D_1 \supset D_2 \supset \ldots \supset D_n, D_n \subset D_1$ is order dense. The following example shows that Theorem 1.44 on the previous page does not hold in greater generality, when we have an infinite decreasing sequence of order dense subsets.

Example 1.45. Let for all $D_N \subset C[0,1]$ be the subspace of continuous functions $f : [0,1] \to \mathbb{R}$ with the property that $f(0) = f\left(\frac{k}{2^N}\right), N \in \mathbb{N}_0, k = 0, 1, \ldots, 2^N$. We will show that $D_{N+1} \subset D_N$ is order dense for all N. Consider $D_\infty := \bigcap_{N=1}^{\infty} D_N$. Then this is a subspace of D_0. Consider $f \in D_\infty$. For all $N \in \mathbb{N}_0$ and all $k \in \{0, 1, \ldots, 2^N\}$ we have $f(0) = f\left(\frac{k}{2^N}\right)$. Since f is continuous and $\{\frac{k}{2^N} : N \in \mathbb{N}_0, k \in \{0, 1, \ldots, 2^N\}\} \subset [0,1]$ is (topologically) dense, we have that f is constant $f(0)$. It is clear that $D_\infty \subset D_0$ is not order dense.
Let $N \in \mathbb{N}_0$ and let $f \in D_N$. Since f is continuous and $[0,1]$ is compact $M := \sup_{x \in [0,1]} f(x) < \infty$. Define a sequence of functions $(f_n)_{n=1}^{\infty} \subset D_{N+1}$ through

$$f_n(x) = \begin{cases} \max\left\{4n2^{N+1}(f(\frac{k}{2^{N+1}} + \frac{1}{4n2^{N+1}}) - M)(x - \frac{k}{2^{N+1}}) + M, f(x)\right\} \\ \text{if } \frac{k}{2^{N+1}} \leq x \leq \frac{k}{2^{N+1}} + \frac{1}{4n2^{N+1}}, k \in \{0, 1, \ldots, 2^{N+1} - 1\} \\ \\ f(x) \text{ if } \frac{k}{2^{N+1}} + \frac{1}{4n2^{N+1}} < x < \frac{k+1}{2^{N+1}} - \frac{1}{4n2^{N+1}}, k \in \{0, 1, \ldots, 2^{N+1} - 1\} \\ \\ \max\left\{4n2^{N+1}(M - f(\frac{k+1}{2^{N+1}} - \frac{1}{4n2^{N+1}}))(x + \frac{1}{4n2^{N+1}} - \frac{k+1}{2^{N+1}}) + \right. \\ \left. f(\frac{k+1}{2^{N+1}} - \frac{1}{4n2^{N+1}}), f(x)\right\} \\ \text{if } \frac{k+1}{2^{N+1}} - \frac{1}{4n2^{N+1}} \leq x \leq \frac{k+1}{2^{N+1}}, , \in \{0, 1, \ldots, 2^{N+1} - 1\} \end{cases}$$

Then $f_n \in D_{N+1}$, for all $n \in \mathbb{N}$, and $f \leq f_n$. Letting $n \to \infty$, we see that $\inf_{D_N}\{f_n : n \in \mathbb{N}_0\} = f$. So $\inf_{D_N}\{g : f \leq g, g \in D_{N+1}\} = f$. We conclude that $D_{N+1} \subset D_N$ is order dense.

Instead of looking at a decreasing sequence, we can also view an increasing sequence $D_1 \subset D_2 \subset \ldots$, such that $D_n \subset D_{n+1}$ is order dense, for all $n \in \mathbb{N}$. Let $X = \cup_{n \in \mathbb{N}} D_n$, then $D_1 \subset X$ is order dense.

Theorem 1.46. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of partially ordered vector spaces such that $D_n \subset D_{n+1}$ is a order dense subspace, for all $n \in \mathbb{N}$. Define $X := \bigcup_{n=1}^{\infty} D_n$. Then X is a partially ordered vector space and $D_1 \subset X$ is an order dense subspace.*

Proof. Let $\{D_n\}_{n=1}^{\infty}$ and X be as in the theorem. For $x, y, z \in X$ there is an $n \in \mathbb{N}$ such that $x, y, z \in D_n$. So all axioms of a vector space are satisfied. Define $\leq$ on X by $x \leq y$, if and only if there is an $n \in \mathbb{N}$ such that $x, y \in D_n$ and $x \leq y$ in D_n. Note that this is well defined, since for $m \in \mathbb{N}$ with $x, y \in D_m$ we either have that $D_n \subset D_m$ or $D_n \supset D_m$, so that in both cases $x \leq y$ in D_m. Note that $\leq$ satisfied all axioms of a partially ordered vector space. Thus $(X, \leq)$ is a well

defined partially ordered vector space.
Suppose $x \in X$. Then there is an $n \in \mathbb{N}$ such that $x \in D_n$. Note that $D_1 \subset D_n$ is order dense; apply Theorem 1.44 on page 16. So $x = \inf_{D_n}\{d \in D_1 : x \leq d\}$. Note that $x \leq \{d \in D_1 : x \leq d\}$ in X. Suppose that $x' \in X$ and $x' \leq \{d \in D_1 : x \leq d\}$. There is an $m > n$ such that $x' \in D_m$. Apply Lemma 1.41 on page 16, $m-n$ times to get that $x = \inf_{D_n}\{d \in D_1 : x \leq d\} = \inf_{D_{n+1}}\{d \in D_1 : x \leq d\} = \ldots = \inf_{D_m}\{d \in D_1 : x \leq d\}$. Thus $x' \leq x$ and hence $x = \inf_X\{d \in D_1 : x \leq d\}$. Therefore $D_1 \subset X$ is order dense. □

Remark 1.47. Note that Theorem 1.44 on page 16 is a special case of Theorem 1.46 on the previous page, let $D_1 = E, D_2 = F$ and for $n \geq 3$ let $D_n = G$.

Theorem 1.48. *Let A be a directed index set and let $\{E_\alpha\}_{\alpha\in A}$ be an increasing net of partially ordered vector spaces, such that $E_\alpha \subset E_\beta$ is order dense for all $\alpha, \beta \in A$ with $\alpha \leq \beta$. Let*

$$X = \bigcup_{\alpha\in A} E_\alpha$$

and define $\leq$ on X through $x \leq y$ if and only if there is an $\alpha \in A$ such that $x, y \in E_\alpha$ and $x \leq y$ in E_α, for all $x, y \in X$. Then X is a well defined vector space and $\leq$ is a well defined vector space ordering on X and $E_\alpha \subset X$ is order dense, for all $\alpha \in A$.

Proof. Let A be a directed index set and let $\{E_\alpha\}_{\alpha\in A}$ be an increasing net of partially ordered vector spaces, such that $E_\alpha \subset E_\beta$ is order dense, for all $\alpha, \beta \in A$ with $\alpha \leq \beta$. Let

$$X = \bigcup_{\alpha\in A} E_\alpha$$

and define $\leq$ on X through $x \leq y$ if and only if there is an $\alpha \in A$ such that $x, y \in E_\alpha$ and $x \leq y$ in E_α, for all $x, y \in X$. Suppose $x, y \in E_\alpha$ and $x \leq y$ in E_α and $x, y \in E_\beta$, then there is a $\gamma \geq \{\alpha, \beta\}$. So $x, y \in E_\gamma \supset E_\alpha$, so $x \leq y$ in E_γ. Since $E_\beta \subset E_\gamma$, we also have that $x \leq y$ in E_β. Thus $\leq$ on X is well defined.
Let $x, y, z \in X$. Since A is directed, there is an $\alpha \in A$ such that $x, y, z \in E_\alpha$. So all axioma's of a vector space are satisfied for X and $\leq$ is a vector space ordering on X.
Let $\alpha \in A$. We will now prove that E_α is an order dense subspace of X. Let $x \in X$. Since A is directed, there is a $\beta \geq \alpha$ such that $x \in E_\beta$. Since $E_\alpha \subset E_\beta$ is order dense, we have that $x = \inf_{E_\beta}\{d \in E_\alpha : x \leq d\}$. Let $x' \in X, x' \leq \{d \in E_\alpha : x \leq d\}$. There is a $\gamma \geq \beta$ such that $x' \in E_\gamma$. Since $E_\beta \subset E_\gamma$ is order dense, we have by Lemma 1.41 on page 16 that $x = \inf_{E_\gamma}\{d \in E_\alpha : x \leq d\}$, so $x' \leq x$. Thus, $x = \inf_X\{d \in E_\alpha : x \leq d\}$ and $E_\alpha \subset X$ is order dense. □

Theorem 1.49. *If F is a partially ordered vector space and E is an order dense subspace of F, then E is integrally closed if and only if F is integrally closed.*

Proof. Let F be a partially ordered vector space and let E be an order dense subspace of F. Clearly, if F is integrally closed, then E is integrally closed. On the other hand, suppose E is integrally closed. Let $x, y \in F$ and suppose that $nx \leq y$, for all $n \in \mathbb{N}_0$. Let $x_0 \in E$ with $x_0 \leq x$ and let $y_0 \in E$ with $y_0 \geq y$. Then $nx_0 \leq y_0$, for all $n \in \mathbb{N}_0$. Since E is integrally closed, we have that $x_0 \leq 0$. Thus for all $x_0 \in E$ with $x_0 \leq x$, we have that $x_0 \leq 0$. Since $x = \sup_F\{x_0 \in E : x_0 \leq x\}$, we have that $x \leq 0$. We conclude that F is integrally closed. □

Remark 1.50. Suppose E is an order dense subspace of a partially ordered vector space F. An interesting question is wether E is Archimedean if and only if F is Archimedean. We could not find a proof.

2 Free spaces

The free vector space over a set A are all linear combinations of elements of A. Thus A is the vector space basis for the free vector space. Something similar can be done for Riesz spaces, normed Riesz

spaces and Banach lattices. We define and study these objects here. We need the free spaces for the construction of various tensor products.
Let B be a set and let $A \subset B$. We define $r_A : \mathbb{R}^B \to \mathbb{R}^A$ to be the restriction map, thus for $f \in \mathbb{R}^B$ we define $r_A(f) = f|_A$. It is clear that r_A is a surjective Riesz homomorphism. Sometimes we write $\xi_A = r_A(\xi)$ for $\xi \in \mathbb{R}^B$. We define $j_A : \mathbb{R}^{\mathbb{R}^A} \to \mathbb{R}^{\mathbb{R}^B}$ by $j_A(f)(\xi) = f(\xi_A) = f(r_A(\xi)) = f(\xi|_A)$, where $f \in \mathbb{R}^{\mathbb{R}^A}$ and $\xi \in \mathbb{R}^B$. Then j_A is an injective Riesz homomorphism. From Proposition 1.15 on page 11 follows that j_A is bipositive.

2.1 Free vector spaces and free Riesz spaces

In this subsection we study the free vector space and the free Riesz space and we will show that the free vector space is a subspace of the free Riesz space.

Definition 2.1. A *free vector space* over a set A is a pair (V, ι) where $\iota : A \to V$ is a map and V is a real vector space, such that for every real vector space W and for every map $\phi : A \to W$ there is a unique linear map $\phi_* : V \to W$ such that $\phi = \phi_* \circ \iota$.

$$\begin{tikzcd} V \arrow[r, "\phi_*"] & W \\ A \arrow[u, "\iota"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Lemma 2.2. *Let A be a set. Suppose (V, ι) and (W, j) are free vector spaces over A. Then there is a unique bijective linear map $\phi : V \to W$ such that $j = \phi \circ \iota$.*

Proof. Let A be a set. Suppose that (V, ι) and (W, j) are free vector spaces over A. By definition there are unique linear maps $\iota_* : W \to V$ and $j_* : V \to W$ such that $\iota = \iota_* \circ j$ and $j = j_* \circ \iota$. Thus $\iota = \iota_* \circ j_* \circ \iota$. Note that also the identity map id_V on V is a linear map such that $\iota = id_V \circ \iota$. From the uniqueness statement it follows that $id_V = \iota_* \circ j_*$. Similarly, we have that the identity map id_W on W is equal to $j_* \circ \iota_*$. Define $\phi = j_* : V \to W$. Then ϕ is the unique isomorphism $\psi : V \to W$ such that $j = \psi \circ \iota$.

$$\begin{tikzcd} V \arrow[rr, bend left, "j_*"] & & W \arrow[ll, bend left, "\iota_*"] \\ & A \arrow[ul, "\iota"] \arrow[ur, "j"'] & \end{tikzcd}$$

□

Definition 2.3. A free Riesz space over a set A is a pair (L, ι) where $\iota : A \to L$ is a map and L is a Riesz space such that for every Riesz space M and every map $\phi : A \to M$ there is a unique Riesz homomorphism $\phi_* : L \to M$ such that $\phi = \phi_* \circ \iota$.

$$\begin{tikzcd} L \arrow[r, "\phi_*"] & M \\ A \arrow[u, "\iota"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

The next lemma can be found in [13, Proposition 3.3].

Lemma 2.4. *A free Riesz space is unique if it exists, in the following sense: let (L, ι) and (M, j) be two free Riesz spaces over a set A, then there is a unique (surjective) Riesz isomorphism $T : L \to M$ such that $T \circ \iota = j$. In particular T is an order isomorphism.*

Proof. Let A be a set and suppose that (L, ι) and (M, j) are two free Riesz spaces over A. There is a unique Riesz homomorphism $j_* : L \to M$ such that $j = j_* \circ \iota$ and a unique Riesz homomorphism $\iota_* : M \to L$ such that $\iota = \iota_* \circ j$. Note that $\iota = \iota_* \circ j = \iota_* \circ j_* \circ j$ and $\iota_* \circ j_* : L \to L$ is a Riesz homomorphism. Note that also that the identity map id_L on L is a Riesz homomorphism such that $\iota = id_L \circ \iota$. From the uniqueness statement follows that $id_L = \iota_* \circ j_*$. Likewise is the identity map id_M on M satisfies $id_M = j_* \circ \iota_*$. Define $T = j_* : L \to M$, then T is an invertible Riesz homomorphism and $T^{-1} = \iota_*$ is a Riesz homomorphism and $T \circ \iota = j_* \circ \iota = j$. Moreover T is the unique Riesz homomorphism with this properties. In particular T is an order isomorphism.

$$\begin{tikzcd}
L \arrow[rr, bend left, "j_*"] & & M \arrow[ll, bend left, "\iota_*"] \\
& A \arrow[ul, "\iota"] \arrow[ur, "j"'] &
\end{tikzcd}$$

□

Theorem 2.5. *([13, Proposition 3.2]). If (L, ι) is a free Riesz space over a set A, then L is generated as Riesz space by $\iota(A)$.*

Proof. Let (L, ι) be a free Riesz space over a set A. Let M be the Riesz subspace of L generated by $\iota(A)$. Define the map $\phi : A \to M$ by $\phi(a) = \iota(a)$. By definition, there is a Riesz homomorphism $\phi_* : L \to M$ such that $\phi = \phi_* \circ \iota$. Let $j : M \to L$ be the inclusion map. Then $j \circ \phi_* : L \to L$ satisfies $j \circ \phi_* \circ \iota = \iota$. By definition, the identity map on L, id_L is the unique Riesz homomorphism $\psi : L \to L$ that satisfies $\iota = \psi \circ \iota$. Thus $j \circ \phi_* = id_L$. But that implies that j is surjective, so $M = L$. We conclude that L is generated as Riesz space by $\iota(A)$.

$$\begin{tikzcd}
L \arrow[rr, bend left, "\phi_*"] & & M \arrow[ll, bend left, "j"] \\
& A \arrow[ul, "\iota"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

□

For every set the the free Riesz space exists, see [13, Proposition 3.7]. The case $A = \emptyset$ is trivial.

Theorem 2.6. *Let A be a set. If $A = \emptyset$, then $(0, \emptyset)$ is the free Riesz space over A. If $A \neq \emptyset$ then $(FRS(A), \iota)$ is the free Riesz space over A, where $FRS(A)$ is the Riesz subspace of $\mathbb{R}^{\mathbb{R}^A}$ generated by elements $\xi_a \in \mathbb{R}^{\mathbb{R}^A}$, defined by $\xi_a(f) = f(a)$ for $a \in A$ and $f \in \mathbb{R}^A$, and where $\iota : A \to FRS(A)$ is defined by $a \mapsto \xi_a$. Moreover ι is injective and $FRS(A)$ is Archimedean.*

Lemma 2.7. *Let A be a set. Let ι be as in Theorem 2.6. Then $\iota(A)$ is a linearly independent set.*

Proof. The case $A = \emptyset$ is trivial, so suppose that $A \neq \emptyset$. Let $a_1, \ldots, a_n \in A$ be finitely many mutually different elements. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ be such that $\sum_{i=1}^n \lambda_i \xi_{a_i} = 0$. Then, for all $f \in \mathbb{R}^A$ we have that

$$\left(\sum_{i=1}^n \lambda_i \xi_{a_i} \right)(f) = \sum_{i=1}^n \lambda_i f(a_i) = 0.$$

Define $f_k \in \mathbb{R}^A$ by $f(a_k) = 1$ and $f(a) = 0$ for $a \in A \backslash \{a_k\}$, where $k \in \{1, \ldots, n\}$. Then

$$\left(\sum_{i=1}^n \lambda_i \xi_{a_i} \right)(f_k) = \sum_{i=1}^n \lambda_i f_k(a_i) = \lambda_k = 0.$$

Thus $\lambda_k = 0$, for $k \in \{1 \ldots, n\}$. It follows that $\xi_1, \ldots, \xi_n$ are linearly independent. □

Theorem 2.8. *Let A be a set. If $A = \emptyset$, then $(0, \emptyset)$ is the free vector space over A. If $A \neq \emptyset$, let ξ_a be as in Theorem 2.6 on the previous page, where $a \in A$. Let $FVS(A) = Span\{\xi_a : a \in A\} \subset FRS(A) \subset \mathbb{R}^{\mathbb{R}^A}$. Define $\iota : A \to FVS(A)$ by $a \mapsto \xi_a$, where $a \in A$. Then $(FVS(A), \iota)$ is the free vector space over A.*

Proof. Let A be a set. The statement is trivial if $A = \emptyset$. So suppose that A has at least one element. Let $\mathrm{FVS}(A)$ and ι be as in the statement of the theorem. By Lemma 2.7 on the preceding page $\{\xi_a : a \in A\}$ is a linearly independent set. Let W be an arbitrary vector space and let $\phi : A \to W$ be a map. Define a linear map $\phi_* : \mathrm{FVS}(A) \to W$ on its basis elements ξ_a by $\xi_a \mapsto \phi(a)$, where $a \in A$. Thus $\phi = \phi_* \circ \iota$. For any other linear map $\psi : \mathrm{FVS}(A) \to W$ that satisfies $\phi = \psi \circ \iota$, we have that $\psi(\xi_a) = \phi(a) = \phi_*(\xi_a)$. Since the ξ_a generate $\mathrm{FVS}(A)$ as vector space, we have that $\psi = \phi_*$ thus ϕ_* is unique. We conclude that $(\mathrm{FVS}(A), \iota)$ is the free vector space over A. ☐

Remark 2.9. We view the free vector space as a subspace of the free Riesz space.

Theorem 2.10. *Let B be a set and let $A \subset B$ be a subset. Let $(FRS(B), \iota_B)$ be the free Riesz space over B. Let $FRS(A)$ be the Riesz subspace of $FRS(B)$ generated by the elements $\iota_B(a)$, where $a \in A$, and let $\iota_A = \iota_B|_A$. Then $(FRS(A), \iota_A)$ is the free Riesz space over A.*

Proof. Let B be a set and let $A \subset B$ be a subset. The map $j_A : \mathbb{R}^{\mathbb{R}^A} \to \mathbb{R}^{\mathbb{R}^B}$ is an injective Riesz homomorphism, and hence $j_A|_{\mathrm{FRS}(A)} \to \mathrm{FRS}(B)$ is an injective Riesz homomorphism. ☐

Proposition 2.11. *Let A be a finite set. Let $(FRS(A), \iota)$ be the free Riesz space over A. Then $\sum_{a \in A} |\iota(a)|$ is a strong order unit for $FRS(A)$.*

Proof. Let A be a finite set with free Riesz space $(\mathrm{FRS}(A), \iota)$. Then the statement is clear from the fact that A is finite and that $\mathrm{FRS}(A)$ is generated by elements $\iota(a)$. ☐

Proposition 2.12. *Let A be a non-empty set and let $\mathcal{F}(A)$ denote the set of all finite subsets of A. Then*

$$FRS(A) = \bigcup_{B \in \mathcal{F}(A)} FRS(B),$$

where we view $FRS(B)$ as a Riesz subspace of $FRS(A)$.

Proof. Let A be a non-empty set with free Riesz space $(\mathrm{FRS}(A), \iota)$. From Theorem 2.10 it is clear that

$$\bigcup_{B \in \mathcal{F}(A)} \mathrm{FRS}(B) \subset \mathrm{FRS}(A).$$

On the other hand every element $x \in \mathrm{FRS}(A)$ is generated by finitely many elements $\iota_{a_1}, \ldots, \iota_{a_n}$, so $x \in \mathrm{FRS}(\{a_1, \ldots, a_n\})$ and hence

$$\mathrm{FRS}(A) = \bigcup_{B \in \mathcal{F}(A)} \mathrm{FRS}(B).$$

☐

Proposition 2.13. *Let A be a set.*

1. *If $\xi \in \mathbb{R}^A$, then $\omega_\xi : FRS(A) \to \mathbb{R}$ defined by $\omega_\xi(f) = f(\xi)$, where $f \in FRS(A)$, is a Riesz homomorphism.*
2. *If $\omega : FRS(A) \to \mathbb{R}$ is a Riesz homomorphism, then there is a $\xi \in \mathbb{R}^A$ such that $\omega = \omega_\xi$.*

Here, we view $FRS(A)$ as a Riesz subspace of $\mathbb{R}^{\mathbb{R}^A}$.

Proof. The first statement is trivial. Suppose $\omega : \mathrm{FRS}(A) \to \mathbb{R}$ is a Riesz homomorphism. Define $\xi \in \mathbb{R}^A$ by $\xi(a) = \omega(\iota(a))$, for $a \in A$. Then for $a \in A$ we have $\omega(\iota(a)) = \xi(a) = \iota(a)(\xi) = \omega_\xi(\iota(a))$. Thus ω and ω_ξ coincide on the set $\{\iota(a) : a \in A\}$ that generates $\mathrm{FRS}(A)$ as Riesz space hence $\omega = \omega_\xi$. ☐

2.2 Free normed Riesz space and free Banach lattice

Something similar to free Riesz space exists for Banach lattices. It is easy to generalize the results of De Pagter and Wickstead for Banach lattices [13] to normed Riesz spaces. We give an overview of the main results.

Definition 2.14. Let A be a non-empty set and let X be a normed space. A map $\phi : A \to X$ is (norm) bounded, if there exists an $M > 0$ such that $||\phi(a)|| \leq M$, for all $a \in A$. We define the norm of a (norm) bounded map $\phi : A \to X$ to be $||\phi|| = \sup\{||\phi(a)|| : a \in A\}$.

Remark 2.15. Note that $||\cdot||$ is norm on the vector space of all norm bounded maps $\phi : A \to X$.

Definition 2.16. Let A be a non-empty set. The free normed Riesz space (free Banach lattice) over A is a pair (L, ι) where L is a normed Riesz space (Banach lattice) and $\iota : A \to L$ is a bounded map such that for any normed Riesz space (Banach lattice) M and for any bounded map $\phi : A \to M$ there is a Riesz homomorphism $\phi_* : L \to M$ with the property $||\phi_*|| = ||\phi||$. Moreover ϕ_* is the unique Riesz homomorphism $\psi : L \to M$ that satisifies $\phi = \psi \circ \iota$.

$$\begin{tikzcd} L \arrow[r, "\phi_*"] & M \\ A \arrow[u, "\iota"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

From the fact that $\iota_* = id_L$ follows that $||\iota|| = ||\iota_*|| = 1$. But even more is true.

Lemma 2.17. *([13, Proposition 4.2].) Let A be a non-empty set, and suppose that (L, ι) is a free Banach lattice or free normed Riesz space over A. Then $||\iota(a)|| = 1$, for every $a \in A$.*

Proof. Let A be a non-empty set, and suppose that (L, ι) is a free Banach lattice over A. Define $j : A \to \mathbb{R}$ by $a \mapsto 1, a \in A$. Then $||j_*|| = ||j|| = 1$. Further, $1 = ||j(a)|| = ||j_*(\iota(a))|| \leq ||j_*||\,||\iota(a)|| = ||\iota(a)||$, thus $||\iota(a)|| \geq 1$, for all $a \in A$. Since $||\iota|| = 1$ we have $||\iota(a)|| \leq 1$, for all $a \in A$. Therefore $||\iota(a)|| = 1$, for all $a \in A$.
The case that (L, ι) is a free normed Riesz space over A is proven similarly.

$$\begin{tikzcd} L \arrow[r, "j_*"] & \mathbb{R} \\ A \arrow[u, "\iota"] \arrow[ur, "j"'] & \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, "\iota_*"] & L \\ A \arrow[u, "\iota"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

□

The free Banach lattice or normed Riesz space is unique if it exists, see [13, Proposition 4.3].

Lemma 2.18. *Let A be a non-empty set, and suppose that (L, ι) and (M, j) are free Banach lattices over A (free normed Riesz spaces over A.) Then there exists a unique isometric order isomorphism $\phi : L \to M$, such that $\phi \circ \iota = j$.*

Proof. Let A be a non-empty set, and suppose that (L, ι) and (M, j) are free Banach lattices over A. Let $j_* : L \to M$ be the unique Riesz homomorphism such that $j = j_* \circ \iota$. Then by Lemma 2.17, $||j_*|| = ||j|| = 1$. Similarly, there is a unique Riesz homomorphism $\iota_* : M \to L$ of norm 1, such that $\iota = \iota_* \circ j$. So $\iota_* \circ j_* : L \to L$ is a Riesz homomorphism and $\iota = \iota_* \circ j = \iota_* \circ j_* \circ \iota$. From the uniqueness statement in the theorem follows that $\iota_* \circ j_*$ is the identity map on L. Similarly, $j_* \circ \iota_*$ is the identity map on M. So ι_* is an isometric order isomorphism.
The case that (L, ι) is a free normed Riesz space over A is proven similarly.

$$\begin{tikzcd} L \arrow[rr, bend left, "j_*"] & & M \arrow[ll, bend left, "\iota_*"] \\ & A \arrow[ul, "\iota"] \arrow[ur, "j"'] & \end{tikzcd}$$

$\square$

For an ordered vector space E we denote by $E^\sim$ the space of all order bounded linear functionals on E. According to [1, Theorem 1.18] $E^\sim$ is equal to the space of regular linear functionals as soon as E is an Archimedean Riesz space. De Pagter and Wickstead define in [13] a lattice norm $||\cdot||_F$ on FRS(A), that turns FRS(A) into the free normed Riesz space over A and its norm completion is the free Banach lattice over A. We will now review this construction.

Definition 2.19. ([13, Definition 4.4]) For a non-empty set A we define a map $||\cdot||^\dagger : \mathrm{FRS}(A)^\sim \to [0,\infty]$ by

$$||\phi||^\dagger = \sup\{|\phi|(|\iota(a)|) : a \in A\}.$$

Let

$$\mathrm{FRS}(A)^\dagger = \{\phi \in \mathrm{FRS}(A)^\sim : ||\phi||^\dagger < \infty\}.$$

The following is clear from the definition.

Lemma 2.20. *FRS$(A)^\dagger$ is a vector lattice ideal in FRS$(A)^\sim$.*

Lemma 2.21. *Let A be a non-empty set. Let $\xi \in \mathbb{R}^A$. Let $\omega_\xi : FRS(A) \to \mathbb{R}$ be defined through $\omega_\xi(x) = x(\xi), x \in FRS(A)$ (See Proposition 2.13 on page 21). Then $||\omega_\xi|| < \infty$ if and only if ξ is bounded. Or, equivalently, $\omega_\xi \in FRS(A)^\dagger$ if and only if ξ is bounded.*

Proof. Let $\xi \in \mathbb{R}^A$. Since $|\xi(a)| = |\iota(a)(\xi)| = |\omega_\xi(\iota(a))| = \omega_\xi(|\iota(a)|) = |\omega_\xi|(|\iota(a)|)$, for all $a \in A$, it follows that $||\omega_\xi||^\dagger < \infty$ if and only if ξ is bounded. $\square$

Lemma 2.22. *If A is a non-empty set, then $||\cdot||^\dagger$ is a Riesz norm on FRS$(A)^\dagger$.*

Proof. Let A be a non-empty set. $||\cdot||^\dagger$ is clearly a Riesz seminorm. Suppose that $||\phi||^\dagger = 0$. Then $|\phi|(|\iota(a)|) = 0$, for all $a \in A$, thus $\phi(|\iota(a)|) = 0$, for all $a \in A$. Let $x \in \mathrm{FRS}(A)$. By Proposition 2.12 on page 21 there are finitely many $a_1, \dots, a_n \in A$ such that $x \in \mathrm{FRS}(\{a_1,\dots,a_n\})$. By Proposition 2.11 on page 21 $e = \sum_{i=1}^n |\iota(a)|$ is a strong order unit for $\mathrm{FRS}(\{a_1,\dots,a_n\})$. Thus there is an $\lambda \in \mathbb{R}^+$ such that $|x| \le \lambda e$. It follows that $|\phi(x)| \le |\phi|(|x|) \le \lambda|\phi|(e) = 0$ and hence $\phi(x) = 0$. So $\phi = 0$. We conclude that $||\cdot||^\dagger$ is a Riesz norm. $\square$

Definition 2.23. For a non-empty set A and for $x \in \mathrm{FRS}(A)$ we define $||x||_F = \sup\{\phi(|x|) : \phi \in (\mathrm{FRS}(A)^\dagger)^+, ||\phi||^\dagger \le 1\}$.

Theorem 2.24. *If A is a non-empty set, then $||\cdot||_F$ is a lattice norm on FRS(A).*

Proof. From the definition it is clear that for $x, y \in \mathrm{FRS}(A)$ with $|x| \le |y|$ we have that $||x||_F \le ||y||_F$ and that $||\cdot||_F$ is positive homogenious and subadditive. First we will show that for all $x \in \mathrm{FRS}(A)$ we have that $||x||_F < \infty$. Let $x \in \mathrm{FRS}(A)$. By Proposition 2.12 on page 21 there are finite $a_1, \dots, a_n \in A$ such that $x \in \mathrm{FRS}(\{a_1,\dots,a_n\})$. By Proposition 2.11 on page 21 $e = \sum_{i=1}^n |\iota(a_i)|$ is a strong order unit of $\mathrm{FRS}(\{a_1,\dots,a_n\})$. So there is a $\lambda > 0$ such that $|x| \le \lambda e$. So $||x||_F \le \lambda||e||_F$. For $\phi \in (\mathrm{FRS}(A)^\dagger)^+, ||\phi||^\dagger \le 1$, we have $\phi(e) = \sum_{i=1}^n \phi(|\iota(a_i)|) \le n$. Thus $||e||_F \le n$ and hence $||x||_F \le \lambda n < \infty$.
It only remains to show that for all $x \in \mathrm{FRS}(A)$ we have that $||x||_F = 0$ implies that $x = 0$. Let $x \in \mathrm{FRS}(A)$ with $||x||_F = 0$. Clearly, for all $\phi \in (\mathrm{FRS}(A)^\dagger)^+$ we have $\phi(x) = 0$. In particular $\omega_\xi(x) = x(\xi) = 0$, for every $\xi \in \mathbb{R}^A$. By Proposition 2.12 on page 21 there is a finite subset B of A such that $x \in \mathrm{FRS}(B)$. In particular for every $\xi \in \mathbb{R}^B \subset \mathbb{R}^A$ we have that $x(\xi) = 0$. But that means that $x = 0$, since $x \in \mathrm{FRS}(B) \subset \mathbb{R}^{\mathbb{R}^B}$. $\square$

Theorem 2.25. *Let A be a non-empty set. Then $((FRS(A), ||\cdot||_F), \iota)$ is the free normed Riesz space over A.*

Proof. Let A be a non-empty set. Consider the free Riesz space over $A, (\mathrm{FRS}(A), \iota)$. Let M be an arbitrary normed Riesz space and let $\phi : A \to M$ a bounded map that maps into M_1, the closed unit ball of M, and suppose that $||\phi|| = 1$. By the definition of the free Riesz space, there is a unique Riesz homomorphism $\phi_* : \mathrm{FRS}(A) \to M$ such that $\phi = \phi_* \circ \iota$. It only remains to prove that $||\phi_*|| = ||\phi||$.
By Lemma 2.17 on page 22 we have for all $a \in A, ||\iota(a)|| = 1$. So $||\phi_*|| \geq ||\phi_*(\iota(a))|| = ||\phi(a)||$, thus $||\phi_*|| \geq ||\phi||$. Suppose that $||\phi_*|| > ||\phi||$. So for some $x \in \mathrm{FRS}(A)$ with $||x|| = 1$ is $||\phi_*(x)|| = ||\phi_*(|x|)|| > ||\phi||$. By Hahn-Banach there exists a positive functional ψ on M of norm at most one and $\psi(\phi_*|x|) > ||\phi||$.
We have $||\psi \circ \phi_*||^\dagger = \sup\{|\psi \circ \phi_*|(|\iota(a)|) : a \in A\} = \sup\{|\psi \circ \phi_*(\iota(a))| : a \in A\} = \sup\{|\psi(\phi(a))| : a \in A\} \leq 1$. Thus $||x||_F \geq ||\psi(\phi_*(|x|))|| > \phi = 1$. This is a contradiction. Therefore $||\phi_*|| = ||\phi||$.
Suppose $\phi : \mathrm{FRS}(A) \to M$ is a bounded map and $||\phi|| > 0$. Then $\bar{\phi} = \phi/||\phi||$ is of norm one and maps into M_1. It is clear that $\bar{\phi}_* = \phi_*/||\phi||$, so $||\phi_*|| = ||\phi||$. Suppose $\phi : \mathrm{FRS}(A) \to M$ is zero. Then $\phi_* = 0$, so $||\phi_*|| = ||\phi||$. This concludes our proof. □

Remark 2.26. We denote the free normed Riesz space over a non-empty set A by $\mathrm{FNRS}(A)$.

For the next theorem see [1, Theorems 4.1 and 4.2].

Theorem 2.27. *Let E be a normed Riesz space and let E' denote its norm dual space. Then E' is a Banach lattice. Consider E as a subspace of its double dual $E'' = (E')'$. Then the norm completion of E is a Banach lattice.*

Theorem 2.28. *Let A be a non-empty set. Let $FBL(A)$ be the $||\cdot||_F$-norm completion of $FRS(A)$. Then $FBL(A)$ is a Banach lattice and $(FBL(A), \iota)$ is the free Banach lattice over A.*

Proof. Let A be a non-empty set and let $\mathrm{FBL}(A)$ be the $||\cdot||_F$-norm completion of $\mathrm{FRS}(A)$. By Theorem 2.27, $\mathrm{FBL}(A)$ is a Banach lattice. Suppose M is Banach lattice and $\phi : A \to M$ a bounded map. By Theorem 2.25 on the previous page there is a Riesz homomorphism $\phi_* : \mathrm{FRS}(A) \to M$ such that $\phi = \phi_* \circ \iota$ and $||\phi_*|| = ||\phi||$, moreover ϕ_* is the unique Riesz homomorphism ψ that satisfies $\phi = \psi \circ \iota$. The Riesz homomorphism ϕ_* extends by continuity to a Riesz homomorphism $\phi_* : \mathrm{FBL}(A) \to M$ and we still have $\phi = \phi_* \circ \iota$ and $||\phi_*|| = ||\phi||$. Suppose $\psi : \mathrm{FBL}(A) \to M$ is also a Riesz homomorphism that satisfies $\phi = \psi \circ \iota$. Then ψ and ϕ_* coincide on $\mathrm{FRS}(A)$. Since the continuous extension is unique, we have that $\psi = \phi_*$. So $(\mathrm{FBL}(A), \iota)$ is the free Banach lattice over A. □

3 Riesz completion

Riesz spaces have many nice properties and there is an abundance of theory about them. That is the reason that one like, to embed partially ordered vector spaces in a nice way into Riesz spaces. It turns out that every pre-Riesz space can be bipositively embedded into a Riesz space, its Riesz completion, such that the embedding is order dense and the Riesz completion is generated as a Riesz space by the pre-Riesz space. Of course we could make use of the so-called 'Dedekind completion' but then we should require that the space is integrally closed. Moreover, the Dedekind completion may be too large. We give a short overview of the construction and theory of the Riesz completion. We follow the Ph.D. thesis of Van Haandel [9].

3.1 The Riesz completion of a directed partially ordered vector space

We start with a definition, taken from [9, Definition 3.1].

Definition 3.1. Let E be a directed partially ordered vector space. A pair (L, ϕ), consisting of a Riesz space L and a Riesz homomorphism $\phi : E \to L$, is called a *Riesz completion* of E if for every Riesz space M and every Riesz homomorphism $\psi : E \to M$ there exists a unique Riesz homomorphism $\psi_* : L \to M$ such that $\psi = \psi_* \circ \phi$.

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\psi"'] & L \arrow[d, "\psi_*"] \\ & M \end{tikzcd} \tag{4}$$

Lemma 3.2. *(See [9, Remark 3.2.1]). The Riesz completion is unique if it exists, in the following sense suppose (L, ϕ) and (M, ψ) are two Riesz completions of a partially ordered vector space E, then there is a unique bijective linear map $T : L \to M$ such that T and T^{-1} are Riesz homomophisms and $\psi = T \circ \phi$.*

Proof. Suppose (L, ϕ) and (M, ψ) are two Riesz completions of a partially ordered vector space E. By definition, there is a unique Riesz homomorphism $\psi_* : L \to M$ such that $\psi = \psi_* \circ \phi$ and a unique Riesz homomorphism $\phi_* : M \to L$ such that $\phi = \phi_* \circ \psi$. Note that the identity map on $L, id_L : L \to L$ is a Riesz homomorphism and $\phi = id_L \circ \phi$. Also note that $\phi_* \circ \psi_* : L \to L$ is a Riesz homomorphism and $\phi = \phi_* \circ \psi = \phi_* \circ \psi_* \circ \phi$. From the uniqueness statement follows that $id_L = \phi_* \circ \psi_*$. Likewise we have that the identity map id_M on M is equal to $\psi_* \circ \phi_*$. Define $T = \psi_* : L \to M$. Then T is a Riesz homomorphism, T is bijective, $T^{-1} = \phi_*$ is a Riesz homomorphism, $\psi = T \circ \phi$ and T is the unique Riesz homomorphism with these properties.

$$\begin{tikzcd} L \arrow[rr, bend left, "\psi_*"] & & M \arrow[ll, bend left, "\phi_*"] \\ & E \arrow[ul, "\phi"] \arrow[ur, "\psi"'] & \end{tikzcd} \qquad \begin{tikzcd} L \arrow[rr, bend left, leftrightarrow, "id_L"] & & L \\ & E \arrow[ul, "\phi"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

□

Remark 3.3. Lemma 3.2 justifies to speak about 'the' Riesz completion of a partially ordered vector space if it exists.

Fortunately, every directed partially ordered vector space has a Riesz completion (see [9, Theorem 3.5]).

Theorem 3.4 (Van Haandel)**.** *Every directed partially ordered vector space E has a Riesz completion. To be precise, let $\Phi(E)$ be the set of all pairs (L, ϕ) where L is a Riesz space and L is a subset of the power set $\mathcal{P}(E)$ of E and $\phi : E \to L$ is a Riesz homomorphism. Let I be an index set such that $\Phi(E) = \{(L_i, \phi_i) : i \in I\}$. Let*

$$M = \prod_{i \in I} L_i.$$

Define $\psi : E \to M$ through

$$\psi(x) = \{\phi_i(x)\}_{i \in I}, x \in E.$$

Then ψ is a Riesz homomorphism. Let E^r be the Riesz subspace of M generated by $\psi(E)$. Let ι_E be the restriction of ψ to E^r. Then (E^r, ι_E) is the Riesz completion of E.

It turns out that pre-Riesz spaces are exactly the spaces that can be bipositively embedded into there Riesz completion. To be precise, we have the following (see [9, Theorem 4.13]).

Theorem 3.5. *Let E be a directed partially ordered vector space with Riesz completion (L, ϕ). Then $\phi : E \to L$ is bipositive if and only if E is a pre-Riesz space.*

3.2 The Riesz completion of a pre-Riesz space

In this subsection we give M.B.J.G. van Haandel's construction of the Riesz completion of a pre-Riesz space (see [9, Chapter 4]). It coincides with the Riesz completion of a general directed partially ordered vector space.

In the rest of this subsection, let E be a pre-Riesz space. Define the *conditional completion* E^δ of E by

$$\begin{cases} E^\delta = \{\{0\}\}, & \text{if } E = 0, \\ E^\delta = \{A \subset E : A^{ul} = A\} \backslash \{\emptyset, E\} & \text{if } E \neq 0. \end{cases}$$

For all $A, B \in E^\delta$ and all $\lambda \in \mathbb{R}$ we define the following operations and relations on E^δ.

(i) $A \oplus B = (A + B)^{ul}$,

(ii) $\ominus A = -(A^u)$,

(iii) if $\lambda > 0$ define $\lambda \star A := \lambda A$,

(iv) if $\lambda = 0$ define $0 \star A = -E^+$, which we denote by 0.

(v) if $\lambda < 0$ define $\lambda \star A = \lambda(A^u)$,[1]

(vi) define a ordering on E^δ by $A \leq B$ if $A \subset B$,

(vii) define a map $\varphi_E : E \to E^\delta$ by $\varphi(x) = \{x\}^{ul} = \{y \in E : y \leq x\}$, for $x \in E$.

Then $(E^\delta, \leq)$ is a lattice with $A \vee B = (A \cup B)^{ul}, A \wedge B = (A \cap B)^{ul} = A \cap B$.
Define the set $E^r = \{A^{ul} \oplus \ominus B^{ul} : A, B \in \text{Fin}(E) \backslash \{\emptyset\}\}$. Then E^r is a subset of E^δ and $\oplus$ restricted to $E^r \times E^r$ maps into $E^r, \ominus$ restricted to E^r maps into E^r and $\star$ restricted to $\mathbb{R} \times E^r$ maps into E^r. It is clear from the definition that φ_E maps into E^r. Moreover, we have the following. (See [9, Corollaries 4.9 and 4.11].)

Theorem 3.6 (Van Haandel)**.** *The addition map $\oplus : E^r \times E^r \to E^r$, the multiplication with -1 map $\ominus : E^r \to E^r$ and the scalar multiplication map $\star : \mathbb{R} \times E^r \to E^r$ are well defined and turn E^r into a real vector space. The ordering $\leq$ on E^δ restricted to E^r is a vector space ordering and turns E^r into a Riesz space. The map $\varphi_E : E \to E^r$ is a bipositive linear map and a complete Riesz homomorphism and $\varphi_E[E]$ is order dense in E^r and generates E^r as a Riesz space. Moreover (E^r, φ_E) is the Riesz completion of E. For $a_1, \ldots, a_n \in E$ we have,*

$$\{a_1, \ldots, a_n\}^{ul} = \bigvee_{i=1}^n \varphi_E(a_i).$$

For the next proof, see [9, the proof of Theorem 4.8].

Proposition 3.7. *Let E be a pre-Riesz space. For $A, B, C, D \in Fin(E) \backslash \{\emptyset\}$ we have the following.*

$$\begin{aligned} \left(A^{ul} \oplus \ominus B^{ul}\right) \vee \left(C^{ul} \oplus \ominus D^{ul}\right) &= (A + D)^{ul} \vee (B + C)^{ul} \oplus \ominus (B + D)^{ul} &\quad (5) \\ &= ((A + D) \cup (B + C))^{ul} \oplus \ominus (B + D)^{ul} &\quad (6) \end{aligned}$$

We have the following nice characterization of pre-Riesz spaces.

Theorem 3.8. *Let E be a directed partially ordered vector space. Then E is pre-Riesz if and only if there exist a Riesz space L and a bipositive linear map $\phi : E \to L$, such that $\phi(E)$ is order dense in L and generates L as a Riesz space. In that case (L, ϕ) is the Riesz completion of E.*

[1] There is a typo in [9, Chapter 4].

Proof. Let E be a directed partially ordered vector space. If E is pre-Riesz, then it follows from [9, Theorem 4.8] that there exist a Riesz space L and a bipositive linear map $\phi : E \to L$ such that $\phi(E)$ is order dense in L and generates L as a Riesz space. On the other hand, if L is a Riesz space and $\phi : E \to L$ is a bipositive linear map such that $\phi(E)$ is order dense in L and generates L as a Riesz space then it follows from [9, Corollary 4.10] that E is pre-Riesz. The last statement follows from [9, Theorem 3.7]. $\square$

The following characterization of a pre-Riesz space is useful.

Corollary 3.9. *Let L be a Riesz space and $E \subset L$ a subspace. If $E \subset L$ is order dense, then E is pre-Riesz.*

Proof. Let L be a Riesz space and let $E \subset L$ be an order dense subspace. Let M be the Riesz subspace of L generated by E. Clearly, $E \subset M$ is order dense. Thus by Theorem 3.8 on the previous page E is a pre-Riesz space. $\square$

The following corollary is straightforward.

Corollary 3.10. *Let E be a directed partially ordered vector space. Then E is a pre-Riesz space if and only if there is a Riesz space L and a bipositive linear map $\phi : E \to L$ such that $\phi(E)$ is order dense in L.*

Note that not every subspace of a Riesz space is pre-Riesz, for instance if it is not directed. Theorem 1.49 on page 18 and Theorem 3.8 on the previous page have the following corollary.

Corollary 3.11. *If E is a pre-Riesz space with Riesz completion (E^r, φ_E), then E is integrally closed if and only if E^r is an Archimedean Riesz space.*

3.3 Morphisms

We give a short summary of results about morphisms that we need in the next section about bimorphisms.

Proposition 3.12. *(See [9, Theorem 5.3(i)]). Let E and F be directed partially ordered vector spaces, with F pre-Riesz and let $\phi : E \to F$ be linear. Then ϕ is a Riesz homomorphism if and only if $phi(X^u)^l = \phi(X)^{ul}$, for all $X \in Fin(E)\backslash\{\emptyset\}$.*

Proposition 3.13. *(See [9, Corollary 5.4]). Let E and F be directed partially ordered vector spaces. Let $\phi : E \to F$ be linear. Then the following statements are equivalent.*

(i) ϕ *is a Riesz* homomorphism,*

(ii) $\phi(\{a, b\}^{ul})^u = \phi(\{a, b\})^u$*, for all* $a, b \in E$*,*

(iii) $0 \in \{a, b\}^{ul}$ *implies* $0 \in \phi(\{a, b\})^{ul}$*, for all* $a, b \in E$*,*

(iv) $\phi(X^{ul}) \subset \phi(X)^{ul}$*, for all* $X \in Fin(E)\backslash\{\emptyset\}$*,*

(v) $\phi(X^{ul})^u = \phi(X)^u$*, for all for all* $X \in Fin(E)\backslash\{\emptyset\}$*,*

(vi) if $0 \in X^{ul}$ *then* $0 \in \phi(X)^{ul}$*, for all* $X \in Fin(E)\backslash\{\emptyset\}$*.*

With Proposition 3.13 we are able to prove that Riesz* homomorphisms between pre-Riesz spaces are exactly the linear maps that can be extended to Riesz homomorphisms between the Riesz completions. Or, to say it differently, Riesz* homomorphisms are restrictions of Riesz homomorphism to the underlaying pre-Riesz spaces.

Theorem 3.14. *Let E and F be pre-Riesz spaces, with Riesz completions (E^r, φ_E) and (F^r, φ_F) respectively. Let $\phi : E \to F$ be a linear map. Then ϕ is a Riesz* homomorphism if and only if there is a Riesz homomorphism $\phi^r : E^r \to F^r$ such that $\varphi_F \circ \phi = \phi^r \circ \varphi_E$.*

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, hook, "\varphi_E"'] & F \arrow[d, hook, "\varphi_F"] \\ E^r \arrow[r, "\phi^r"'] & F^r \end{tikzcd}$$

Proof. We will only give the proof of one implication, because we use the ideas of this proof in the proof of Theorem 4.17 on page 32. The proof is taken from [9, Theorem 5.6]. Let $\phi : E \to F$ be a Riesz* homomorphism. Define $\phi^r : E^r \to F^r$ as follows

$$\phi^r(A^{ul} \oplus \ominus B^{ul}) = \phi(A)^{ul} \oplus \ominus \phi(B)^{ul}, A, B \in \mathrm{Fin}(E)\backslash\{\emptyset\}.$$

Claim 3.15. *ϕ^r is well defined.*

Proof of Claim 3.15. Let $A, B, C, D \in \mathrm{Fin}(E)\backslash\{\emptyset\}$ be such that $A^{ul} \oplus \ominus B^{ul} = C^{ul} \oplus \ominus D^{ul}$. Then $(A + D)^{ul} = (B + C)^{ul}$. Thus we have

$$\begin{aligned}
(\phi(A) + \phi(D))^{ul} &= \text{(by linearity of } \phi) \\
\phi(A + D)^{ul} &= \text{(by Proposition 3.13 on the previous page(v))} \\
(\phi((A + D)^{ul}))^{ul} &= \text{(by assumption)} \\
(\phi((B + C)^{ul}))^{ul} &= \text{(by Proposition 3.13 on the preceding page(v))} \\
\phi(B + C)^{ul} &= \text{(by linearity of } \phi) \\
(\phi(B) + \phi(C))^{ul}.
\end{aligned}$$

So $\phi(A)^{ul} \oplus \ominus \phi(B)^{ul} = \phi(C)^{ul} \oplus \ominus \phi(D)^{ul}$. Thus ϕ^r is well defined. □

Claim 3.16. *ϕ^r is linear.*

Proof of Claim 3.16. Let $A, B, C, D \in \mathrm{Fin}(E)\backslash\{\emptyset\}$. Then

$$\begin{aligned}
\phi^r(A^{ul} \oplus \ominus B^{ul} \oplus C^{ul} \oplus \ominus D^{ul}) &= \phi^r((A + C)^{ul} \oplus \ominus (B + D)^{ul}) \\
&= \phi(A + C)^{ul} \oplus \ominus \phi(B + D)^{ul} \\
&= (\phi(A) + \phi(C))^{ul} \oplus \ominus (\phi(B) + \phi(D))^{ul} \\
&= \phi(A)^{ul} \oplus \phi(C)^{ul} \oplus \ominus \phi(B)^{ul} \oplus \ominus \phi(D)^{ul} \\
&= \phi(A)^{ul} \oplus \ominus \phi(B)^{ul} \oplus \phi(C)^{ul} \oplus \ominus \phi(D)^{ul} \\
&= \phi^r(A^{ul} \oplus \ominus B^{ul}) \oplus \phi^r(C^{ul} \oplus \ominus D^{ul}).
\end{aligned}$$

Thus ϕ^r is additive.
Let $\lambda > 0$, then $\lambda \star (A^{ul} \oplus \ominus B^{ul}) = (\lambda A)^{ul} \oplus \ominus (\lambda B)^{ul}$. So $\phi^r(\lambda \star (A^{ul} \oplus \ominus B^{ul})) = \phi(\lambda A)^{ul} \oplus \ominus \phi(\lambda B)^{ul} = \lambda \star (\phi(A)^{ul} \oplus \ominus \phi(B)^{ul}) = \lambda \star \phi^r(A^{ul} \oplus \ominus B^{ul})$. Note that $\phi^r(0) = \phi(0)^{ul} = -F^+ = 0$. For $\lambda < 0$ we have $\lambda \star (A^{ul} \oplus \ominus B^{ul}) = -\lambda \star \ominus (A^{ul} \oplus \ominus B^{ul}) = -\lambda \star (B^{ul} \oplus \ominus A^{ul}) = (-\lambda B)^{ul} \oplus \ominus (-\lambda A)^{ul}$. Thus $\phi^r(\lambda \star (A^{ul} \oplus \ominus B^{ul})) = \phi(-\lambda B)^{ul} \oplus \ominus \phi(-\lambda A)^{ul} = \lambda \star (\phi(A)^{ul} \oplus \ominus \phi(B)^{ul}) = \lambda \star \phi(A^{ul} \oplus \ominus B^{ul})$. □

Claim 3.17. *ϕ^r is a Riesz homomorphism.*

Proof. We have by Proposition 3.7 on page 26

$$\left(A^{ul} \oplus \ominus B^{ul}\right) \vee \left(C^{ul} \oplus \ominus D^{ul}\right) = ((A + D) \cup (B + C))^{ul} \oplus \ominus (B + D)^{ul}. \quad (7)$$

It follows that

$$\begin{aligned}
\phi^r((A^{ul} \oplus \ominus B^{ul}) \vee (C^{ul} \oplus \ominus D^{ul})) &= \phi((A+B) \cup (B+C))^{ul} \oplus \ominus \phi(B+D)^{ul} \\
&= \phi(A+B)^{ul} \vee \phi(B+C)^{ul} \oplus \ominus \phi(B+D)^{ul} \\
&= \left(\phi(A)^{ul} \oplus \ominus \phi(B)^{ul}\right) \vee \left(\phi(C)^{ul} \oplus \ominus \phi(D)^{ul}\right) \\
&= \phi^r(A^{ul} \oplus \ominus B^{ul}) \vee \phi^r(C^{ul} \oplus \ominus D^{ul}).
\end{aligned}$$

We conclude that ϕ^r is a Riesz homomorphism. □

For the proof of the converse statement we refer to [9, the proof of Theorem 5.6]. □

Complete Riesz* homomorphisms between pre-Riesz spaces can be extended to complete Riesz homomorphisms between the Riesz completions. Even more is true, every complete Riesz* homomorphism is also a complete Riesz homomorphism (see [9, Theorem 5.12]).

Theorem 3.18. *Let E and F be pre-Riesz spaces and let $\phi : E \to F$ be linear. Let (E^r, φ_E) and (F^r, φ_F) be the Riesz completions of E and F respectively. Then the following statements are equivalent.*

1. *ϕ is a complete Riesz* homomorphism,*
2. *ϕ is a complete Riesz homomorphism,*
3. *there exists an order continuous Riesz homomorphism $\phi^r : E^r \to F^r$ such that $\phi^r \circ \varphi_E = \varphi_F \circ \phi$.*

$$\begin{tikzcd}
E \arrow[r, "\phi"] \arrow[d, hook, "\varphi_E"'] & F \arrow[d, hook, "\varphi_F"] \\
E^r \arrow[r, "\phi^r"'] & F^r
\end{tikzcd}$$

4 Riesz* bimorphisms

In the previous section we have seen that Riesz* homomorphisms between pre-Riesz spaces are exactly the restrictions of Riesz homomorphisms between the Riesz completions. One would expect that something similar will hold for bimorphisms. Unfortunately it is not that simple. Only in a few cases we can prove that a Riesz* bimorphism between pre-Riesz spaces extends to a Riesz bimorphism between the Riesz completions.

Definition 4.1. Let E, F and G be directed partially ordered vector spaces. Let $b : E \times F \to G$ be a bilinear map. We say that b is a

(i) (complete) Riesz* bimorphism, if $b(x, \cdot)$ is a (complete) Riesz* homomorphism, for every $x \in E^+$, and $b(\cdot, y)$ is a (complete) Riesz* homomorphism, for every $y \in F^+$.

(ii) (complete) Riesz bimorphism, if $b(x, \cdot)$ is a (complete) Riesz homomorphism, for every $x \in E^+$, and $b(\cdot, y)$ is a (complete) Riesz homomorphism, for every $y \in F^+$. (See [3, Definition 3.1].)

The following lemma is useful. (See [12, Lemma 3.1].)

Lemma 4.2. *Let L, M and N be Riesz spaces and let $\phi : L \times M \to N$ be a bilinear map. Then ϕ is a Riesz bimorphism if and only if $|\phi(x, y)| = \phi(|x|, |y|)$, for all $x \in L$ and $y \in M$.*

Lemma 4.3. *Let L, M and N be Riesz spaces and let $\phi : L \times M \to N$ be a Riesz bimorphism. If $a, b \in L$ are orthogonal, then $\phi(a, c) \perp \phi(b, c)$, for all $c \in M$. Likewise, for all $a \in L$ and all orthogonal elements $b, c \in M$ we have $\phi(a, b) \perp \phi(a, c)$.*

Proof. If $a, b \in L$ are orthogonal, then $|a| \wedge |b| = 0$ thus $0 = \phi(|a| \wedge |b|, |c|) = \phi(|a|, |c|) \wedge \phi(|b|, |c|) = |\phi(a, c)| \wedge |\phi(b, c)|$. Therefore $\phi(a, c) \perp \phi(b, c)$. The other statement is proven similarly. □

In order to try to extend a Riesz* bimorphism between pre-Riesz spaces to a Riesz bimorphism between the Riesz completions we need some definitions.

Definition 4.4. A pre-Riesz space E with Riesz completion (E^r, φ) has the finite Van Haandel property if for every $y \in (E^r)^+$ there are finitely many $x_1, \ldots, x_n \in E^+$ such that

$$y = \bigvee_{i=1}^{n} \varphi(x_i).$$

Remark 4.5. All Riesz spaces have the finite Van Haandel property.

Definition 4.6. A pre-Riesz space E with Riesz completion (E^r, φ) has the countable Van Haandel property, if for all $y \in (E^r)^+$ there exist a finite or countable set $A \subset E^+$ such that $y = \sup\{\varphi(x) : x \in A\}$.

Example 4.7. The space $C^1[0, 1]$ of all continuously differentiable fuctions on $\mathbb{R}$ has the countable Van Haandel property.

Proof. The Riesz completion of $C^1[0, 1]$ is the space R of all continuous piecewise differentiable functions on $[0, 1]$. Let $f \in R^+$. Let $x_1, \ldots, x_m \in [0, 1]$ be the points where f is not differentiable. For every $n \in \mathbb{N}$ there is a $g_n \in C^1[0, 1]$ such that $g_n \leq f$ and $f(x) = g_n(x)$ for $x \in [0, 1] \setminus \left(\cup_{i=1}^{m} (x_i - \frac{1}{n}, x_i + \frac{1}{n}) \right)$. Hence $\sup_{n \in \mathbb{N}} g_n = f$. □

Definition 4.8. A pre-Riesz space E with Riesz completion (E^r, φ) has the (usual) Van Haandel property if for every $y \in (E^r)^+$ we have that $y = \sup\{\varphi(x) : x \in E^+, \varphi(x) \leq y\}$.

Example 4.9. Consider the space of affine functions aff $[0, 1]$ on $[0, 1]$, with the pointwise ordering. Its Riesz completion is the space R of all continuous piecewise linear functions f on $[0, 1]$. Then, aff $[0, 1]$ does not have the Van Haandel property. Take for example $f \in R^+$ defined by $f(x) = 1 - 2x$ for $x \in [0, 0.5)$ and $f(x) = -1 + 2x$ for $x \in [0.5, 1]$. Then for every $g \in$ aff $[0, 1]$ with $g \leq f$ we have that $g = 0$ or $g \ngeq 0$.

The benefit of pre-Riesz spaces with the finite, countable or usual Van Haandel property respectively is that every positive element in the Riesz completion is the supremum of a finite, countable or maybe uncountable set of positive elements from the pre-Riesz space, respectively.

Lemma 4.10. *Let E be a pre-Riesz space with the countable Van Haandel property, with Riesz completion (E^r, φ). Let $A^{ul} \oplus \ominus B^{ul} \geq 0$. Let $C \subset E^+$ be a countable set such that $A^{ul} \oplus \ominus B^{ul} = \sup\{\varphi(x) : x \in C\}$. Then $A^{ul} \oplus \ominus B^{ul} = C^{ul}$.*

Proof. Let $E, A^{ul} \oplus \ominus B^{ul}$ and C be as in the Lemma. By definition

$$\begin{aligned} A^{ul} \oplus \ominus B^{ul} &= \sup\{\{x\}^{ul} : x \in C\} \\ &= \left(\bigcup_{x \in C} \{x\}^{ul} \right)^{ul} \\ &= \left(\bigcup_{x \in C} \{y \in E : y \leq x\} \right)^{ul} \\ &= C^{ul}. \end{aligned}$$

□

Lemma 4.11. *Let E a pre-Riesz space with the Van Haandel property, with Riesz completion (E^r, φ_E). Let $A^{ul} \oplus \ominus B^{ul} \geq 0$ be a positive element of the Riesz completion of E. Then $A^{ul} \oplus \ominus B^{ul} = \{x \in E^+ : \{x\}^{ul} \leq A^{ul} \oplus \ominus B^{ul}\}^{ul}$. That is, for every positive element $A^{ul} \oplus \ominus B^{ul} \in (E^r)^+$ there is a bounded set $C \subset E^+$ such that $A^{ul} \oplus \ominus B^{ul} = C^{ul}$.*

Proof. Let E be a pre-Riesz space with the Van Haandel property and Riesz completion (E^r, φ_E). Let $A^{ul} \oplus \ominus B^{ul} \in (E^r)^+$. By definition we have

$$\begin{aligned}
A^{ul} \oplus \ominus B^{ul} &= \sup\{\{x\}^{ul} : x \in E^+, \{x\}^{ul} \leq A^{ul} \oplus \ominus B^{ul}\} \\
&= \left(\bigcup \left\{\{x\}^{ul} : x \in E^+, \{x\}^{ul} \leq A^{ul} \oplus \ominus B^{ul}\right\}\right)^{ul} \\
&= \left(\bigcup \left\{\{y \in E : y \leq x\} : x \in E^+, \{x\}^{ul} \leq A^{ul} \oplus \ominus B^{ul}\right\}\right)^{ul} \\
&= \left\{x \in E^+ : \{x\}^{ul} \leq A^{ul} \oplus \ominus B^{ul}\right\}^{ul}.
\end{aligned}$$

□

Remark 4.12. Clearly, a pre-Riesz space with the finite Van Haandel property has the countable Van Haandel property and a pre-Riesz space with the countable Van Haandel property has the Van Haandel property. The space of all differentiable functions on $[0,1]$ has the countable Van Haandel property, but not the finite Van Haandel property. We do not know whether there is a pre-Riesz space with the Van Haandel property, but not with the countable Van Haandel property. We do also not know whether there is a pre-Riesz space E that is not a Riesz space with the finite Van Haandel property.

In the theory about pre-Riesz spaces there is also a notion of a 'pervasive' pre-Riesz space. It turns out that for integrally closed pre-Riesz spaces having the Van Haandel property is equivalent to being pervasive.

Definition 4.13. A pre-Riesz space E with Riesz completion (E^r, φ_E) is pervasive if for all $y \in E^r$ with $y > 0$ there is an $x \in E$ such that $0 < \varphi_E(x) \leq y$.

Remark 4.14. It is immediately clear that every pre-Riesz space with the finite, countable or usual Van Haandel property is pervasive.

Theorem 4.15. *Let E be an integrally closed pre-Riesz space. If E is pervasive, then E has the Van Haandel property.*

Proof. Let E be an integrally closed pervasive pre-Riesz space. Assume that E is a subspace of its Riesz completion E^r. We argue by contradiction. Suppose E has not the Van Haandel property. Then there is a $y \in (E^r)^+$ such that y is not the supremum of $S = \{x \in E^+ : x \leq y\}$. Note that $y > 0$. Since E is pervasive we have that S is not empty. Thus there is a $z \in E^r$ such that $S \leq z$ and $y \not\leq z$. Let $w = y \wedge z \in E^r$. Clearly we have that $S \leq w$ and $w < y$ thus $0 < y - w$. Since E is pervasive there is a $u \in E$ such that $0 < u \leq y - w$. Thus $w + u \leq y$. Therefore $S + u \leq w + u \leq y$. Thus $S + u \subset S$. Inductively, it follows that $S + nu \subset S$, for all $n \in \mathbb{N}_0$. Let $x \in S$, then $x + nu \leq y$, for all $n \in \mathbb{N}_0$. Thus $nu \leq y - x$, for all $n \in \mathbb{N}_0$. Since E^r is integrally closed we have that $u \leq 0$. Thus $u > 0$ and $u \leq 0$ and that is a contradiction. We conclude that y is the supremum of S. □

Remark 4.14 and Theorem 4.15 have the following corollary.

Corollary 4.16. *Let E be an integrally closed pre-Riesz space. Then E has the Van Haandel property if and only if E is pervasive.*

Corollary 4.16 is very useful in determining whether an integrally closed pre-Riesz space has the Van Haandel property. It is enough to show that for every $y \in E^r$ with $y > 0$ there is an $x \in E$ with $0 < x \leq y$. In general that is much easier than to show that y is the supremum of $\{x \in E^+ : x \leq y\}$ for every $y \in (E^r)^+$.
We try to extend a Riesz* bimorphism between pre-Riesz space to a Riesz homomorphism between the Riesz completions. Under some conditions we can prove that. The theorems and proofs are an analogue to Theorem 3.14 on page 28.

Theorem 4.17. *Let E, F and G be pre-Riesz spaces with Riesz completions $(E^r, \varphi_E), (F^r, \varphi_F)$ and (G^r, φ_G) respectively. Let $b : E \times F \to G$ be a Riesz* bimorphism. Then the map $b^r : E^r \times F^r \to G^r$ defined by*

$$b^r(A^{ul} \oplus \ominus B^{ul}, X^{ul} \oplus \ominus Y^{ul}) = b(A, X)^{ul} \oplus \ominus b(A, Y)^{ul} \oplus \ominus b(B, X)^{ul} \oplus b(B, Y)^{ul},$$

where A, B are non-empty finite subsets of E^+ and X, Y are non-empty finite subsets of F^+, is well defined and satisfies $b^r \circ (\varphi_E, \varphi_F) = \varphi_G \circ b$. Moreover, if either

(i) E and F have have the finite Van Haandel property, or

(ii) E and F are integrally closed, and have the countable Van Haandel property, and E^r and F^r have the sequentially order continuous property, or

(iii) E and F are integrally closed, and have the Van Haandel property, and E^r and F^r have the order continuous property, or

(iv) E and F are integrally closed, and have the Van Haandel property, and b is a complete Riesz homomorphism,*

then b^r is a Riesz bimorphism. In cases (iii) and (iv) b^r is an order continuous Riesz homomorphism.

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[d, hook, "\varphi_E \times \varphi_F"'] & G \arrow[d, hook, "\varphi_G"] \\ E^r \times F^r \arrow[r, "b^r"'] & G^r \end{tikzcd} \tag{8}$$

Proof. Let E, F and G be pre-Riesz spaces with Riesz completion $(E^r, \varphi_E), (F^r, \varphi_F)$ and (G^r, φ_G) respectively. Let $b : E \times F \to G$ be a Riesz* bimorphism. For sets A in E and X in F we let $b(A, X) = \{b(x, y) : x \in A, y \in X\}$.
We will see that

$$b^r(A^{ul} \oplus \ominus B^{ul}, X^{ul} \oplus \ominus Y^{ul}) := b(A, X)^{ul} \oplus \ominus b(A, Y)^{ul} \oplus \ominus b(B, X)^{ul} \oplus b(B, Y)^{ul},$$

with $A, B \in \mathrm{Fin}(E^+)\backslash\{\emptyset\}$ and $X, Y \in \mathrm{Fin}(F^+)\backslash\{\emptyset\}$, is a well defined bilinear map $E^r \times F^r \to G^r$ such that diagram (8) commutes. To do this we prove a sequence of claims. Define, for all $x \in E^+$ and for all $X, Y \in \mathrm{Fin}(F)\backslash\{\emptyset\}$,

$$b^r_x(X^{ul} \oplus \ominus Y^{ul}) = b(\{x\}, X)^{ul} \oplus \ominus b(\{x\}, Y)^{ul}.$$

Since, for every $x \in E^+, b(x, \cdot)$ is a Riesz* homomorphism, by Theorem 3.14 on page 28 $b^r_x : F^r \to G^r$ is a Riesz homomorphism.

Let $A = \{a_1, \ldots, a_n\} \subset E^+$ be a finite set of positive elements of E. Define $b^r_A : F^r \to G^r$ by

$$b^r_A(X^{ul}) = \bigvee_{i=1}^{n} b^r_{a_i}(X^{ul}), X \in \mathrm{Fin}(F)\backslash\{\emptyset\}$$

and

$$b^r_A(X^{ul} \oplus \ominus Y^{ul}) = b^r_A(X^{ul}) \oplus \ominus b^r_A(Y^{ul}), X, Y \in \mathrm{Fin}(F)\backslash\{\emptyset\}.$$

Then

$$b^r_A(X^{ul}) = \bigvee_{i=1}^{n} b(a_i, X)^{ul} = b(A, X)^{ul}$$

and

$$b^r_A(X^{ul})^u = [b(A, X)^{ul}]^u = b(A, X)^u, \tag{9}$$

where the last equality holds by Proposition 1.4 on page 7.

Claim 4.18. *b_A^r is well defined.*

Proof. Let $W, X, Y, Z \in \mathrm{Fin}(F)\backslash\{\emptyset\}$ such that $W^{ul} \oplus \ominus X^{ul} = Y^{ul} \oplus \ominus Z^{ul}$. Then $(W+Z)^{ul} = (X+Y)^{ul}$. So

$$\begin{aligned}
(b(A,W)+b(A,Z))^{ul} &= \text{(by bilinearity of } b)\\
b(A,W+Z)^{ul} &= \text{(by equation (9 on the previous page))}\\
b_A^r((W+Z)^{ul})^{ul} &= \text{(by assumption)}\\
b_A^r((X+Y)^{ul})^{ul} &= \text{(by equation (9 on the preceding page))}\\
b(A,X+Y)^{ul} &= \text{(by bilinearity of } b)\\
(b(A,X)+b(A,Y))^{ul}.
\end{aligned}$$

So

$$b_A^r(W^{ul} \oplus \ominus X^{ul}) = b_A^r(Y^{ul} \oplus \ominus Z^{ul}).$$

□

Claim 4.19. *b_A^r is linear.*

Proof. Let $W, X, Y, Z \in \mathrm{Fin}(F)\backslash\{\emptyset\}$. Then we have

$$\begin{aligned}
b_A^r(W^{ul} \oplus \ominus X^{ul} \oplus Y^{ul} \oplus \ominus Z^{ul}) &=\\
b_A^r((W+Y)^{ul} \oplus \ominus (X+Z)^{ul}) &=\\
b_A^r((W+Y)^{ul}) \oplus \ominus b_A^r((X+Z)^{ul}) &=\\
b(A,W+Y)^{ul} \oplus \ominus b(A,X+Z)^{ul} &=\\
b(A,W)^{ul} \oplus b(A,Y)^{ul} \oplus \ominus b(A,X)^{ul} \oplus \ominus b(A,Z)^{ul} &=\\
b_A^r(W^{ul}) \oplus \ominus b_A^r(X^{ul}) \oplus b_A^r(Y^{ul}) \oplus \ominus b_A^r(Z^{ul}) &=\\
b_A^r(W^{ul} \oplus \ominus X^{ul}) \oplus b_A^r(Y^{ul} \oplus \ominus Z^{ul}).
\end{aligned}$$

So b_A^r is additive.
Let $\lambda > 0$, then $\lambda \star (W^{ul} \oplus \ominus X^{ul}) = (\lambda W)^{ul} \oplus \ominus (\lambda X)^{ul}$. So $b_A^r(\lambda \star (W^{ul} \oplus \ominus X^{ul})) = b(A, \lambda W)^{ul} \oplus \ominus b(A, \lambda X)^{ul} = \lambda \star \left(b(A,W)^{ul} \oplus \ominus b(A,X)^{ul}\right) = \lambda \star b_A^r(W^{ul} \oplus \ominus X^{ul})$.
Note that $b_A(0) = b(A,0)^{ul} = \{0\}^{ul} = 0$.
Let $\lambda < 0$. Then $\lambda \star (W^{ul} \oplus \ominus X^{ul}) = (-\lambda) \star (X^{ul} \oplus \ominus W^{ul})$. So $b_A^r(\lambda \star (W^{ul} \oplus \ominus X^{ul})) = b_A^r((-\lambda) \star (X^{ul} \oplus \ominus W^{ul})) = (-\lambda) \star \left(b_A^r(X^{ul}) \oplus \ominus b_A^r(W^{ul})\right) = \lambda \star \left(b_A^r(W^{ul}) \oplus \ominus b_A^r(X^{ul})\right) = \lambda \star b_A^r(W^{ul} \oplus \ominus X^{ul})$. □

Claim 4.20. *b_A^r is a Riesz homomorphism.*

Proof. We have by Proposition 3.7 on page 26

$$(W^{ul} \oplus \ominus X^{ul}) \vee (Y^{ul} \oplus \ominus Z^{ul}) = ((W+Z) \cup (X+Y))^{ul} \oplus \ominus (X+Z)^{ul}.$$

So

$$\begin{aligned}
b_A^r((W^{ul} \oplus \ominus X^{ul}) \vee (Y^{ul} \oplus \ominus Z^{ul})) &=\\
b(A, (W+Z) \cup (X+Y))^{ul} \oplus \ominus b(A, X+Z)^{ul} &=\\
\left(b(A,W+Z)^{ul} \vee b(A,X+Y)^{ul}\right) \oplus \ominus b(A,X+Z)^{ul} &=\\
\left(b(A,W)^{ul} \oplus \ominus b(A,X)^{ul}\right) \vee \left(b(A,Y)^{ul} \oplus \ominus b(A,Z)^{ul}\right) &=\\
b_A^r(W^{ul} \oplus \ominus X^{ul}) \vee b_A^r(Y^{ul} \oplus \ominus Z^{ul}).
\end{aligned}$$

□

Let $A, B \in \mathrm{Fin}(E)\backslash\{\emptyset\}$. Since A and B are finite and E is directed there is an $h \in E^+$ such that $h \geq -x$, for all $x \in A \cup B$. Thus $h + x \geq 0$, for all $x \in A \cup B$. Note that $A^{ul} \oplus \ominus B^{ul} = A^{ul} \oplus \ominus B^{ul} \oplus \{h\}^{ul} \oplus \ominus \{h\}^{ul} = A^{ul} \oplus \{h\}^{ul} \oplus \ominus (B^{ul} \oplus \{h\}^{ul}) = (A+h)^{ul} \oplus \ominus (B+h)^{ul}$. Define

$$b^r(A^{ul} \oplus \ominus B^{ul}, \cdot) = b^r_{A+h}(\cdot) - b^r_{B+h}(\cdot).$$

Claim 4.21. *$b^r(A^{ul} \oplus \ominus B^{ul}, \cdot)$ is independent of the choice of $h \in E^+$.*

Proof. Let $h, h' \in E^+$ be such that $h + x, h' + x \in E^+$, for all $x \in A \cup B$. Since E is directed, there is an $g \in E^+$ such that $h \leq g$ and $h' \leq g$. Note that $b_{A+g}(X^{ul}) - b_g(X^{ul}) = b_{A+h}(X^{ul}) + b_{g-h}(X^{ul}) - b_g(X^{ul}) = b_{A+h}(X^{ul}) - b_h(X^{ul})$, for $X \in \mathrm{Fin}(F)\backslash\{\emptyset\}$. Likewise $b_{A+h'} - b_{h'}(X^{ul}) = b_{A+g}(X^{ul}) - b_g(X^{ul})$, for $X \in \mathrm{Fin}(F)\backslash\{\emptyset\}$. Thus $b_{A+h}(\cdot) - b_{B+h}(\cdot) = b_{A+h'}(\cdot) - b_{h'}(\cdot)$. □

Claim 4.22. *$b^r(A^{ul} \oplus \ominus B^{ul}, \cdot)$ is well defined.*

Proof. Let $A, B, C, D \in \mathrm{Fin}(E)\backslash\{\emptyset\}$ be such that $A^{ul} \oplus \ominus B^{ul} = C^{ul} \oplus \ominus D^{ul}$ and let $W, X, Y, Z \in \mathrm{Fin}(F)\backslash\{\emptyset\}$ be such that $W^{ul} \oplus \ominus X^{ul} = Y^{ul} \oplus \ominus Z^{ul}$. From our previous claim we may assume that $A, B, C, D \in \mathrm{Fin}(E^+)\backslash\{\emptyset\}$ and $W, X, Y, Z \in \mathrm{Fin}(F^+)\backslash\{\emptyset\}$. So

$$\begin{array}{rcl}
b^r(A^{ul} \oplus \ominus B^{ul}, W^{ul} \oplus \ominus X^{ul}) & = & \\
b^r(A^{ul} \oplus \ominus B^{ul}, W^{ul}) \oplus \ominus b^r(A^{ul} \oplus \ominus B^{ul}, X^{ul}) & = & (\text{ by Claim 4.18 on the preceding page}) \\
b^r(C^{ul} \oplus \ominus D^{ul}, W^{ul}) \oplus \ominus b^r(C^{ul} \oplus \ominus D^{ul}, X^{ul}) & = & \\
b^r(C^{ul} \oplus \ominus D^{ul}, W^{ul} \oplus \ominus X^{ul}) & = & \\
b^r(C^{ul}, W^{ul} \oplus \ominus X^{ul}) \oplus \ominus b^r(D^{ul}, W^{ul} \oplus \ominus X^{ul}) & = & (\text{by Claim 4.18 on the previous page}) \\
b^r(C^{ul}, Y^{ul} \oplus \ominus Z^{ul}) \oplus \ominus b^r(D^{ul}, Y^{ul} \oplus \ominus Z^{ul}) & = & \\
b^r(C^{ul} \oplus \ominus D^{ul}, Y^{ul} \oplus \ominus Z^{ul}). & &
\end{array}$$

So b^r is well defined. □

It is clear from the definition that b^r satisfies $b^r \circ (\varphi_E, \varphi_F) = \varphi_G \circ b$ and that proves the first part of the theorem. Moreover, if E and F have the finite Van Haandel property, then b^r is a Riesz bimorphism.

Now assume further that E, F and G are integrally closed, and that either

(ii) E and F have the countable Van Haandel property and the Riesz completions E^r and F^r have the countable order continuous property, or

(iii) E and F have the Van Haandel property and the Riesz completions E^r and F^r have the order continuous property, or

(iv) E and F have the Van Haandel property and b is a complete Riesz* homomorphism.

Let $(E^\delta, \psi_E), (F^\delta, \psi_F)$ and (G^δ, ψ_G) be the Dedekind completions of E, F and G respectively. Then $H^r \subset H^\delta$ and φ_H is the restriction of ψ_H to H^r, for $H = E, F$ or G.

Let $A \subset E^+$ be a non-empty bounded set, such that $A^{ul} \in E^r$. Define $b^\delta_A : F^r \to G^\delta$ by

$$b^\delta_A(X^{ul}) = \bigvee_{x \in A} b^r_x(X^{ul}), X \in \mathrm{Fin}(E)\backslash\{\emptyset\}$$

and

$$b^\delta_A(X^{ul} \oplus \ominus Y^{ul}) = b^\delta_A(X^{ul}) \oplus \ominus b^\delta_A(Y^{ul}), X, Y \in \mathrm{Fin}(F)\backslash\{\emptyset\}.$$

Then

$$b^\delta_A(X^{ul}) = \bigvee_{x \in A} b(x, X)^{ul} = b(A, X)^{ul}$$

and

$$b^\delta_A(X^{ul})^u = [b(A, X)^{ul}]^u = b(A, X)^u, \tag{10}$$

where the last equality holds by Proposition 1.4 on page 7.

Claim 4.23. *b^δ_A is well defined.*

Proof. Let $W, X, Y, Z \in \mathrm{Fin}(F)\backslash\{\emptyset\}$ such that $W^{ul} \oplus \ominus X^{ul} = Y^{ul} \oplus \ominus Z^{ul}$. Then $(W+Z)^{ul} = (X+Y)^{ul}$. So

$$\begin{array}{rcl} (b(A,W)+b(A,Z))^{ul} & = & \text{(by bilinearity of } b) \\ b(A,W+Z)^{ul} & = & \text{(by equation (10 on the previous page))} \\ b^\delta_A((W+Z)^{ul})^{ul} & = & \text{(by assumption)} \\ b^\delta_A((X+Y)^{ul})^{ul} & = & \text{(by equation (10 on the preceding page))} \\ b(A,X+Y)^{ul} & = & \text{(by bilinearity of } b) \\ (b(A,X)+b(A,Y))^{ul}. & & \end{array}$$

Thus

$$b^\delta_A(W^{ul} \oplus \ominus X^{ul}) = b^\delta_A(Y^{ul} \oplus \ominus Z^{ul}).$$

□

Claim 4.24. *b^δ_A is linear.*

Proof. Let $W, X, Y, Z \in \mathrm{Fin}(F)\backslash\{\emptyset\}$. Then we have

$$\begin{array}{rcl} b^\delta_A(W^{ul} \oplus \ominus X^{ul} \oplus Y^{ul} \oplus \ominus Z^{ul}) & = & \\ b^\delta_A((W+Y)^{ul} \oplus \ominus (X+Z)^{ul}) & = & \\ b^\delta_A((W+Y)^{ul}) \oplus \ominus b^\delta_A((X+Z)^{ul}) & = & \\ b(A,W+Y)^{ul} \oplus \ominus b(A,X+Z)^{ul} & = & \\ b(A,W)^{ul} \oplus b(A,Y)^{ul} \oplus \ominus b(A,X)^{ul} \oplus \ominus b(A,Z)^{ul} & = & \\ b^\delta_A(W^{ul}) \oplus \ominus b^\delta_A(X^{ul}) \oplus b^\delta_A(Y^{ul}) \oplus \ominus b^\delta_A(Z^{ul}) & = & \\ b^\delta_A(W^{ul} \oplus \ominus X^{ul}) \oplus b^\delta_A(Y^{ul} \oplus \ominus Z^{ul}). & & \end{array}$$

Thus b^δ_A is additive.
Let $\lambda > 0$, then $\lambda \star (W^{ul} \oplus \ominus X^{ul}) = (\lambda W)^{ul} \oplus \ominus (\lambda X)^{ul}$. So $b^\delta_A(\lambda \star (W^{ul} \oplus \ominus X^{ul})) = b(A, \lambda W)^{ul} \oplus \ominus b(A, \lambda X)^{ul} = \lambda \star \left(b(A,W)^{ul} \oplus \ominus b(A,X)^{ul}\right) = \lambda \star b^\delta_A(W^{ul} \oplus \ominus X^{ul})$.
Note that $b_A(0) = b(A,0)^{ul} = \{0\}^{ul} = 0$.
Let $\lambda < 0$. Then $\lambda \star (W^{ul} \oplus \ominus X^{ul}) = (-\lambda) \star (X^{ul} \oplus \ominus W^{ul})$. So $b^\delta_A(\lambda \star (W^{ul} \oplus \ominus X^{ul})) = b^\delta_A((-\lambda) \star (X^{ul} \oplus \ominus W^{ul})) = (-\lambda) \star \left(b^\delta_A(X^{ul}) \oplus \ominus b^\delta_A(W^{ul})\right) = \lambda \star \left(b^\delta_A(W^{ul}) \oplus \ominus b^\delta_A(X^{ul})\right) = \lambda \star b^\delta_A(W^{ul} \oplus \ominus X^{ul})$. □

Claim 4.25. *b^δ_A is a Riesz homomorphism.*

Proof. We have by Proposition 3.7 on page 26

$$(W^{ul} \oplus \ominus X^{ul}) \vee (Y^{ul} \oplus \ominus Z^{ul}) = ((W+Z) \cup (X+Y))^{ul} \oplus \ominus (X+Z)^{ul}.$$

Thus

$$\begin{array}{rcl} b^\delta_A((W^{ul} \oplus \ominus X^{ul}) \vee (Y^{ul} \oplus \ominus Z^{ul})) & = & \\ b(A, (W+Z) \cup (X+Y))^{ul} \oplus \ominus b(A, X+Z)^{ul} & = & \\ \left(b(A,W+Z)^{ul} \vee b(A,X+Y)^{ul}\right) \oplus \ominus b(A,X+Z)^{ul} & = & \\ \left(b(A,W)^{ul} \oplus \ominus b(A,X)^{ul}\right) \vee \left(b(A,Y)^{ul} \oplus \ominus b(A,Z)^{ul}\right) & = & \\ b^\delta_A(W^{ul} \oplus \ominus X^{ul}) \vee b^\delta_A(Y^{ul} \oplus \ominus Z^{ul}). & & \end{array}$$

□

Now we consider the several cases.

(ii) Now assume that E and F have the countable Van Haandel property, and that E^r and F^r have the countable continuous property. Let $A, B \in \mathrm{Fin}(E^+)\backslash\{\emptyset\}$ be such that $A^{ul} \oplus \ominus B^{ul} \geq 0$. Since E has the countable Van Haandel property, by Lemma 4.10 on page 30 there is a *countable* bounded set $C \subset E^+$ such that $A^{ul} \oplus \ominus B^{ul} = C^{ul}$.
Let $X \in \mathrm{Fin}(F^+)\backslash\{\emptyset\}$. Then $b^r(\cdot, X^{ul})$ is a Riesz homomorphism. Since E^r has the σ-order continuous property, $b^r(\cdot, X^{ul})$ is σ-order continuous. Thus $b^r(A^{ul} \oplus \ominus B^{ul}, X^{ul}) = b^r(\bigvee_{x\in C}\{x\}^{ul}, X^{ul}) = \bigvee_{x\in C} b^r(\{x\}^{ul}, X^{ul}) = b(C, X)^{ul}$. Hence $b^r(A^{ul} \oplus \ominus B^{ul}, \cdot) = b^\delta_C(\cdot)$. It follows that $b^r(A^{ul} \oplus \ominus B^{ul}, \cdot)$ is a Riesz homomorphism. Likewise, for all $X, Y \in \mathrm{Fin}(F^+)\backslash\{\emptyset\}$ with $X^{ul} \oplus \ominus Y^{ul} \geq 0$ we have that $b^r(\cdot, X^{ul} \oplus \ominus Y^{ul})$ is a Riesz homomorphism. We conclude that b^r is a Riesz bimorphism.

(iii) and (iv) Suppose that E and F have the Van Haandel property and that E^r and F^r have the order continuous property or b is a complete Riesz* bimorphism. Let $A, B \in \mathrm{Fin}(E^+)\backslash\{\emptyset\}$ be such that $A^{ul} \oplus \ominus B^{ul} \geq 0$. Since E has the Van Haandel property, by Lemma 4.11 on page 31 there exists a bounded set $C \subset E^+$ such that $A^{ul} \oplus \ominus B^{ul} = C^{ul}$.
Let $X \in \mathrm{Fin}(F^+)\backslash\{\emptyset\}$. Then $b^r(\cdot, X^{ul})$ is a Riesz homomorphism. If E^r has the order continuous property, then $b^r(\cdot, X^{ul})$ is order continuous. If b is a complete Riesz* bimorphism, then by [9, Theorem 5.12] $b^r(\cdot, X^{ul})$ is order continuous. So $b^r(A^{ul} \oplus \ominus B^{ul}, X^{ul}) = b^r(\bigvee_{x\in C}\{x\}^{ul}, X^{ul}) = \bigvee_{x\in C} b^r(\{x\}^{ul}, X^{ul}) = b(C, X)^{ul}$. Thus $b^r(A^{ul} \oplus \ominus B^{ul}, \cdot) = b^\delta_C(\cdot)$. It follows that $b^r(A^{ul} \oplus \ominus B^{ul}, \cdot)$ is a Riesz homomorphism. Similarly, for $X, Y \in \mathrm{Fin}(F^+)\backslash\{\emptyset\}$ with $X^{ul} \oplus \ominus Y^{ul} \geq 0$, we have that $b^r(\cdot, X^{ul} \oplus \ominus Y^{ul})$ is a Riesz homomorphism. We conclude that b^r is a Riesz bimorphism. If fact b^r is an order continuous Riesz homomorphism.

□

We need restrictive assumptions on E and F to prove that a Riesz* bimorphism $b : E \times F \to G$ extends to a Riesz bimorphism between the Riesz completions. The converse is true in general, even if the spaces are not integrally closed.

Theorem 4.26. *Let E, F and G be pre-Riesz spaces with Riesz completions $(E^r, \varphi_E), (F^r, \varphi_F)$ and (G^r, φ_G) respectively. Let $b : E \times F \to G$ be a bilinear map and $b^r : E^r \times F^r \to G^r$ a (complete) Riesz bimorphism such that $\varphi_G \circ b = b^r \circ (\varphi_E, \varphi_F)$. Then b is a (complete) Riesz* bimorphism.*

$$\begin{tikzcd}
E \times F \arrow[r, "b"] \arrow[d, hook, "\varphi_E \times \varphi_F"'] & G \arrow[d, hook, "\varphi_G"] \\
E^r \times F^r \arrow[r, "b^r"'] & G^r
\end{tikzcd}$$

Proof. Let E, F and G be pre-Riesz spaces with Riesz completions $(E^r, \varphi_E), (F^r, \varphi_F)$ and (G^r, φ_G) respectively. Let $b : E \times F \to G$ be a bilinear map and let $b^r : E^r \times F^r \to G^r$ be a (complete) Riesz bimorphism such that $\varphi_G \circ b = b^r \circ (\varphi_E, \varphi_F)$.
Let $x \in E^+$. Then $\varphi_G(b(x, \cdot)) = b^r(\varphi_E(x), \varphi_F(\cdot))$. Since $b^r(\varphi_E(x), \cdot) : F^r \to G^r$ is a (complete) Riesz homomorphism, by Theorem 3.14 on page 28 (Theorem 3.18 on page 29) $b(x, \cdot) : F \to G$ is a (complete) Riesz* homomorphism. Likewise, for every $y \in F^+ : b(\cdot, y) : E \to G$ is a (complete) Riesz* homomorphism. So b is a (complete) Riesz* bimorphism. □

A combination of Theorem 4.17 on page 32 and Theorem 4.26 gives the following theorem.

Theorem 4.27. *Let E, F and G be pre-Riesz spaces with Riesz completions $(E^r, \varphi_E), (F^r, \varphi_F)$ and (G^r, φ_G) respectively. Let $b : E \times F \to G$ be bilinear. Suppose that either*

(i) E and F have have the finite Van Haandel property, or

(ii) E and F are integrally closed, and have the countable Van Haandel property, and E^r and F^r have the sequentially order continuous property, or

(iii) E and F are integrally closed, and have the Van Haandel property, and E^r and F^r have the order continuous property,

then b is a Riesz bimorphism if and only if there exists a Riesz bimorphism $b^r : E^r \times F^r \to G^r$ such that $\varphi_G \circ b = b^r \circ (\varphi_E, \varphi_F)$. Moreover, if E and F are integrally closed and have the Van Haandel property, then b is a complete Riesz* bimorphism if and only if b^r is a complete Riesz bimorphism.*

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[d, hook, "\varphi_E \times \varphi_F"'] & G \arrow[d, hook, "\varphi_G"] \\ E^r \times F^r \arrow[r, "b^r"'] & G^r \end{tikzcd}$$

5 The Archimedean completion

The main idea of Riesz completions is to make a Riesz space of a directed partially ordered vector space, since we understand the first better. Something similar can be done with non-Archimedean partially ordered vector spaces to make them Archimedean.

5.1 The Archimedean completion of a partially ordered vector space

Definition 5.1. Let E be a partially ordered vector space. The Archimedean completion of E is a pair (F, ι) where F is an Archimedean partially ordered vector space and $\iota : E \to F$ is a positive linear map such that for all Archimedean partially ordered vector spaces G and for any positive linear map $\phi : E \to G$ there exists a unique positive linear map $\phi_* : F \to G$ with $\phi = \phi_* \circ \iota$.

$$\begin{tikzcd} E \arrow[r, "\iota"] \arrow[dr, "\phi"'] & F \arrow[d, "\phi_*"] \\ & G \end{tikzcd} \tag{11}$$

Proposition 5.2. *Let E be a partially ordered vector space and suppose (F, ι) and (G, j) are Archimedean completions of E. Then there exists a unique order isomorphism $\phi : F \to G$ with $j = \phi \circ \iota$.*

Proof. The proof is similar to the proof of Lemma 3.2 on page 25. ☐

Remark 5.3. If E is an Archimedean partially ordered vector space and id_E is the identity map on E, then (E, id_E) is the Archimedean completion of E.

For every partially ordered vector space E, the Archimedean completion exists.

Theorem 5.4. *Let E be a partially ordered vector space. Let I be the intersection of all relatively uniformly closed order ideals that contain $\{0\}$. Let $E^a = E/I$ and let $\iota = q : E \to E^a$ be the quotient map. Then (E^a, ι) is the Archimedean completion of E.*

Proof. Let E be a partially ordered vector space. Let I be the intersection of all ru-closed order ideals that contain $\{0\}$. Let $E^a = E/I$ and let $\iota = q : E \to E^a$ be the quotient map. Note that ι is positive. By Theorem 1.23 on page 13 E^a is Archimedean. Let G be an arbitrary Archimedean partially ordered vector space and let $\phi : E \to G$ be positive. By Proposition 1.16 on page 11 $\ker\phi$ is an order ideal and by Proposition 1.25 on page 14 $\ker\phi$ is relatively uniformly closed. Clearly $0 \in \ker\phi$, so $I \subset \ker\phi$. Define $\phi_* : E^a \to G$ by $\phi_*(q(x)) = \phi(x)$. Suppose for $x, y \in E$ we have $q(x) = q(y)$, then $x - y \in I \subset \ker\phi$, so $\phi(x) = \phi(y)$. So ϕ_* is well defined and clearly a positive linear map, moreover $\phi = \phi_* \circ \iota$. Let $\psi : E^a \to G$ be any positive linear map such that

$\phi = \psi \circ \iota$. Thus $\psi(q(x)) = \phi(x) = \phi_*(q(x))$, for $x \in E$. It follows that $\psi = \phi_*$. Hence (E^a, ι) is the Archimedean completion of E.

$$\begin{tikzcd} E^a = E/I \arrow[r, "\phi_*"] & G \\ & E \arrow[u, "\phi"'] \arrow[ul, "\iota=q"] \end{tikzcd}$$

□

Example 5.5. Consider $\mathbb{R}^2$ with the following vector space ordering, $x = (x_1, x_2) \le y = (y_1, y_2)$ if and only if $x_1 < y_1$ or $x = y$. Thus $K = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\} \cup \{0\}$ is the positive cone. We have for all $n \in \mathbb{Z}$ that $n(0,1) = (0,n) \le (1,0)$, but $(0,1) \neq 0$ thus $E = (\mathbb{R}^2, K)$ is not Archimedean. Let us calculate the Archimedean completion of E. Since E is not Archimedean $\{0\}$ is not relatively uniformly closed. Clearly, $\mathbb{R}^2$ is a relatively uniformly closed ideal. Let $(x_1, x_2) \in K \backslash \{0\}$ and consider $\ell = \operatorname{Span}\{(x_1, x_2)\}$. Note that $(2x_1, 2x_2 + 1)$ is in the order interval $[(x_1, x_2), (3x_1, 3x_2)]$ and that $(x_1, x_2), (3x_1, 3x_2) \in \ell$, but $(2x_1, 2x_2 + 1) \notin \ell$, and hence ℓ is not an order ideal. Note that any two mutually different elements of $\ell' = \operatorname{Span}\{(0,1)\}$ are not comparable, and hence it is trivial, that ℓ' is an order ideal. Let $\{(0, x_n)\}_{n=1}^{\infty}$ be any sequence in ℓ' that relatively uniformly converges to some $y = (y_1, y_2) \in E$. So there is a $u = (u_1, u_2) \in K$ and a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ such that $\varepsilon_n \downarrow 0$ and $-\varepsilon_n u \le (0, x_n) - (y_1, y_2) = (-y_1, x_n - y_2) \le \varepsilon_n u$, for all $n \in \mathbb{N}$. Thus for any $n \in \mathbb{N}$ we either have that $-\varepsilon_n u_1 < -y_1 < \varepsilon_n u_1$ or $\pm\varepsilon_n u = (-y_1, x_n - y_2)$. It follows that $-\varepsilon_n u_1 \le -y_1 \le \varepsilon_n u_1$, for all $n \in \mathbb{N}$. We conclude that $y_1 = 0$, thus $y \in \ell'$, and hence ℓ' is a relatively uniformly closed order ideal. Moreover let $\phi : E \to \mathbb{R}$ be defined through $(x_1, x_2) \mapsto x_1$. From Theorem 5.4 on the preceding page follows that $(\mathbb{R}, \phi)$ is the Archimedean completion of E.

Remark 5.6. An Archimedean partially ordered vector space need not be integrally closed. An interesting question is whether there is something like an 'integrally closed completion' of a partially ordered vector space.

5.2 The Archimedean completion of a Riesz space

In this section we consider the Archimedean completion of a Riesz space.

Definition 5.7. Let L be a Riesz space. The Archimedean completion of L is a pair (M, ι) where M is an Archimedean Riesz space and $\iota : L \to M$ is a Riesz homomorphism such that for every Archimedean Riesz space N and for every Riesz homomorphism $\phi : L \to N$ there is a unique Riesz homomorphism $\phi_* : M \to N$ with $\phi = \phi_* \circ \iota$.

$$\begin{tikzcd} L \arrow[r, "\iota"] \arrow[dr, "\phi"'] & M \arrow[d, "\phi_*"] \\ & N \end{tikzcd}$$

Proposition 5.8. *Let L be a Riesz space and suppose that (M, ι) and (N, j) are Archimedean completions of L. Then there is a unique surjective Riesz isomorphism $\phi : M \to N$ with $j = \phi \circ \iota$.*

Proof. The proof is similar to the proof of Lemma 3.2 on page 25. □

Remark 5.9. Let L be an Archimedean Riesz space and let id_L be the identity map on L. Then, (L, id_L) is the Archimedean completion of L.

Theorem 5.10. *Let L be a Riesz space. Let I be the intersection of all relatively uniformly closed Riesz ideals that contain $\{0\}$. Let $L^a = L/I$ and let $\iota = q$ be the surjective quotient Riesz homomorphism. Then (L^a, ι) is the Archimedean completion of L.*

Proof. Let L be a Riesz space. Let I be the intersection of all relatively uniformly closed Riesz ideals that contain $\{0\}$. Then I is a relatively uniformly closed Riesz ideal. Let $L^a = L/I$ and let $\iota = q$ be the quotient Riesz homomorphism. Let N be an Archimedean Riesz space and let $\phi : L \to N$ be a Riesz homomorphism. Define $\phi_* : L^a \to N$ by $\phi_*(q(x)) = \phi(x)$.

Claim 5.11. *ϕ_* is well defined.*

Proof of claim 5.11. Suppose that $q(x) = q(y)$, for $x, y \in L$. Then $x - y \in I$. Since N is Archimedean and ϕ is a Riesz homomorphism, we have by Propositions 1.25 on page 14 and 1.16 on page 11 that $\ker \phi$ is a relatively uniformly closed ideal. Since $0 \in \ker \phi$ we have $I \subset \ker \phi$. Thus $\phi(x - y) = 0$, so $\phi(x) = \phi(y)$ and $\phi_*(q(x)) = \phi_*(q(y))$. □

It is clear from the definition that ϕ_* is linear. We have $\phi_*(|q(x)|) = \phi_*(q(|x|)) = \phi(|x|) = |\phi(x)| = |\phi_*(q(x))|$, for all $x \in L$. Thus ϕ_* is a well defined Riesz homomorphism that satisfies $\phi = \phi_* \circ \iota$. It remains to prove is that ϕ_* is the unique Riesz homomorphism ψ that satisfies $\phi = \psi \circ \iota$. Let $\psi : L^a \to N$ be such a Riesz homomorphism. Then $\psi(q(x)) = \phi(x) = \phi_*(q(x))$, for all $x \in L$. Thus $\psi = \phi_*$. We conclude that (L^a, ι) is the Archimedean completion of L.

$$\begin{tikzcd}
L^a = L/I \arrow[r, "\phi_*"] & N \\
& L \arrow[ul, "\iota = q"] \arrow[u, "\phi"']
\end{tikzcd}$$

□

Corollary 5.12. *Let L be a Riesz space. Then the Archimedean completion as partially ordered vector space and the Archimedean completion as Riesz space coincide.*

6 The Archimedean Riesz tensor product

The usual vector space tensor product can be constructed as a quotient of a free space. In this thesis we study the tensor product of Banach lattices and integrally closed pre-Riesz spaces as a quotient of a free space. The tensor product of Riesz spaces as a quotient space of a free space is constructed by B. de Pagter [12]. We give the main ideas and prove some properties of the Archimedean Riesz tensor product that will be usefull later on.
The following definition is taken from [12].

Definition 6.1. Let L and M be (Archimedean) Riesz spaces. The (Archimedean) Riesz tensor product of L and M is a pair (T, b) where T is an (Archimedean) Riesz space and $b : L \times M \to T$ a Riesz bimorphism, such that for every (Archimedean) Riesz space N and every Riesz bimorphism $\phi : L \times M \to N$ there is a unique Riesz homomorphism $\phi_* : T \to N$ such that $\phi = \phi_* \circ b$.

$$\begin{tikzcd}
L \times M \arrow[r, "b"] \arrow[dr, "\phi"'] & T \arrow[d, "\phi_*"] \\
& N
\end{tikzcd}$$

The next theorem can be found in [12]

Theorem 6.2. *The (Archimedean) Riesz tensor product is unique if it exists, in the following sense: let L and M be (Archimedean) Riesz spaces and suppose that (S, b) and (T, c) are two (Archimedean) Riesz tensor products of L and M, then there is a unique bijective linear map $\phi : S \to T$ such that ϕ and ϕ^{-1} are Riesz homomorphisms and $\phi \circ b = c$. In particular, ϕ is an order isomorphism.*

Proof. Let L and M be (Archimedean) Riesz spaces and suppose that (S, b) and (T, c) are two (Archimedean) Riesz tensor products of L and M. By definition there is unique Riesz homomorphism $c_* : S \to T$ such that $c = c_* \circ b$ and a unique Riesz homomorphism $b_* : T \to S$ such that $b = b_* \circ c$. So $b = b_* \circ c = b_* \circ c_* \circ b$. Note that $b_* \circ c_* : S \to S$ is a Riesz homomorphism and that also the identity map id_S on S is a Riesz homomorphism such that $b = id_S \circ b$. From the uniqueness statement it follows that $b_* \circ c_* = id_S$. Likewise we have that $c_* \circ b_*$ is the identity map on T. Define $\phi = c_* : S \to T$. Then ϕ is bijective, ϕ and $\phi^{-1} = b_* : T \to S$ are Riesz homomorphisms, $c = \phi \circ b$ and ϕ is the unique Riesz homomorphism with these properties. In particular ϕ is an order isomorphism.

$$\begin{tikzcd} L \times M \arrow[r, "b"] \arrow[dr, "c"'] & S \arrow[d, bend right, "c_*"'] \\ & T \arrow[u, bend right, "b_*"'] \end{tikzcd}$$

□

Theorem 6.3. *Let L and M be (Archimedean) Riesz spaces. If (T, b) is the (Archimedean) Riesz tensor product of L and M, then T is generated as Riesz space by elements $b(x, y), x \in L, y \in M$.*

Proof. Let L, M and (T, b) be as in the theorem. Let S be the Riesz subspace of T generated by the elements $b(x, y), x \in L, y \in M$. Let N be an arbitrary (Archimedean) Riesz space and $\phi : L \times M \to N$ a Riesz bimorphism. Let $\phi'_* : T \to N$ be the unique Riesz homomorphism with $\phi = \phi'_* \circ b$. Note that b maps into S. Let $\phi_* : S \to N$ be the restriction of ϕ'_* to S. Then ϕ_* is a Riesz homomorphism and $\phi = \phi_* \circ b$. Let $\psi : S \to N$ be any Riesz homomorphism with $\phi = \psi \circ b$. For all $x \in L$ and $y \in M$ we have that $\phi_*(b(x, y)) = \psi(b(x, y))$. Since S is generated as Riesz space by elements $b(x, y), x \in L, y \in M$, we have that $\phi_* = \psi$. Thus (S, b) is also the (Archimedean) Riesz tensor product of L and M. From Theorem 6.2 on the preceding page follows that $S = T$. This concludes our proof. □

With thanks to B. de Pagter [12] we have the following nice representation of the (Archimedean) Riesz tensor product.

Theorem 6.4. *Let L and M be (Archimedean) Riesz spaces. Let J be the ideal in $FRS(L \times M)$ generated by the elements*

$$\begin{cases} \iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z), & x, y \in L, z \in M, \alpha, \beta \in \mathbb{R}, \\ \iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z), & x \in L, y, z \in M, \alpha, \beta \in \mathbb{R}, \\ |\iota(x, y)| - \iota(|x|, |y|), & x \in L, y \in M. \end{cases}$$

Let $J_1 = J$, and in the Archimedean case, let J_1 be the uniform closure of J in $FRS(L \times M)$. Let $T := FRS(L \times M)/J_1$ and let $q : FRS(L \times M) \to T$ be the quotient map. Let $b : L \times M \to T$ be defined by $b(x, y) = q(\iota(x, y))$. Then b is a Riesz bimorphism and (T, b) is the (Archimedean) Riesz tensor product of L and M.

Proof. Let L and M be (Archimedean) Riesz spaces and let J, J_1, T, q and b be as in the theorem.

Claim 6.5. *b is a Riesz bimorphism.*

Proof of Claim 6.5. Note that for all $x, y \in L$ and $z \in M$ and for all $\alpha, \beta \in \mathbb{R}$, we have that

$$\begin{aligned} b(\alpha x + \beta y, z) &= q(\iota(\alpha x + \beta y, z)) \\ &= q(\alpha\iota(x, z) + \beta\iota(y, z)) \\ &= \alpha q(\iota(x, z)) + \beta q(\iota(y, z)) \\ &= \alpha b(x, z) + \beta b(y, z). \end{aligned}$$

Likewise, for all $x \in L$ and $y, z \in M$ and for all $\alpha, \beta \in \mathbb{R}$ we have that $b(x, \alpha y + \beta z) = \alpha b(x, y) + \beta b(y, z)$. Thus b is bilinear.
Note that for all $x \in L$ and $y \in M$ we have

$$\begin{aligned} b(|x|, |y|) &= q(\iota(|x|, |y|)) \\ &= q(|\iota(x, y)|) \\ &= |q(\iota(x, y))| \\ &= |b(x, y)| \end{aligned}$$

By Lemma 4.2 on page 29 we have that b is a Riesz bimorphism. □

Let N be an arbitrary (Archimedean) Riesz space. Let $\phi : L \times M \to N$ be a Riesz bimorphism. By definition, there is a unique Riesz homomorphism $\psi : \mathrm{FRS}(L \times M) \to N$ such that $\phi = \psi \circ \iota$.

Claim 6.6. $J_1 \subset \ker \psi$.

Proof of Claim 6.6. Since ϕ is a Riesz homomorphism, we have that $\phi(\alpha x + \beta y, z) - \alpha\phi(x, z) - \beta\phi(y, z) = 0$ for all $x, y \in L, z \in M$ and $\alpha, \beta \in \mathbb{R}$. Thus

$$\begin{aligned} 0 &= \phi(\alpha x + \beta y, z) - \alpha\phi(x, z) - \beta\phi(y, z) \\ &= \psi(\iota(\alpha x + \beta y, z)) - \alpha\psi(\iota(x, z)) - \beta\psi(\iota(y, z)) \\ &= \psi(\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z)) \end{aligned}$$

So $\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z) \in \ker\psi$. Likewise, for all $x \in L$ and for all $y, z \in M$ and for all $\alpha, \beta \in \mathbb{R}$ we have that $\iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z) \in \ker\psi$. Since ϕ is a Riesz bimorphism, by Proposition 4.2 on page 29 we have

$$\begin{aligned} 0 &= |\phi(x, y)| - \phi(|x|, |y|) \\ &= |\psi(\iota(x, y))| - \psi(\iota(|x|, |y|)) \\ &= \psi(|\iota(x, y)|) - \psi(\iota(|x|, |y|)) \\ &= \psi(|\iota(x, y)| - \iota(|x|, |y|)), \end{aligned}$$

for all $x \in L$ and $y \in M$. Thus $|\iota(x, y)| - \iota(|x|, |y|) \in \ker\psi$. By Proposition 1.16 on page 11 $\ker\psi$ is an ideal and it contains the set that generates $J = J_1$, so $J_1 \subset \ker\psi$, and in the Archimedean case by Proposition 1.25 on page 14, we have that $\ker\psi$ is relatively uniformly closed too, thus $J_1 \subset \ker\psi$. □

Define $\phi_* : T \to N$ by $\phi_*(q(x)) = \psi(x), x \in \mathrm{FRS}(L \times M)$. Suppose that $q(x) = q(y)$, for $x, y \in \mathrm{FRS}(L \times M)$, then $x - y \in J_1 \subset \ker\psi$, so $\psi(x) = \psi(q)$. Thus ϕ_* is well defined and clearly a Riesz homomorphism.
Suppose $\chi : T \to N$ is a Riesz homomorphism that satisfies $\phi = \chi \circ b$. Then, for all $x \in L, y \in M$, we have $\chi(b(x, y)) = \chi(q(\iota(x, y)) = \phi(x, y) = \phi_*(q(\iota(x, y)) = \psi(\iota(x, y))$. By Theorem 2.5 on page 20, $\mathrm{FRS}(L \times M)$ is generated as Riesz space, by $\{\iota(x, y) : x \in L, y \in M\}$ and $\chi \circ q$ is a Riesz homomorphism. Thus we have that $\chi \circ q = \psi$. Thus $\chi = \phi_*$. It follows that (T, b) is the (Archimedean) Riesz tensor product of L and M.

$$\begin{tikzcd}
L \times M \arrow[r, "\phi"] \arrow[d, hook, "\iota"'] & N \\
\mathrm{FRS}(L \times M) \arrow[ur, "\psi"] \arrow[r, "q"'] & T = \mathrm{FRS}(L \times M)/J_1 \arrow[u, "\phi_*"']
\end{tikzcd}$$

□

Remark 6.7.
1. If L and M are Archimedean Riesz spaces, then both the Riesz tensor product and the Archimedean Riesz tensor product exists, but are in general not isomorphic, see [12].
2. For (Archimedean) Riesz spaces L and M we denote the Riesz tensor product by $L\tilde{\otimes}M$ and the Archimedean Riesz tensor product by $L\bar{\otimes}M$. The constructed Riesz bimorphism b in the previous theorem is denoted by $\otimes$. For $x \in L$ and $y \in M$ we define $x \otimes y = b(x, y)$.
3. From the construction follows that $L\tilde{\otimes}M$ is Riesz isomorphic to $M\tilde{\otimes}L$ and that $L\bar{\otimes}M$ is Riesz isomorphic to $M\bar{\otimes}L$.

Theorem 6.8. *Suppose L and M are Riesz spaces, L' is a Riesz subspace of L and M' is a Riesz subspace of M. Let (T, b) be the (Archimedean) Riesz tensor product of L and M. Let S be the Riesz subspace of T generated by elements $b(x, y), x \in L', y \in M'$. Let $c = b|_{L' \times M'}$. Then (S, c) is the (Archimedean) Riesz tensor product of L' and M'.*

Proof. Let L and M be (Archimedean) Riesz spaces. Let L' a Riesz subspace of L and M' a Riesz subspace of M. Let (T', b') the (Archimedean) Riesz tensor product of L' and M' and let (T, b) be the (Archimedean) Riesz tensor product of L and M. Note that $c = b|_{L'\times M'} : L' \times M' \to T$ is a Riesz bimorphism. By definition there is a Riesz homomorphism $c_* : T' \to T$ such that $c_* \circ b' = c$. Thus $b(x, y) = c_*(b'(x, y))$, for all $x \in L', y \in M'$. Thus $c_* : T' \to T$ is injective. It follows that we can view T' via c_* as a Riesz subspace of T generated by elements $b(x, y), x \in L', y \in M'$.

$$
\begin{tikzcd}
L' \times M' \arrow[r, "b'"] \arrow[d, hook] \arrow[dr, "c"] & T' \arrow[d, "c_*"] \\
L \times M \arrow[r, "b"'] & T
\end{tikzcd}
$$

□

In the rest of this section let E and F denote integrally closed pre-Riesz spaces. Let (E^r, ι_E) and (F^r, ι_F) be the Riesz completions of E and F respectively.

In the following we will prove some results that we need for the construction of the integrally closed Riesz* tensor product in the next section.

The free Riesz space over the set $E^r \times F^r$ exists and is given by the pair $(\mathrm{FRS}(E^r \times F^r), \iota)$ where $\mathrm{FRS}(E^r \times F^r)$ is the Riesz subspace of $\mathbb{R}^{\mathbb{R}^{E^r\times F^r}}$ generated by elements $\iota(x, y)$ defined through $\iota(x, y)(f) = f((x, y)), (x, y) \in E^r \times F^r, f \in \mathbb{R}^{E^r\times F^r}$ and $\iota : E^r \times F^r \to \mathrm{FRS}(E^r \times F^r)$ is given by $(x, y) \mapsto \iota(x, y)$. Then ι is injective. The Archimedean Riesz tensor product T^r of E^r and F^r exists. Let J be the order ideal in the free Riesz space $\mathrm{FRS}(E^r \times F^r)$ generated by the elements

$$
\begin{cases}
\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z), & x, y \in E^r, z \in F^r, \alpha, \beta \in \mathbb{R}, \\
\iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z), & x \in E^r, y, z \in F^r, \alpha, \beta \in \mathbb{R}, \\
|\iota(x, y)| - \iota(|x|, |y|), & x \in E^r, y \in F^r.
\end{cases}
\tag{12}
$$

Define J_1 to be the uniform closure of J in $\mathrm{FRS}(E^r \times F^r)$). Let $T^r = \mathrm{FRS}(E^r \times F^r)/J_1$. Let $q : \mathrm{FRS}(E^r \times F^r) \to T^r$ be the quotient map. Then q is a Riesz homomorphism. Let $b^r : E^r \times F^r \to T^r$ be defined as follows $b^r(x, y) = q(\iota(x, y))$. Then b^r is a Riesz bimorphism and (T^r, b^r) is the Archimedean Riesz tensor product of E^r and F^r. Let for an element $x \in \mathrm{FRS}(E^r \times F^r), [x] := q(x)$ denotes the equivalence class of x. For $a, b \in T^r$ $a \leq b$ if and only if there are $x, y \in \mathrm{FRS}(E^r \times F^r)$ such that $x \leq y$ and $a = [x]$ and $b = [y]$.

Let T be the linear span of elements $b^r(\iota_E(x), \iota_F(y)) \in T^r, x \in E, y \in F$. Note that $b^r(\iota_E(x), \iota_F(y)) = q(\iota(\iota_E(x), \iota_F(y))) = [\iota(\iota_E(x), \iota_F(y))]$.

Later we will prove that T is the positive tensor product of E and F, and we need it for the construction of the Riesz* tensor product. We will see that T^r is the Riesz completion of T.

Remark 6.9. Note that T is directed. Since T is a subspace of an integrally closed partially ordered vector space, T is integrally closed. Thus by Proposition 1.7(iv) on page 8 T is an integrally closed pre-Riesz space.

Let $G = q^{-1}[T]$.

Theorem 6.10. *G is an integrally closed directed partially ordered vector space and $[G] = T$. In particular G is pre-Riesz.*

Proof. G is a subspace of an integrally closed partially ordered vector space and hence integrally closed. It is clear that $[G] = T$. Let $x, y \in G$. Since T is directed, there is an $Z \in T^+$ such that $[x] \leq Z$ and $[y] \leq Z$. So there are $z_1, z_2 \in G$ such that $[z_1] = [z_2] = Z$ and $x \leq z_1$ and $y \leq z_2$.

Since $Z \geq 0$ there are $h_1, h_2 \in J_1$ such that $z_1 + h_1 \geq 0$ and $z_2 + h_2 \geq 0$. Since J_1 is directed there is an $h \in J_1$ such that $h \geq h_1$ and $h \geq h_2$. Thus $z_1 + h \geq 0$ and $z_2 + h \geq 0$. It follows that $x \leq z_1 \leq z_1 + z_2 + h$, and $y \leq z_2 \leq z_1 + z_2 + h$ and $z_1 + z_2 + h \in G$. So G is directed. So G is a directed integrally closed partially ordered vector space and by Proposition 1.7(iv) on page 8 G is a pre-Riesz space. $\square$

Theorem 6.11. *The inclusion map $\iota : T \to T^r, x \mapsto x$ is a Riesz homomorphism.*

Proof. Let $X, Y \in T$. Let $x, y \in G$ such that $X = [x]$ and $Y = [y]$. Then $A := \{Z \in T : X \leq Z, Y \leq Z\} = \{[z] : z \in G, x \vee y \leq z\}$. So $B := \{Z \in T^r : Z \leq X \text{ for all } X \in A\} = \{X \in T^r : X \leq [x \vee y]\}$. Since T^r is Riesz are all lower bounds of the upper bounds of $\{X, Y\}$ in T^r equal to $C := \{Z \in T^r : Z \leq X \vee Y = [x \vee y]\}$. So $B = C$. So ι is a Riesz homomorphism. $\square$

Theorem 6.12 (Van Haandel)**.** *Let L be a Riesz space and H be a directed partially ordered vector space. Let $\varphi : H \to L$ be a Riesz homomorphism, then $\varphi(H)$ is order dense in the Riesz subspace M of L generated by $\varphi(H)$. Moreover $\varphi(H)$ is pre-Riesz. Let $\iota : \varphi(H) \to M$ be the inclusion map, then is (M, ι) the Riesz completion of $\varphi(H)$.*

Proof. Let L be a Riesz space and H be a directed partially ordered vector space. Let $\varphi : H \to L$ be a Riesz homomorphism. Let M be the Riesz subspace of L generated by $\varphi(H)$. Since φ is a Riesz homomorphism, is by [9, Theorem 2.12] $\varphi(H)$ order dense in M. By [9, Corollary 2.13] is $\varphi(H)$ pre-Riesz. The last statement follows from [9, Theorem 3.7]. $\square$

Theorem 6.13. *T is dense in T^r and generates T as a Riesz space and (T^r, ι) is the Riesz completion of T.*

Proof. $\iota_E(E)$ generates E^r as a Riesz space, $\iota_F(F)$ generates F^r as a Riesz space and T^r is generated as a Riesz space by elements $b^r(x, y), (x, y) \in E^r \times F^r$. So T^r is generated by T as a Riesz space. Since ι is a Riesz homomorphism, is by Theorem 6.12 T order dense in T^r and (T^r, ι) is the Riesz completion of T. $\square$

7 The integrally closed Riesz* tensor product

Kalauch and Van Gaans give in [7] the construction of the tensor product of arbitrary integrally closed pre-Riesz spaces E and F. They use positive linear maps as universal mappings. It seems that Riesz* homomorphisms are more natural, since Riesz* homomorphisms extends to Riesz homomorphisms between the Riesz completions and one would like that the Riesz completion of the tensor product, is the Archimedean Riesz space tensor product of the Riesz completions. Naively one can think that a Riesz* bimorphism $\phi : E \times F \to G$ extends to a Riesz bimorphism $\phi^r : E^r \times F^r \to G^r$ and that the restriction $\phi_* : E \times F \to G$ of ϕ^r_* is the universal map that satisfies $\phi = \phi_* \circ b$, where $b : E \times F \to T$ is a Riesz* bimorphism, T is an integrally closed pre-Riesz space and (T, b) is the so-called 'Riesz* tensor product' of E and F such that b extends to a Riesz bimorphism $b^r : E^r \times F^r \to T^r$ in such a way that (T^r, b^r) is the Archimedean Riesz tensor product of E^r and F^r.

$$\begin{tikzcd}
E^r \times F^r \arrow[rrrr, "b^r"] \arrow[rrddd, bend right, "\phi^r"'] & & & & T^r \arrow[llddd, bend left, "\phi^r_*"] \\
& E \times F \arrow[ul, hook', "\iota_E \times \iota_F"'] \arrow[rr, "b"] \arrow[dr, "\phi"'] & & T \arrow[ur, hook, "\iota_T"'] \arrow[dl, "\phi_*"] & \\
& & G \arrow[d, hook, "\iota_G"] & & \\
& & G^r & &
\end{tikzcd}$$

The reality is not so easy; only under some conditions on the spaces E and F we can prove that the Riesz* tensor product of E and F exists. But what is more important, in Section 8, we will prove that the tensor product (T, b) of Van Gaans and Kalauch is the one we are interested in. That is, b is the restriction of a Riesz bimorphism b^r and (T^r, b^r) is the Archimedean Riesz tensor product of E^r and F^r, where T^r is the Riesz completion of T.

Definition 7.1. Let E and F be (integrally closed) pre-Riesz spaces. An (integrally closed) Riesz* tensor product of E and F is a pair (T, b), where T is an (integrally closed) pre-Riesz space and $b : E \times F \to T$ is a Riesz* bimorphism, such that for every (integrally closed) pre-Riesz space G and every Riesz* bimorphism $\phi : E \times F \to G$ there is a *unique* Riesz* homomorphism $\phi_* : T \to G$ with $\phi = \phi_* \circ b$.

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[dr, "\phi"'] & T \arrow[d, "\phi_*"] \\ & G \end{tikzcd}$$

If an (integrally closed) Riesz* tensor product exists, then it is unique. In the following sense.

Theorem 7.2. *Let E and F be (integrally closed) pre-Riesz spaces and suppose that (S, b) and (T, c) are (integrally closed) Riesz* tensor products of E and F. Then there is a unique bijective linear map $\phi : S \to T$ such that ϕ and ϕ^{-1} are Riesz* homomorphisms and $\phi \circ b = c$. Moreover, ϕ is an order isomorphism.*

Proof. The proof is similar to the proof of Theorem 6.2 on page 39. □

Theorem 7.3. *Let E and F be integrally closed pre-Riesz spaces with either*

(i) E and F have the finite Van Haandel property, or

(ii) E and F have the countable Van Haandel property and the Riesz completions of E and F have the σ-order continuous property, or

(iii) E and F have the Van Haandel property and the Riesz completions of E and F have the order continuous property.

Let (E^r, ι_E) and (F^r, ι_F) be the Riesz completions of E and F respectively. Let (T^r, b^r) be the Archimedean Riesz tensor product of E^r and F^r. Define $T := Span\{b^r(\iota_E(x), \iota_F(y)) : x \in E, y \in F\}$ and define $b : E \times F \to T$ by $b(x, y) = b^r(\iota_E(x), \iota_F(y))$. Let $\iota_T : T \to T^r$ be the inclusion map. Then, b is a Riesz bimorphism, (T^r, ι_T) is the Riesz completion of T, and (T, b) is the integrally closed Riesz* tensor product of E and F.*

Proof. Let E and F be as in the theorem. Thus we can apply Theorem 4.17 on page 32. Let (E^r, ι_E) and (F^r, ι_F) be the Riesz completions of E and F respectively. Let (T^r, b^r) be the Archimedean Riesz tensor product of E^r and F^r. Define $T := \mathrm{Span}\{b^r(\iota_E(x), \iota_F(y)) : x \in E, y \in F\}$ and define $b : E \times F \to T$ by $b(x, y) = b^r(\iota_E(x), \iota_F(y))$. Let $\iota_T : T \to T^r$ be the inclusion map.

Claim 7.4. *(T^r, ι_T) is the Riesz completion of T, and b is a Riesz* bimorphism.*

Proof of Claim 7.4. Note that b is well defined and bilinear. By Theorem 6.13 on the preceding page is (T^r, ι_T), the Riesz completion of T. Note that b^r is a Riesz bimorphism and $\iota_T \circ b = b^r \circ (\iota_E, \iota_F)$. Thus by Theorem 4.26 on page 36 b is a Riesz*-bimorphism. □

Let G be an arbitrary integrally closed pre-Riesz space. Let (G^r, ι_G) be the Riesz completion of G. Let $\phi : E \times F \to G$ be a Riesz*-bimorphism. By Theorem 4.17 on page 32 there exist a Riesz bimorphism $\phi^r : E^r \times F^r \to G^r$ such that $\phi^r \circ (\iota_E, \iota_F) = \iota_G \circ \phi$. By Theorem 6.4 on page 40 there exist a (unique) Riesz homomorphism $\phi^r_* : T^r \to G^r$ such that $\phi^r_* \circ b^r = \phi^r$.

Claim 7.5. *$\phi^r_*|_T : T \to G^r$ maps into $\iota_G(G)$.*

Proof of Claim 7.5. Let $t \in T$. Then there are $x_1, \dots, x_n \in E, y_1, \dots, y_n \in F$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$t = \sum_{i=1}^{n} \alpha_i b^r(\iota_X(x_i), \iota_Y(y_i)) = \sum_{i=1}^{n} \alpha_i b(x_i, y_i).$$

Thus

$$\begin{aligned}
\phi^r_*(t) &= \sum_{i=1}^{n} \alpha_i \phi^r_*(b^r(\iota_X(x_i), \iota_Y(y_i))) \\
&= \sum_{i=1}^{n} \alpha_i \phi^r(\iota_X(x_i), \iota_Y(y_i)) \\
&= \sum_{i=1}^{n} \alpha_i \iota_G(\phi(x_i, y_i)) \in \iota_G(G).
\end{aligned}$$

□

So $\phi_* : T \to G$ defined by $\phi_*(t) = \iota_G^{-1}(\phi^r_*(t))$ is well defined and linear. Moreover, $\iota_G \circ \phi_* = \phi^r_* \circ \iota_T$. Hence, by Theorem 3.14 on page 28 ϕ_* is a Riesz* homomorphism.

Claim 7.6. *ϕ_* is the unique linear map ψ such that $\psi \circ b = \phi$.*

Proof of Claim 7.6. Let $\psi : T \to G$ be a bilinear map such that $\psi \circ b = \phi$. Let $t \in T$. Then there are $x_1, \dots, x_n \in E, y_1, \dots, y_n \in F$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that $t = \sum_{i=1}^{n} \alpha_i b(x_i, y_i)$. So

$$\psi(t) = \sum_{i=1}^{n} \alpha_i \psi(b(x_i, y_i)) = \sum_{i=1}^{n} \alpha_i \phi(x_i, y_i)$$

and

$$\phi_*(t) = \sum_{i=1}^{n} \alpha_i \phi_*(b(x_i, y_i)) = \sum_{i=1}^{n} \alpha_i \phi(x_i, y_i).$$

It follows that $\psi(t) = \phi_*(t)$ for all $t \in T$. Thus $\psi = \phi_*$. □

We conclude that (T, b) is the integrally closed Riesz* tensor product of E and F.

$$\begin{tikzcd}
E^r \times F^r \arrow[rrrr, "b^r"] \arrow[dddrr, bend right, "\phi^r"'] & & & & T^r \arrow[dddll, bend left, "\phi^r_*"] \\
& E \times F \arrow[ul, hook, "\iota_E \times \iota_F"'] \arrow[rr, "b"] \arrow[dr, "\phi"'] & & T \arrow[ur, hook, "\iota_T"'] \arrow[dl, "\phi_*"] & \\
& & G \arrow[d, hook, "\iota_G"] & & \\
& & G^r & &
\end{tikzcd}$$

□

Corollary 7.7. *If L and M are Archimedean Riesz spaces, then the integrally closed Riesz* tensor product of L and M exists.*

Proof. If L and M are Riesz, then L and M have the finite Van Haandel property, so we can apply Theorem 7.3 on the previous page. □

8 The positive tensor product

There exists also another notion of tensor products of integrally closed pre-Riesz spaces, it considers positive linear maps and positive bimorphisms, instead of Riesz* homomorphisms and bimorphisms. To distinguish the two notions of tensor products, we call the notion with positive linear maps and positive bimorphisms the 'positive tensor product'. In this section we define the positive tensor product, prove some elementary properties and give three constructions of the positive tensor product. One construction is due to Van Gaans and Kalauch and the other two constructions are new.
Positive bimorphisms are just what one thinks they are.

Definition 8.1. Let E, F and G be partially ordered vector spaces and let $b : E \times F \to G$ be bilinear. Then we call b a positive bimorphism or positive bilinear map, if $b(x, \cdot)$ is positive, for all $x \in E^+$, and $b(\cdot, y)$ is positive, for all $y \in F^+$. This is equivalent to $b(x, y) \geq 0$ for all $x \in E^+$ and $y \in F^+$.

Definition 8.2. Let E and F be integrally closed pre-Riesz spaces. A pair (T, b), where T is an integrally closed pre-Riesz space and $b : E \times F \to T$ is a positive bimorphism, is a positive tensor product of E and F if for all integrally closed pre-Riesz spaces G and positive bimorphisms $\phi : E \times F \to G$ there is a unique positive linear map $\phi_* : T \to G$ such that $\phi = \phi_* \circ b$.

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[dr, "\phi"'] & T \arrow[d, "\phi_*"] \\ & G \end{tikzcd} \tag{13}$$

The positive tensor product is unique, if it exists. In the following sense.

Theorem 8.3. *Let E and F be integrally closed pre-Riesz spaces and suppose that (S, b) and (T, c) are positive tensor products of E and F. Then there is a unique order isomorphism $\phi : S \to T$ such that $\phi \circ b = c$.*

Proof. The proof is similar to the proof of Theorem 6.2 on page 39. □

Theorem 8.4. *Let E and F be integrally closed pre-Riesz spaces. If (T, b) is the positive tensor product of E and F, then T is generated as vector space by elements $b(x, y), x \in E, y \in F$.*

Proof. Let E, F and (T, b) as in the theorem. Let S be the subspace of T generated by the elements $b(x, y), x \in E, y \in F$. Clearly, S is integrally closed and generated by positive elements $b(x, y), x \in E^+, y \in F^+$ and hence directed. Thus S is an integrally closed pre-Riesz space. Let G be an arbitrary integrally closed pre-Riesz space and $\phi : E \times F \to G$ a positive bimorphism. Let $\phi'_* : T \to G$ be the unique positive linear map with $\phi = \phi'_* \circ b$. Note that b maps into S. Let $\phi_* : S \to G$ be the restriction of ϕ'_* to S. Then ϕ_* is a positive linear map and $\phi = \phi_* \circ b$. Let $\psi : S \to G$ be any positive linear map with $\phi = \psi \circ b$. For all $x \in E$ and $y \in F$ we have that $\phi_*(b(x, y)) = \psi(b(x, y))$. Since S is generated as vector space by elements $b(x, y), x \in E, y \in F$, we have that $\phi_* = \psi$. Thus (S, b) is also the positive tensor product of L and M. From Theorem 8.3 follows that $S = T$. This concludes our proof. □

Example 8.5. Let E be an integrally closed pre-Riesz space. We calculate the positive tensor product of $\mathbb{R}$ and E. Define $b : \mathbb{R} \times E \to E$ through $b(r, x) = rx$. Let F be an arbitrary integrally closed pre-Riesz space and let $\phi : \mathbb{R} \times E \to F$ be a positive bimorphism. Note that $\phi(r, x) = \phi(1, rx)$, for all $r \in \mathbb{R}$ and $x \in E$. Define $\phi_* : E \to F$ through $\phi_*(x) = \phi(1, x)$. Then ϕ_* is a positive linear map and for all $r \in \mathbb{R}$ and $x \in E$ we have $\phi(r, x) = \phi(1, rx) = \phi_*(rx) = \phi_*(b(r, x))$. Thus $\phi = \phi_* \circ b$. Let $\psi : E \to F$ be any positive linear map with $\phi = \psi \circ b$, then $\psi(x) = \psi(b(1, x)) = \phi(1, x) = \phi_*(x)$, for all $x \in E$. Thus $\phi_* = \psi$. It follows that (E, b) is the positive tensor product of $\mathbb{R}$ and E.

8.1 Tensor cones

In this section, we give a short overview of the results of Van Gaans and Kalauch [7]. We skip most of the proofs.

Definition 8.6. Let E and F be partially ordered vector spaces and let $E \otimes F$ be the usual vector space tensor product of E and F. We define the projective tensor cone K_T to be

$$K_T = \left\{ \sum_{i=1}^{n} \lambda_i x_i \otimes y_i : n \in \mathbb{N} \text{ and } \lambda_i \in \mathbb{R}^+, x_i \in E^+, y_i \in F^+, \text{ for } i \in \{1, \ldots, n\} \right\}.$$

It is just the wedge generated by positive elements $x \otimes y$. But it is in fact a cone and generates $E \otimes F$ according to the following lemma.

Lemma 8.7. *(See [7, Theorem 2.5].) Let E and F be partially ordered vector spaces, then the projective cone K_T is a cone and $(E \otimes F, K_T)$ is directed.*

Definition 8.8. (See [7, Definition 4.1].) Let E and F be integrally closed pre-Riesz spaces. A cone K_I in $E \otimes F$ is called an *integrally closed tensor cone* if the projective cone K_T is contained in K_I, $(E \otimes F, K_I)$ is an integrally closed pre-Riesz space and the following universal mapping is satisfied. For every integrally closed pre-Riesz space G and for every positive bilinear map $\phi : E \times F \to G$ the unique linear map $\phi_* : (E \otimes F, K) \to G$, with $\phi = \phi_* \circ \otimes$, is positive.

Remark 8.9. Van Gaans and Kalauch call the integrally closed tensor cone the Archimedean tensor cone.

Theorem 8.10. *Let E and F be integrally closed pre-Riesz spaces. Suppose K_1 and K_2 are integrally closed tensor cones of $E \otimes F$, then $K_1 = K_2$.*

Proof. Let E and F be integrally closed pre-Riesz spaces. Suppose K_1 and K_2 are integrally closed tensor cones of $E \otimes F$. Note that $\phi : E \times F \to E \otimes F$ defined by $\phi(x, y) = x \otimes y$ is positive with cones K_1 and K_2, since both contain K_T.
By definition there is a unique positive linear map $\phi_1 : (E \otimes F, K_1) \to (E \otimes F, K_2)$ such that $\phi_1(x \otimes y) = \phi(x, y) = x \otimes y$, for all $x \in E, y \in F$, and a unique positive linear map $\phi_2 : (E \otimes F, K_2) \to (E \otimes F, K_1)$ such that $\phi_2(x \otimes y) = \phi(x, y) = x \otimes y$, for all $x \in E, y \in F$. Thus $\phi_1 = \phi_2$ is the identity map on $E \otimes F$, and the identity map is bipositive hence $K_1 = K_2$. $\square$

Definition 8.11. Let E and F be integrally closed pre-Riesz spaces. Let (E^r, ι_E) and (F^r, ι_F) be the Riesz completions of E and F respectively. Consider the Archimedean Riesz tensor product $E^r \bar{\otimes} F^r$ of E^r and F^r. We can view the usual vector space tensor product $E^r \otimes F^r$ as a subspace of $E^r \bar{\otimes} F^r$ with the induced ordering. Define $\rho : E \times F \to E^r \otimes F^r$ by $\rho(x, y) = (\otimes \circ (\iota_E, \iota_F))(x, y) = \iota_E(x) \otimes \iota_F(y)$. Then ρ is positive bilinear and induces a unique linear map $\rho_* : E \otimes F \to E^r \otimes F^r$ with $\rho(x, y) = \rho_*(x \otimes y)$ for all $x \in E$ and $y \in F$. By [7, Lemma 2.4] ρ_* is injective. We define the *Fremlin tensor cone* to be

$$K_F = \{x \in E \otimes F : \rho_*(x) \geq 0\}.$$

$$\begin{tikzcd}
E^r \times F^r \arrow[drr, "\otimes"'] & E \times F \arrow[l, hook', "\iota_E \times \iota_F"'] \arrow[r, "\otimes"] \arrow[dr, "\rho"'] & E \otimes F \arrow[d, "\rho_*"] \\
 & & E^r \otimes F^r \arrow[d, hook] \\
 & & E^r \bar{\otimes} F^r
\end{tikzcd}$$

Proposition 8.12. *(See [7, Lemmas 4.2 and 4.3].) The Fremlin tensor cone is a generating integrally closed cone. Moreover $K_T \subset K_F$ and K_F is relatively uniformly closed in $(E \otimes F, K_T)$.*

Theorem 8.13. *(See [7, Theorem 4.4].) Let E and F be integrally closed pre-Riesz spaces. For a cone K in $E \otimes F$ the following four statements are equivalent.*

(i) K is the integrally closed tensor cone.

(ii) For all integrally closed pre-Riesz spaces (S, K_S) and for any linear map $\phi : E \otimes F \to S$ with $\phi(x) \in K_S$ for all $x \in K_T$, we also have that $\phi(x) \in K_S$ for all $x \in K$.

(iii) K is the intersection of all integrally closed cones in $E \otimes F$ that contain K_T.

(iv) K is the relatively uniformly closure of K_T in $(E \otimes F, K_T)$.

Proposition 8.12 on the preceding page and Theorem 8.13 yield the following.

Theorem 8.14. *For any pair of integrally closed pre-Riesz spaces E and F, the relatively uniformly closure K_I of K_T in $(E \otimes F, K_T)$ is a cone. That is, K_I is the integrally closed tensor cone in $E \otimes F$ and $(E \otimes F, K_I)$ is the positive tensor product of E and F.*

Proof. Let E and F be integrally closed pre-Riesz spaces and let K_I be the relatively uniformly closure of K_T in $(E \otimes F, K_T)$. By Proposition 8.12 on the preceding page we have that $K_T \subset K_F$ and that K_F is a relatively uniformly closed cone in $(E \otimes F, K_T)$. Since K_I is the relatively uniformly closure of K_T in $(E \otimes F, K_T)$, it follows that $K_I \subset K_F$. Thus K_I is a cone. From Theorem 8.13 follows that K_I is the integrally closed tensor cone in $E \otimes F$ thus $(E \otimes F, K_I)$ is the positive tensor product of E and F. □

From Theorem 8.13(iv) and Proposition 8.12 on the previous page it follows that $K_I \subset K_F$. An open question until now is whether $K_I = K_F$ (see [7, page 16]). Corollary 8.23 on page 50 gives a positive answer to this question; the cones are in fact equal.

From the construction follows clearly the following two theorems.

Theorem 8.15. *Let E', E, F' and F be integrally closed pre-Riesz spaces, such that $E' \subset E, F' \subset F$ then there exists a bipositive linear map $\phi : E' \otimes F' \to E \otimes F$ with $\phi(x \otimes y) = x \otimes y$.*

Theorem 8.16. *Let E and F be integrally closed pre-Riesz spaces. Then there is an order isomorphism $\phi : E \otimes F \to F \otimes E, \phi(x \otimes y) = y \otimes x, x \in E, y \in F$.*

8.2 Construction of the positive tensor product via a free Riesz space

Now we give another construction of the positive tensor product of two arbitrary integrally closed pre-Riesz spaces. This construction does not make use of the results of Fremlin.

Let E and F be two arbitrary integrally closed pre-Riesz spaces. We construct the positive tensor product (T, b) of E and F out the free Riesz space $(\mathrm{FRS}(E \times F), \iota)$ over $E \times F$, by choosing a suitable ideal J in $\mathrm{FRS}(E \times F)$ and by choosing a vector subspace T of $T' = \mathrm{FRS}(E \times F)/J$ generated by elements $q(\iota(x, y)), x \in E, y \in F$, where q is the quotient map. The positive bimorphism b is chosen to be $q \circ \iota$ restricted to T.

Theorem 8.17. *Let E and F be integrally closed pre-Riesz spaces. Consider the free Riesz space over $E \times F$ (seen as a set), $(FRS(E \times F), \iota)$. Let J be the intersection of all relatively uniformly closed order ideals that contain the following set.*

$$\begin{cases} \iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z), & x, y \in E, z \in F, \alpha, \beta \in \mathbb{R}, \\ \iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z), & x \in E, y, z \in F, \alpha, \beta \in \mathbb{R}, \\ \iota(x, y) - |\iota(x, y)| & x \in E^+, y \in F^+. \end{cases} \tag{14}$$

Let $T' = FRS(E \times F)/J$ and let $q : FRS(E \times F) \to T'$ be the quotient Riesz homomorphism. Define $b' : E \times F \to T'$ by $b'(x, y) = q(\iota(x, y)) = (q \circ \iota)(x, y)$. Let $T = Span\{b'(x, y) : x \in E, y \in F\}$ and let $b : E \times F \to T$ be the restriction of b'. Then T is an integrally closed pre-Riesz space, b is a positive bilinear map and (T, b) is the positive tensor product of E and F.

Proof. Let E, F, J, T', T, b', b and q be as in the theorem. Since $\mathrm{FRS}(E\times F)$ is a Riesz space, J is a relatively uniformly closed ideal, by Theorem 1.23 on page 13 we have that T' is an Archimedean Riesz space.

Claim 8.18. *b' is a positive bimorphism.*

Proof of Claim 8.18. Let $x, y \in E$ and $z \in F$. Let $\alpha, \beta \in \mathbb{R}$. Then $0 = q(\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z)) = q(\iota(\alpha x + \beta y, z)) - \alpha q(\iota(x, z)) - \beta q(\iota(y, z)) = b'(\alpha x + \beta y, z) - \alpha b'(x, z) - \beta b'(y, z)$. Therefore $b'(\alpha x + \beta y, z) = \alpha b'(x, z) + \beta b'(y, z)$. Likewise, we have for all $x \in E, y, z \in F$ and $\alpha, \beta \in \mathbb{R}$: $b'(x, \alpha y + \beta z) = \alpha b'(x, y) + \beta b'(x, z)$. Thus b' is bilinear. Let $x \in E^+, y \in F^+$. Then $b'(x, y) = q(\iota(x, y)) = q(|\iota(x, y)|) = |q(\iota(x, y))| = |b'(x, y)| \geq 0$. Hence b' is a positive bimorphism. □

Note that T is generated as subspace of T' by positive elements $b'(x, y), x \in E^+, y \in F^+$, thus T is a directed integrally closed partially ordered vector space and hence an integrally closed pre-Riesz space. Since $b'(E \times F) \subset T$, we have that b is well defined and clearly a positive bilinear map. Let G be an arbitrary integrally closed pre-Riesz space and let $\phi : E \times F \to G$ be an arbitrary positive bimorphism. Let G^r be the Riesz completion of G and we may assume that $G \subset G^r$. We view ϕ as a positive bimorphism from $E \times F$ to G^r. Let $\psi : \mathrm{FRS}(E \times F) \to G^r$ be the unique Riesz bimorphism that satisfies $\phi = \psi \circ \iota$.

$$\begin{tikzcd} \mathrm{FRS}(E\times F) \arrow[r, "\psi"] & G^r \\ E\times F \arrow[u, "\iota"] \arrow[ur, "\phi"'] & \end{tikzcd} \tag{15}$$

Note that $\ker(\psi)$ is a relatively uniformly closed ideal of $\mathrm{FRS}(L \times M)$. We will show that $J \subset \ker(\psi)$. To do that, it is sufficient to show that the set that generates J is contained in $\ker(\psi)$. Note that for all $\alpha, \beta \in \mathbb{R}$ and $x, y \in E$ and $z \in F$ we have $\psi(\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z)) = \psi(\iota(\alpha x + \beta y, z)) - \alpha\psi(\iota(x, z)) - \beta\psi(\iota(y, z)) = \phi(\alpha x + \beta y, z) - \alpha\phi(x, z) - \beta\phi(y, z) = 0$. So $\iota(\alpha x + \beta y, z) - \alpha\iota(x, z) - \beta\iota(y, z) \in \ker(\psi)$. Likewise $\iota(x, \alpha y + \beta z) - \alpha\iota(x, y) - \beta\iota(x, z) \in \ker(\psi)$, for all $x \in E, y, z \in F$ and $\alpha, \beta \in \mathbb{R}$. Let $x \in E^+, y \in F^+$. Since $\phi(x, y) \geq 0$, we have that $\psi(\iota(x, y) - |\iota(x, y)|) = \psi(\iota(x, y)) - |\psi(\iota(x, y))| = \phi(x, y) - |\phi(x, y)| = 0$. Thus $\iota(x, y) - |\iota(x, y)| \in \ker(\psi)$ for all $x \in E^+, y \in F^+$. We conclude that $J \subset \ker(\psi)$. Now define $\phi'_* : T' \to G^r$ by $\phi'_*(q(x)) = \psi(x)$. Suppose $q(x) = q(y)$ for $x, y \in \mathrm{FRS}(E \times F)$. Then $y - x \in J \subset \ker(\psi)$. Thus $\phi_*(q(x)) = \psi(x) = \psi(x + y - x) = \psi(y) = \phi_*(q(y))$. It follows that ϕ'_* is well defined. Clearly, ϕ'_* is linear. Let $x \in \mathrm{FRS}(E \times F)$. Then $\phi'_*(|q(x)|) = \phi'_*(q(|x|)) = \psi(|x|) = |\psi(x)| = |\phi'_*(q(x))|$, since ψ is a Riesz homomorphism. Hence ϕ'_* is a Riesz homomorphism and in particular a positive linear map and $\phi'_*(b'(x, y)) = \phi'_*(q(\iota(x, y)) = \psi(\iota(x, y)) = \phi(x, y)$.
It is clear that $\phi = \phi_* \circ b$ and that b is positive. It remains to show that ϕ_* is the unique positive linear map with this property. Let $\chi : T \to G$ be a positive linear map that satisfies $\phi = \chi \circ b$. Let $t \in T$. Then there are $x_1, \ldots, x_n \in E, y_1, \ldots, y_n \in F$ such that $t = \sum_{i=1}^n b(x_i, y_i)$. So $\chi(t) = \sum_{i=1}^n \chi(b(x_i, y_i)) = \sum_{i=1}^n \phi(x_i, y_i) = \sum_{i=1}^n \phi_*(b(x_i, y_i)) = \phi_*(t)$. In fact, we proved that ϕ_* is the unique *linear map* χ with the property that $\phi = \chi \circ b$. Hence (T, b) is the positive tensor product of E and F. □

We have shown something stronger. Remark that we only use the fact that E and F are partially ordered vector spaces, that G is pre-Riesz, but not that it is integrally closed. Note that T is directed if E and F are directed. Therefore we have the following.

Theorem 8.19. *Let E and F be partially ordered vector spaces. Then there exists a pair (T, b) where T is an integrally closed partially ordered vector space and $b : E \times F \to T$ is a positive bimorphism, such that for any pre-Riesz space G (either integrally closed or not integrally closed) and for any positive bimorphism $\phi : E \times F \to G$ there exist a unique linear map $\phi_* : T \to G$, such* *that $\phi = \phi_* \circ b$. Moreover ϕ_* is positive. If E and F are directed, then T is an integrally closed pre-Riesz space.*

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[dr, "\phi"'] & T \arrow[d, "\phi_*"] \\ & G \end{tikzcd} \tag{16}$$

8.3 Construction of the positive tensor product via the Archimedean Riesz tensor product

Suppose E' is a directed subspace of an integrally closed pre-Riesz space E and F' is a directed subspace of an integrally closed pre-Riesz space F. Then E' and F' are directed integrally closed partially ordered vector spaces, and hence integrally closed pre-Riesz spaces. We have seen that $E' \otimes F'$ can be viewed as a subspace of $E \otimes F$, Theorem 8.15 on page 48.

As said before (Remark 6.7 on page 41), we denote the Archimedean Riesz tensor product of two Archimedean Riesz spaces L and M by $L\bar{\otimes}M$.

Theorem 8.20. *We can view the ordinary, vector space tensor product of L and $M, L \otimes M$ as a subspace of $L\bar{\otimes}M$ generated by the elements $x \otimes y, x \in L, y \in M$.*

According to Fremlin [3, Proposition 5.1] we have the next result.

Theorem 8.21. *Let L, M and N be Archimedean Riesz spaces. Let $\phi : L \times M \to N$ be positive. Then the unique linear map $\phi_* : L \otimes M \to N$ with $\phi = \phi_* \circ \otimes$ is positive.*

Theorem 8.22. *Let E and F be integrally closed pre-Riesz spaces. Let (E^r, φ_E) and (F^r, φ_F) be the Riesz completions of E and F respectively. From Theorem 8.21 follows that $E^r \otimes F^r$ with the induced ordering of $E^r\bar{\otimes}F^r$ is the positive tensor product of E^r and F^r. Let $T = Span\{\varphi_E(x) \otimes \varphi_F(y) : x \in E, y \in F\}$, and let $b' = \otimes|_{\varphi_E(E)\times\varphi_F(F)}$. By Theorem 8.15 on page 48 (T, b') is the positive tensor product of $\varphi_E(E)$ and $\varphi_F(F)$. Let $b : E \times F \to T$ be defined trough $b(x, y) = b'(\varphi_E(x), \varphi_F(y))$. Since $\iota_X : X \to \varphi_X(X), X = E, F$, is an order isomorphism, we have that (T, b) is the positive tensor product of E and F. Let $\iota_T : T \to T^r$ be the inclusion map. By Theorem 6.13 on page 43 T is dense in T^r and generates T^r as a Riesz space and (T^r, ι_T) is the Riesz completion of T. By Theorem 4.26 on page 36 b is a Riesz* bimorphism.*

It follows that every positive tensor product is in a nice way a subspace of the Archimedean Riesz tensor product of the Riesz completions and the choice for positive linear maps as universal mappings was a good choice. But most interesting is the following corollary.

Corollary 8.23. *Let E and F be integrally closed pre-Riesz spaces. Let K_I be the integrally closed tensor cone of $E \otimes F$ and let K_F be the Fremlin tensor cone of $E \otimes F$. Then $K_I = K_F$ and for a cone $K \subset E \otimes F$ the following five statements are equivalent.*

(i) *K is the integrally closed tensor cone.*

(ii) *K is the Fremlin tensor cone.*

(iii) *For all integrally closed pre-Riesz spaces (S, K_S) and for any linear map $\phi : E \otimes F \to S$ with $\phi(x) \in K_S$ for all $x \in K_T$, we also have that $\phi(x) \in K_S$ for all $x \in K$.*

(iv) *K is the intersection of all integrally closed cones in $E \otimes F$ that contain K_T.*

(v) *K is the relatively uniformly closure of K_T in $(E \otimes F, K_T)$.*

Moreover, if K satisfies one of the five properties, we have that $(E \otimes F, K)$ is positive tensor product of E and F.

In conclusion, this sections shows that the positive tensor products $E \otimes F$ of integrally closed pre-Riesz spaces E and F is very beautiful. It has the following nice properties.

1. $\otimes : E \times F \to E \otimes F$ is a Riesz* bimorphism.
2. $\otimes : E \times F \to E \otimes F$ extends to the Riesz bimorphism $\otimes : E^r \times F^r \to E^r \bar{\otimes} F^r$. Or to say it differently, $\otimes$ is the restriction of the Riesz bimorphism $\otimes : E^r \times F^r \to E^r \bar{\otimes} F^r$.
3. The Riesz completion of $E \otimes F$ is the Archimedean Riesz tensor product $E^r \bar{\otimes} F^r$.

9 Examples

In this section we compute the positive tensor product of two examples. The examples illustrate how the theory can be used in calculations.

9.1 Tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$

The most basic example is the tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$.

Theorem 9.1. *Let $m, n \in \mathbb{N}$. Define $\phi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{mn}$ through $\phi(a, b) = ab^t$, where we view $\mathbb{R}^{mn}$ as the space of real $m \times n$-matrices with coordinate-wise ordering and the vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ as column vectors. Then ϕ is a Riesz bimorphism and $(\mathbb{R}^{mn}, \phi)$ is the usual linear space, positive and Archimedean Riesz tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$.*

Proof. Let $m, n \in \mathbb{N}$. Denote the standard basis of $\mathbb{R}^m$ by $(e_1^1, \ldots, e_m^1)$ and the standard basis of $\mathbb{R}^n$ by $(e_1^2, \ldots, e_n^2)$. Define $\phi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{mn}$ through $\phi(a, b) = ab^t$. First we will show that $(\mathbb{R}^{mn}, \phi)$ is the usual vector space tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$. Note that ϕ is a bilinear map. Let V be an arbitrary real vector space. Let $\psi : \mathbb{R}^m \times \mathbb{R}^n \to V$ be a bilinear map. Define $\psi_* : \mathbb{R}^{mn} \to V$ by $\psi_*((a_{ij})_{ij}) = \sum_{i=1}^m \sum_{j=1}^n a_{ij} \psi(e_i^1, e_j^2)$, then ψ_* is clearly a linear map with $\phi = \phi_* \circ \psi$. Let $T : \mathbb{R}^{mn} \to V$ be a linear map with $\psi = T \circ \phi$. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, then $e_i^1 e_j^{2t} = \phi(e_i^1, e_j^2)$, therefore $T(e_i^1 e_j^{2t}) = (T \circ \phi)(e_i^1, e_j^2) = \psi(e_i^1, e_j^2) = (\psi_* \circ \phi)(e_i^1, e_j^2) = \psi_*(e_i^1 e_j^{2t})$. Since $\{e_i^1 e_j^{2t} : i = 1, \ldots, m, j = 1, \ldots, n\}$ generates $\mathbb{R}^{mn}$ as a vector space, we have that $T = \psi_*$. Thus $\psi_* : \mathbb{R}^{mn} \to V$ is the unique linear map $T : \mathbb{R}^{mn} \to V$ with $\psi = T \circ \phi$. Thus $(\mathbb{R}^{mn}, \phi)$ is the vector space tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$. Note that the projective tensor cone $K_T = \{x = (x_{ij})_{ij} \in \mathbb{R}^{mn} : x_{ij} \geq 0, \text{ for all } i = 1, \ldots, m, j = 1, \ldots, n\}$ (this cone generates the point-wise ordering). Note that $(\mathbb{R}^{mn}, K_T)$ is an Archimedean Riesz space, thus by Theorem 8.13 on page 48 $((\mathbb{R}^{mn}, K_T), \phi)$ is the positive tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$. Since $(\mathbb{R}^{mn}, K_T)$ is an Archimedean Riesz space, by Theorem 8.22 on the preceding page $((\mathbb{R}^{mn}, K_T), \phi)$ is the Archimedean Riesz space tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$.

$$
\begin{tikzcd}
\mathbb{R}^m \times \mathbb{R}^n \arrow[r, "\phi"] \arrow[dr, "\psi"'] & \mathbb{R}^{mn} \arrow[d, "\psi_*"] \\
& V
\end{tikzcd}
$$

□

Corollary 9.2. *$(\mathbb{R}^{mn}, \phi)$ is also the integrally closed Riesz* tensor product of $\mathbb{R}^m$ and $\mathbb{R}^n$.*

Proof. Since $\mathbb{R}^{mn}$ is generated as vector space by elements $\phi(e_i^1, e_j^2)$ and $\mathbb{R}^m$ and $\mathbb{R}^n$ are Riesz spaces, the conclusion follows by Corollary 7.7 on page 45 and Theorem 7.3 on page 44. □

9.2 Tensor product of polyhedral cones

In this section we study the positive tensor product of partially ordered vector spaces with a polyhedral cone. Van Gaans, Kalauch and Lemmens gave a nice representation of the Riesz completion of such spaces.

Definition 9.3. ([8, Page 12]) Let $k \geq n \geq 1$ be natural numbers. Let $u \in \mathbb{R}^n$. Let $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be k mutually different linear functionals, with $f_i(u) = 1, i = 1, \ldots, k$. Let

$$K = \{x \in \mathbb{R}^n : f_i(x) \geq 0, \text{ for } i = 1, \ldots, k\}.$$

Then we call K a k-sided polyhedral cone in $\mathbb{R}^n$ generated by $f_1, \ldots, f_k$.

Remark 9.4. If the cone has non-empty interior, then $(\mathbb{R}^n, K)$ is an integrally closed pre-Riesz space and u is an order unit for $(\mathbb{R}^n, K)$.

Theorem 9.5 (Van Gaans-Kalauch-Lemmens)**.** *([8, Proposition 13].) Let K be a polyhedral cone in $\mathbb{R}^n$ with $k \geq n$ facets generated by functionals $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ and let $u \in K$ be such that $f_i(u) = 1$, for all $i = 1, \ldots, k$. Define $\Phi : (\mathbb{R}^n, K) \to \mathbb{R}^k$ through*

$$\Phi(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_k(x) \end{pmatrix}.$$

Then Φ is bipositive and $\Phi(u) = (1, \ldots, 1)^t$. If K has non-empty interior, then $\Phi(\mathbb{R}^n)$ is order dense in $\mathbb{R}^k$ (with the standard ordering) and $(\mathbb{R}^k, \Phi)$ is the Riesz completion of $(\mathbb{R}^n, K)$.

With this nice representation of the Riesz completion of a polyhedral cone, we can calculate the positive tensor product of two polyhedral cones.
Let K be a polyhedral cone of $\mathbb{R}^m$ with $k \geq m$ facials generated by functionals $f_1, \ldots, f_k : \mathbb{R}^m \to \mathbb{R}$ and let $u \in K$ be such that $(f_1(u), \ldots, f_n(u)) = (1, \ldots, 1)$ and let L be a polyhedral cone of $\mathbb{R}^n$ with $l \geq n$ facials generated by linear functionals $g_1, \ldots, g_l : \mathbb{R}^n \to \mathbb{R}$ and let $v \in L$ be such that $(g_1(v), \ldots, g_n(v)) = (1, \ldots, 1)$ and assume that K and L have non-empty interior. Let $(\mathbb{R}^k, \Phi_1)$ be the Riesz completion of $(\mathbb{R}^m, K)$ and let $(\mathbb{R}^l, \Phi_2)$ be the Riesz completion of $(\mathbb{R}^n, L)$ with Φ_1 and Φ_2 as in Theorem 9.5. Let $\phi : \mathbb{R}^k \times \mathbb{R}^l \to \mathbb{R}^{kl}$ be defined through $\phi(a, b) = ab^t$. By Theorem 9.1 on the preceding page $(\mathbb{R}^{kl}, \phi)$ is the Archimedean Riesz tensor product of $\mathbb{R}^k$ and $\mathbb{R}^l$.
Let

$$\begin{aligned} T &= \mathrm{Span}\{\phi(\Phi_1(x), \Phi_2(y)) : x \in \mathbb{R}^m, y \in \mathbb{R}^n\} \\ &= \mathrm{Span}\{\phi((f_1(x), \ldots, f_k(x))^t, (g_1(y), \ldots, g_l(y))^t) : x \in \mathbb{R}^m, y \in \mathbb{R}^n\} \\ &= \mathrm{Span}\{(f_i(x)g_j(y))_{i=1,\ldots,k, j=1,\ldots,l} : x \in \mathbb{R}^m, y \in \mathbb{R}^n\} \end{aligned}$$

with the ordering induced from $\mathbb{R}^{mn}$ (so the pointwise ordering). Define $b : (\mathbb{R}^m, K) \times (\mathbb{R}^n, L) \to T$ by $b(x, y) = \Phi_1(x)\Phi_2(y)^t$. Clearly $b(u, v) = (1)_{i=1,\ldots,m, j=1,\ldots,n}$. By Theorem 8.22 on page 50 we have that (T, b) is the positive tensor product of $(\mathbb{R}^k, K)$ and $(\mathbb{R}^l, L)$, T is order dense in $\mathbb{R}^{kl}$, T generates $\mathbb{R}^{kl}$ as a Riesz space, and if $\iota : T \to \mathbb{R}^{kl}$ is the inclusion map, then $(\mathbb{R}^{kl}, \iota)$ is the Riesz completion of T.

$$\begin{tikzcd} \mathbb{R}^k \times \mathbb{R}^l \arrow[r, "\phi"] & \mathbb{R}^{kl} \\ (\mathbb{R}^m, K) \times (\mathbb{R}^n, L) \arrow[u, "\Phi_1 \times \Phi_2"] \arrow[r, "b"] & T \arrow[u, hook, "\iota"'] \end{tikzcd}$$

10 Tensor product of partially ordered vector spaces with Fremlin norm and closed cone

If two partially ordered vector spaces have topologies, it would be natural if its tensor product has a compatible topological structure too. That is possible with partially ordered vector spaces with

Fremlin norm and norm closed positive cone, the so-called Fremlin spaces (see Definition 1.33 on page 15). Recall that a norm ρ on a partially ordered vector space E is called a Fremlin norm provided that if $-y \leq x \leq y$ then $\rho(x) \leq \rho(y)$, for all $x, y \in E$ (Definition 1.28 on page 14). The Fremlin spaces are exactly the subspaces of Banach lattices (Theorem 1.35 on page 15) hence they are integrally closed. This fact is very useful in the construction of a tensor product of Fremlin spaces.

10.1 Definition and properties

Definition 10.1. A Fremlin space tensor product of Fremlin spaces E and F is a pair (T, b) where T is a Fremlin space and $b : E \times F \to T$ is a continuous positive bimorphism with $||b|| \leq 1$, such that for every Fremlin space G and for every continuous positive bimorphism $\phi : E \times F \to G$ there is a continuous positive linear map $\phi_* : T \to G$ such that $\phi = \phi_* \circ b$ and $||\phi_*|| \leq ||\phi||$. Moreover, ϕ_* is the unique positive linear map $\chi : T \to G$ such that $\phi = \chi \circ b$.

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[dr, "\phi"'] & T \arrow[d, "\phi_*"] \\ & G \end{tikzcd}$$

Theorem 10.2. *Let E and F be Fremlin spaces and suppose that (S, b) and (T, c) are Fremlin space tensor products of E and F. Then there exists a unique order isomorphism $\phi : S \to T$ such that $c = \phi \circ b$. Moreover ϕ is an isometric isomorphism.*

Proof. By definition there are unique continuous positive linear maps $c_* : S \to T$ such that $c = c_* \circ b$ and $b_* : T \to S$ such that $b = b_* \circ c$. Thus $b = b_* \circ c_* \circ b$ and $b_* \circ c_* : S \to S$ are positive linear maps. Note that the identity map on S, id_S is also a positive linear map with $b = id_S \circ b$. By the uniqueness statement in the definition it follows that $b_* \circ c_* = id_S$. Likewise $c_* \circ b_*$ is the identity map on T. Define $\phi = c_* : S \to T$. Then ϕ is the unique continuous order isomorphism such that $c = \phi \circ b$. Note that $||c_*|| \leq ||c|| \leq 1$ and $||b_*|| \leq ||b|| \leq 1$. Hence $||\phi(x)|| \leq ||x||$, for all $x \in S$. Suppose that for some $x \in S$ we have $||\phi(x)|| = ||c_*(x)|| < ||x||$, then $||x|| = ||b_*(c_*(x))|| \leq ||b_*|| \, ||c_*(x)|| < ||x||$ and that is a contradiction. Thus $||\phi(x)|| = ||x||$, for all $x \in S$. Hence ϕ is an isometric isomorphism too.

$$\begin{tikzcd} E \times F \arrow[r, "b"] \arrow[dr, "c"'] & S \arrow[d, bend right, "c_*"'] \\ & T \arrow[u, bend right, "b_*"'] \end{tikzcd}$$

□

10.2 Construction of Fremlin space tensor product

Theorem 10.3. *For all Fremlin spaces E_1 and E_2 the Fremlin space tensor product exists. If $E_1 = 0$, or $E_2 = 0$, then $(0, 0)$ is the Fremlin space tensor product of E_1 and E_2. If E_1 and E_2 are both nonzero let $S_i = \{x \in E_i : ||x|| = 1\}$, let J be the $|| \cdot ||_F$-norm closed ideal of $FBL(S_1 \times S_2)$*

generated by the elements

$$\begin{cases} \|x+y\|\,\|z\|\iota\left(\frac{x+y}{\|x+y\|},\frac{z}{\|z\|}\right)-\|x\|\,\|z\|\iota\left(\frac{x}{\|x\|},\frac{z}{\|z\|}\right)-\|y\|\,\|z\|\iota\left(\frac{y}{\|y\|},\frac{z}{\|z\|}\right), \\ x,y\in E_1\backslash\{0\}, z\in E_2\backslash\{0\}, \text{ and } x+y\neq 0, \\ \|\alpha x\|\,\|y\|\iota\left(\frac{\alpha x}{\|\alpha x\|},\frac{z}{\|y\|}\right)-\alpha\|x\|\,\|y\|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right), \\ x\in E_1\backslash\{0\}, y\in E_2\backslash\{0\}, \alpha\in\mathbb{R}\backslash\{0\}, \\ \|x\|\,\|y+z\|\iota\left(\frac{x}{\|x\|},\frac{y+z}{\|y+z\|}\right)-\|x\|\,\|y\|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)-\|x\|\,\|z\|\iota\left(\frac{x}{\|x\|},\frac{z}{\|z\|}\right), \\ x\in E_1\backslash\{0\}, y,z\in E_2\backslash\{0\}, \text{ and } y+z\neq 0, \\ \|x\|\,\|\alpha y\|\iota\left(\frac{x}{\|x\|},\frac{\alpha y}{\|\alpha y\|}\right)-\alpha\|x\|\,\|y\|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right), \\ x\in E_1\backslash\{0\}, y\in E_2\backslash\{0\}, \alpha\in\mathbb{R}\backslash\{0\}, \\ \|x\|\,\|y\|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)-\|x\|\,\|y\|\left|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)\right|, \\ x\in E_1^+\backslash\{0\}, y\in E_2^+\backslash\{0\}. \end{cases}$$

Let $T' = FBL(S_1\times S_2)/J$, then T' is a Banach lattice. Let $q : FBL(S_1\times S_2)\to T'$ be the quotient homomorphism. Define $b' : E_1\times E_2\to T'$ through

$$b'(x,y)=\begin{cases} 0 & \text{if } x=0, \text{ or } y=0, \\ q\left(\|x\|\,\|y\|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)\right) & \text{otherwise.}\end{cases}$$

Let $T = Span\{b'(x,y) : x\in E_1, y\in E_2\}$, and let $b : E_1\times E_2\to T$ be the restriction of b' to T, then b' and hence b are well defined continuous positive bilinear maps, T is a Fremlin space and (T,b) is the Fremlin space tensor product of E_1 and E_2. Moreover $\|b\| = 1$.

Proof. The case that E_1 or E_2 are zero is trivial, so suppose that E_1 and E_2 are non-zero. Let $E_1, E_2, S_1, S_2, J, T', q, b', T$ and b be as in the theorem.

Claim 10.4. *b' is a positive bimorphism.*

Proof of Claim 10.4. Let $x\in E_1, y\in E_2$ and let $\alpha\in\mathbb{R}$.
Suppose $\alpha = 0$, then $b'(\alpha x,y) = b'(0,y) = 0 = \alpha b'(x,y)$. Suppose $\alpha\neq 0$. Clearly, if $x = 0$ or $y = 0$ or both, then $b'(\alpha x,y) = 0 = \alpha b'(x,y)$. If $x\neq 0, y\neq 0$ and $\alpha\neq 0$, then

$$\begin{aligned} b'(\alpha x,y) &= q\left(\|\alpha x\|\,\|y\|\iota\left(\frac{\alpha x}{\|\alpha x\|},\frac{y}{\|y\|}\right)\right) \\ &= q\left(\alpha\|x\|\,\|y\|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)\right) \\ &= \alpha q\left(\|x\|\,\|y\|\iota\left(\frac{x}{\|x\|},\frac{y}{\|y\|}\right)\right) \\ &= \alpha b'(x,y). \end{aligned}$$

Thus $b'(\alpha x,y) = \alpha b'(x,y)$, for all $x\in E_1, y\in E_2$. Let $x,y\in E_1, z\in E_2$. If $z = 0$ then $b'(x+y,z) = 0 = b'(x,z)+b'(y,z)$. Suppose $z\neq 0$. If $x = 0$ or $y = 0$ or both are 0, then $b'(x+y,z) = b'(x,z)+b'(y,z)$. If $x\neq 0$ and $y\neq 0$, but $x+y = 0$, then $x = -y$. Thus $b'(x,z) = b'(-y,z) = -b'(y,z)$. It follows that $b'(x+y,z) = 0 = b'(x,z)+b'(y,z)$. Finely, if $x\neq 0, y\neq 0$ and $x+y\neq 0$, then

$$\begin{aligned} b'(x+y,z) &= q\left(\|x+y\|\,\|z\|\iota\left(\frac{x+y}{\|x+y\|},\frac{z}{\|z\|}\right)\right) \\ &= q\left(\|x\|\,\|z\|\iota\left(\frac{x}{\|x\|},\frac{z}{\|z\|}\right)+\|y\|\,\|z\|\iota\left(\frac{y}{\|y\|},\frac{z}{\|z\|}\right)\right) \\ &= q\left(\|x\|\,\|z\|\iota\left(\frac{x}{\|x\|},\frac{z}{\|z\|}\right)\right)+q\left(\|y\|\,\|z\|\iota\left(\frac{y}{\|y\|},\frac{z}{\|z\|}\right)\right) \\ &= b'(x,z)+b'(y,z). \end{aligned}$$

We conclude that $b'(\cdot, z)$ is linear, for all $z \in E_2$. Likewise $b'(x, \cdot)$ is linear, for all $x \in E_1$. Thus b' is bilinear.
Let $x \in L_1^+ \backslash \{0\}$ and $y \in L_2^+ \backslash \{0\}$. Then

$$\begin{aligned} b'(x,y) &= q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right) \\ &= q\left(||x||\,||y||\left|\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right|\right) \\ &= \left|q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right)\right| \\ &= |b'(x,y)| \geq 0. \end{aligned}$$

Hence b' is a positive bilinear map. □

By Proposition 1.18 on page 12 T' is a Banach lattice with the quotientnorm and -ordering. By Theorem 1.35 on page 15 it follows that T is a Fremlin space. Clearly, b is a positive bimorphism. Let F be an arbitrary Fremlin space and let $\phi : E_1 \times E_2 \to F$ be a continuous positive bimorphism. According to Theorem 1.35 on page 15 we may assume that F is a subspace of a Banach lattice $\mathbb{F}$. View ϕ as a continuous positive bilinear map into $\mathbb{F}$.
Let $\phi_1 : S_1 \times S_2 \to \mathbb{F}$ be the restriction of ϕ to $S_1 \times S_2$. Thus ϕ_1 is a bounded map. Let $\psi : \mathrm{FBL}(S_1 \times S_2) \to \mathbb{F}$ be the unique Riesz homomorphism that satisfies $\phi_1 = \psi \circ \iota$ and $||\psi|| = ||\phi_1|| = ||\phi||$. We will prove that $J \subset \ker(\psi)$. Since $\ker(\psi)$ is a $||\cdot||_F$-norm closed order ideal, we only need to prove that the set that generates J is contained in $\ker(\psi)$. Let $\alpha \in \mathbb{R}\backslash\{0\}, x \in E_1\backslash\{0\}, y \in E_2\backslash\{0\}$. Then

$$\begin{aligned} & \psi\left(||\alpha x||\,||y||\iota\left(\tfrac{\alpha x}{||\alpha x||}, \tfrac{y}{||y||}\right) - \alpha||x||\,||y||\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right) \\ = \; & ||\alpha x||\,||y||\psi\left(\iota\left(\tfrac{\alpha x}{||\alpha x||}, \tfrac{y}{||y||}\right)\right) - \alpha||x||\,||y||\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right) \\ = \; & ||\alpha x||\,||y||\phi_1\left(\tfrac{\alpha x}{||\alpha x||}, \tfrac{y}{||y||}\right) - \alpha||x||\,||y||\phi_1\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right) \\ = \; & \phi(\alpha x, y) - \alpha\phi(x,y) \\ = \; & 0. \end{aligned}$$

Thus $||\alpha x||\,||y||\iota\left(\frac{\alpha x}{||\alpha x||}, \frac{y}{||y||}\right) - \alpha||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right) \in \ker(\psi)$. Likewise $||x||\,||\alpha y||\iota\left(\frac{x}{||x||}, \frac{\alpha y}{||\alpha y||}\right) - \alpha||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right) \in \ker(\psi)$, for all $x \in E_1\backslash\{0\}, y \in E_2\backslash\{0\}$ and $\alpha \in \mathbb{R}\backslash\{0\}$.
For all $x, y \in E_1\backslash\{0\}$ and $z \in E_2\backslash\{0\}$ with $x + y \neq 0$ we have

$$\begin{aligned} & \psi\left(||x+y||\,||z||\iota\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right) - ||x||\,||z||\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right) - ||y||\,||z||\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\ = \; & ||x+y||\,||z||\psi\left(\iota\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right)\right) - ||x||\,||z||\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right)\right) - ||y||\,||z||\psi\left(\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\ = \; & ||x+y||\,||z||\phi_1\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right) - ||x||\,||z||\phi_1\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right) - ||y||\,||z||\phi_1\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right) \\ = \; & \phi(x+y, z) - \phi(x,z) - \phi(y,z) \\ = \; & 0. \end{aligned}$$

Thus $||x+y||\,||z||\iota\left(\frac{x+y}{||x+y||}, \frac{z}{||z||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||}, \frac{z}{||z||}\right) - ||y||\,||z||\iota\left(\frac{y}{||y||}, \frac{z}{||z||}\right) \in \ker(\psi)$. Likewise $||x||\,||y+z||\iota\left(\frac{x}{||x||}, \frac{y+z}{||y+z||}\right) - ||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||}, \frac{z}{||z||}\right) \in \ker(\psi)$, for all $x \in E_1\backslash\{0\}$ and $y, z \in E_2\backslash\{0\}$ with $y + z \neq 0$.
For all $x \in E_1^+\backslash\{0\}$ and $y \in E_2^+\backslash\{0\}$ we have

$$\begin{aligned} & \psi\left(||x||\,||y||\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right) - ||x||\,||y||\left|\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right|\right) \\ = \; & ||x||\,||y||\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right) - ||x||\,||y||\left|\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right)\right| \\ = \; & ||x||\,||y||\phi_1\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right) - ||x||\,||y||\left|\phi_1\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right| \\ = \; & \phi(x,y) - |\phi(x,y)| \\ = \; & 0, \end{aligned}$$

since ϕ is positive, and x and y are positive. It follows that

$$||x||\,||y||\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)-||x||\,||y||\left|\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right|\in\ker(\psi).$$

Thus $J\subset\ker(\psi)$.
Define $\phi_*':T'\to\mathbb{F}$ through $\phi_*(q(x))=\psi(x), x\in\mathrm{FBL}(S_1\times S_2)$. Then ϕ_* is a well defined Riesz homomorphism, since $\phi_*(|q(x)|)=\phi_*(q(|x|))=\psi(|x|)=|\psi(x)|=|\phi_*(q(x))|$. Note that for all $x\in E_1\backslash\{0\}$ and $y\in E_2\backslash\{0\}$ we have

$$\begin{aligned}
\phi_*(b(x,y)) &= \phi_*\left(q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\right)\\
&= ||x||\,||y||\psi\left(\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\\
&= ||x||\,||y||\phi_1\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\\
&= \phi(x,y).
\end{aligned}$$

If $x\in E_1, y\in E_2$ and $x=0$ or $y=0$ or both, then $\phi_*'(b(x,y))=\phi_*(q(0))=\psi(0)=0=\phi(x,y)$. Thus $\phi=\phi_*'\circ b$.
In particular ϕ_*' maps T into F. Let $\phi_*:T\to F$ be the restriction of ϕ_*' to T.
It is clear that $\phi=\phi_*\circ b$ and that ϕ_* is positive. We have to show that ϕ_* is the unique positive linear map with these properties. Let $\chi:T\to F$ be a positive linear map that satisfies $\phi=\chi\circ b$. Let $t\in T$. Then there are $x_1,\dots,x_n\in E_1, y_1,\dots,y_n\in E_2$ with $t=\sum_{i=1}^n b(x_i,y_i)$. Thus $\chi(t)=\sum_{i=1}^n\chi(b(x_i,y_i))=\sum_{i=1}^n\phi(x_i,y_i)=\sum_{i=1}^n\phi_*(b(x_i,y_i))=\phi_*(t)$. In fact, we proved that ϕ_* is the unique *linear map* $\chi:T\to F$ with the property that $\phi=\chi\circ b$. Note that $||\phi_*||\le||\phi_*'||$, and for every $x\in FBL(S_1\times S_2)$ of norm one, we have $||\phi_*'(q(x))||=||\psi(x)||$. Therefore $||\phi_*'||=||\psi||=||\phi_1||=||\phi||$. Thus $||\phi_*||\le||\phi||$. Note that for all $x\in S_1, y\in S_2$ we have $||b(x,y)||=||q(\iota(x,y))||\le||\iota(x,y)||=1$ thus $||b||\le1$. We conclude that (T,b) is the Fremlin space tensor product of E_1 and E_2. Note that b is non-trivial. Thus $||b||>0$. We have already seen that $||b||\le1$. Suppose $||b||=1-\varepsilon<1$ for some $0<\varepsilon<1$. Then for all $x\in S_1, y\in S_2$ we have that $||b(x,y)||=||b_*(b(x,y))||\le||b_*||\,||b(x,y)||\le(1-\varepsilon)||b_*||\le(1-\varepsilon)||b||$ thus $||b||\le(1-\varepsilon)||b||$ and that is a contradiction with the fact that $||b||>0$. We conclude that $||b||=1$. This concludes the proof of the theorem.

$$\begin{tikzcd}
\mathrm{FBL}(S_1\times S_2) \arrow[r, "q"] \arrow[dr, "\psi"] & T' \arrow[d, "\phi_*'"] & T \arrow[l, hook'] \arrow[d, "\phi_*"] \\
S_1\times S_2 \arrow[u, hook, "\iota"] \arrow[r, "\phi_1"'] & \mathbb{F} & F \arrow[l, hook']
\end{tikzcd}$$

□

Remark 10.5. From the construction follows that T is generated as vector space by elements $b(x,y), x\in E_1, y\in E_2$ and that the Fremlin space tensor product of E_1 and E_2 is order isomorphic to the Fremlin space tensor product of E_2 and E_1.

Theorem 10.6. *Let E', F', E and F be Fremlin spaces such that E' is a subspace of E and F' is a subspace of F. Let (T,b') be the Fremlin space tensor product of E' and F' and (T,b) the Fremlin space tensor product of E and F. Then there is a continuous bipositive linear map $\iota:T'\to T$, such that for all $x\in E'$ and $y\in F'$: $\iota(b'(x,y))=b(x,y)$.*

Proof. The proof is similar to the proof of Theorem 6.8 on page 41. Note that $||\iota||\le||b||\le1$. Hence ι is continuous. □

11 The normed Riesz space tensor product and Banach lattice tensor product

If Riesz spaces L and M have a Riesz norm, then it will be natural if the Archimedean Riesz tensor product also has a Riesz norm, likewise the tensor product of Banach lattices should be a Banach lattice. In this section we define and study the Banach lattice tensor product. We give a construction of the Banach lattice tensor product as a quotient of a free Banach lattice. Finally we consider the relation between the Fremlin space tensor product and the Banach lattice tensor product. We will prove that the Fremlin space tensor product is also the positive tensor product. Recall that a norm $||\cdot||$ on a Riesz space L is called a Riesz norm when $|x| \leq |y|$ implies that $||x|| \leq ||y||$, for all $x, y \in L$ (Definition 1.29 on page 14) and that a normed Riesz space is Archimedean (Proposition 1.32 on page 15).

11.1 Definitions and properties

Definition 11.1. Let L and M be normed Riesz spaces (Banach lattices). The normed Riesz space (Banach lattice) tensor product of L and M is a pair (T, b), where T is a normed Riesz space (Banach lattice) and $b : L \times M \to T$ is a continuous Riesz bimorphism with $||b|| \leq 1$ and the property that for every normed Riesz space (Banach lattice) N and for every continuous Riesz bimorphism $\phi : L \times M \to N$ there is a continuous Riesz homomorphism $\phi_* : T \to N$ such that $\phi = \phi_* \circ b$ and $||\phi_*|| \leq ||\phi||$. Moreover, we require that ϕ_* is the unique Riesz homomorphism χ with the property that $\phi = \chi \circ b$.

$$\begin{tikzcd} L \times M \arrow[r, "b"] \arrow[dr, "\phi"'] & T \arrow[d, "\phi_*"] \\ & N \end{tikzcd}$$

Remark 11.2. Later on we will see that $||\phi_*|| = ||\phi||$ and when the spaces are non-trivial, that $||b|| = 1$.

If the normed Riesz space (Banach lattice) tensor product exists, then it is unique as Riesz space and as normed space.

Theorem 11.3. *Let L and M be normed Riesz spaces (Banach lattices). Suppose (S, b) and (T, c) are normed Riesz space (Banach lattice) tensor products of L and M. Then there is a unique Riesz homomorphism $\phi : S \to T$ such that $\phi \circ b = c$. Moreover ϕ is invertible, $\phi^{-1} : T \to S$ is a Riesz homomorphism and ϕ is isometric. In particular, ϕ is an order isomorphism.*

Proof. The proof is similar to the proof of Theorem 10.2 on page 53. □

Theorem 11.4. *Let L and M be normed Riesz spaces (Banach lattices). If (T, b) is the normed Riesz space (Banach lattice) tensor product of L and M, then T is generated as Riesz space (Banach lattice) by elements $b(x, y)$.*

Proof. Let L and M be normed Riesz spaces (Banach lattices). Suppose (T, b) is the normed Riesz space (Banach lattice) tensor product of L and M. Let S be the Riesz subspace (Banach sublattice) of T generated by elements $b(x, y)$. Note that b maps into S. Let N be an arbitrary normed Riesz space (Banach lattice) and let $\phi : L \times M \to N$ be a continuous Riesz homomorphism. Let $\phi'_* : T \to N$ be the unique continuous Riesz homomorphism with $\phi = \phi'_* \circ b$. Note that $||\phi_*|| \leq ||\phi'_*|| \leq ||\phi||$. Let $\phi_* : S \to N$ be the restriction of ϕ'_* to S. Then ϕ_* is a Riesz homomorphism and $\phi = \phi_* \circ b$. Let $\psi : S \to N$ be any Riesz homomorphism such that $\phi = \psi \circ b$. Then $\phi_*(b(x, y)) = \psi(b(x, y))$, for all $x \in L$ and $y \in M$. Thus ϕ_* and ψ coincide on S. Hence $\phi_* = \psi$. We conclude that (S, b) is also the normed Riesz space (Banach lattice) tensor product of L and M. From Theorem 11.3 follows that $S = T$. This concludes our proof. □

Theorem 11.5. *Let L and M be Banach lattices. Suppose (T', b) is the normed Riesz space tensor product of L and M. Let T be the norm completion of T'. Then (T, b) is the Banach lattice tensor product of L and M.*

Proof. Let L and M be Banach lattices. Suppose (T', b) is the normed Riesz space tensor product of L and M. Let T be the norm completion of T'. Then by Theorem 2.27 on page 24 T is a Banach lattice. Note that $b : L \times M \to T$ is a continuous Riesz bimorphism. Let N be any Banach lattice and $\phi : L \times M \to N$ a continuous Riesz bimorphism. Then there exist a unique continuous Riesz homomorphism $\psi : T' \to N$ such that $\phi = \psi \circ b$. Let ϕ_* be the continuous extension of ψ to T. Then ϕ_* is a Riesz homomorphism and $\phi = \phi_* \circ b$. Let $\chi : T \to N$ be an arbitrary Riesz homomorphism such that $\phi = \chi \circ b$. Since b maps into T', we have that $\phi = \chi|_{T'} \circ b$. From the uniqueness of ψ follows that $\chi|_{T'} = \psi$. The continuous extensions of $\chi|_{T'}$ and ψ coincide thus $\chi = \psi_*$. Note that $||\phi_*|| = ||\psi|| \leq ||\phi||$ and $||b|| \leq 1$. We conclude that (T, b) is the Banach lattice tensor product of L and M. $\square$

11.2 Construction of the normed Riesz space and Banach lattice tensor product via a free normed Riesz space.

Theorem 11.6. *Let L_1 and L_2 be normed Riesz spaces (Banach lattices). If $L_1 = 0$ or $L_2 = 0$, then $(0, 0)$ is the normed Riesz space (Banach lattice) tensor product of L_1 and L_2. Suppose L_1 and L_2 are non-zero, let $S_i = \{x \in L_i : ||x|| = 1\}$, for $i \in \{1, 2\}$. Consider the free normed Riesz space $(FNRS(S_1 \times S_2), \iota)$ (free Banach lattice $FBL(S_1 \times S_2)$) with Riesz norm $||\cdot||_F$. Let J be the intersection of all $||\cdot||_F$-norm closed order ideals that contain*

$$\left\{\begin{array}{l}
||x+y||\,||z||\iota\left(\frac{x+y}{||x+y||}, \frac{z}{||z||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||}, \frac{z}{||z||}\right) - ||y||\,||z||\iota\left(\frac{y}{||y||}, \frac{z}{||z||}\right), \\
x, y \in L_1\backslash\{0\}, z \in L_2\backslash\{0\}, \text{ and } x+y \neq 0, \\
||\alpha x||\,||y||\iota\left(\frac{\alpha x}{||\alpha x||}, \frac{y}{||y||}\right) - \alpha||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right), \\
x \in L_1\backslash\{0\}, y \in L_2\backslash\{0\}, \alpha \in \mathbb{R}\backslash\{0\}, \\
||x||\,||y+z||\iota\left(\frac{x}{||x||}, \frac{y+z}{||y+z||}\right) - ||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||}, \frac{z}{||z||}\right), \\
x \in L_1\backslash\{0\}, y, z \in L_2\backslash\{0\}, \text{ and } y+z \neq 0, \\
||x||\,||\alpha y||\iota\left(\frac{x}{||x||}, \frac{\alpha y}{||\alpha y||}\right) - \alpha||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right), \\
x \in L_1\backslash\{0\}, y \in L_2\backslash\{0\}, \alpha \in \mathbb{R}\backslash\{0\}, \\
||x||\,||y|| \left|\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right)\right| - ||x||\,||y||\iota\left(\frac{|x|}{||x||}, \frac{|y|}{||y||}\right), \\
x \in L_1\backslash\{0\}, y \in L_2\backslash\{0\}.
\end{array}\right. \tag{17}$$

Let $T = FNRS(S_1 \times S_2)/J$, and let $q : FNRS(S_1 \times S_2) \to T$ be the quotient homomorphism, define $b : L_1 \times L_2 \to T$ through

$$b(x, y) = \begin{cases} 0 & \text{if } x = 0, \text{ or } y = 0, \\ q\left(||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right)\right) & \text{otherwise.} \end{cases} \tag{18}$$

Then b is a continuous Riesz bimorphism and (T, b) is the normed Riesz space (Banach lattice) tensor product of L_1 and L_2. Moreover for any normed Riesz space (Banach lattice) M and any continuous Riesz bimorphism $\phi : L_1 \times L_2 \to M$, the induced continuous Riesz homomorphism $\phi_* : T \to M$ that satisfies $\phi = \phi_* \circ b$ also satisfies $||\phi_*|| = ||\phi||$. Furthermore if L_1 and L_2 are non-trivial then $||b|| = 1$.

Proof. Since the proof of the Banach lattice case is similar to the proof of the normed Riesz space case, we only do the latter. Let L_1 and L_2 be normed Riesz spaces. The case that L_1 or L_2 is 0 is trivial, so suppose that both L_1 and L_2 are non-trivial. Let S_1, S_2, J, T, q and b be as in the theorem. We will show that b is a Riesz bimorphism and that (T, b) is the normed Riesz space tensor product of L_1 and L_2.

Claim 11.7. *b is a continuous Riesz bimorphism.*

Proof of Claim 11.7 on the previous page. Note that $b(x,y)=0$, if $x=0$ or $y=0$. Let $\alpha\in\mathbb{R}$. If $\alpha=0$, then $b(\alpha x,y)=b(0,y)=0=\alpha b(x,y)$. Suppose $\alpha\neq 0$. Clearly, if $x=0$ or $y=0$ or both, then $b(\alpha x,y)=\alpha b(x,y)$. If $x\neq 0, y\neq 0$ and $\alpha\neq 0$, then

$$\begin{aligned} b(\alpha x,y) &= q\left(||\alpha x||\,||y||\iota\left(\tfrac{\alpha x}{||\alpha x||},\tfrac{y}{||y||}\right)\right)\\ &= q\left(\alpha||x||\,||y||\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\\ &= \alpha q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\\ &= \alpha b(x,y). \end{aligned}$$

Let $x,y\in L_1, z\in L_2$. If $z=0$, then $b(x+y,z)=0=b(x,z)+b(y,z)$. If $z\neq 0$, and $x=0$ or $y=0$ or both are 0, then, clearly, $b(x+y,z)=b(x,z)+b(y,z)$. If $z\neq 0, x\neq 0$ and $y\neq 0$ but $x+y=0$, then $x=-y$. Thus $b(x,z)=b(-y,z)=-b(y,z)$. Hence $b(x+y,z)=0=b(x,z)+b(y,z)$. Finally, if $x\neq 0, y\neq 0, z\neq 0$ and $x+y\neq 0$, then

$$\begin{aligned} b(x+y,z) &= q\left(||x+y||\,||z||\iota\left(\tfrac{x+y}{||x+y||},\tfrac{z}{||z||}\right)\right)\\ &= q\left(||x||\,||z||\iota\left(\tfrac{x}{||x||},\tfrac{z}{||z||}\right)+||y||\,||z||\iota\left(\tfrac{y}{||y||},\tfrac{z}{||z||}\right)\right)\\ &= q\left(||x||\,||z||\iota\left(\tfrac{x}{||x||},\tfrac{z}{||z||}\right)\right)+q\left(||y||\,||z||\iota\left(\tfrac{y}{||y||},\tfrac{z}{||z||}\right)\right)\\ &= b(x,z)+b(y,z). \end{aligned}$$

Hence $b(\cdot,z)$ is linear, for all $z\in L_2$. Likewise $b(x,\cdot)$ is linear, for all $x\in L_1$. Thus b is bilinear. Let $x\in L_1, y\in L_2$. Suppose $x=0$ or $y=0$ or both, then $|b(x,y)|=0=b(|x|,|y|)$. Suppose $x\neq 0$ and $y\neq 0$. Then

$$\begin{aligned} |b(x,y)| &= \left|q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\right|\\ &= q\left(||x||\,||y||\left|\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right|\right)\\ &= q\left(||x||\,||y||\iota\left(\tfrac{|x|}{||x||},\tfrac{|y|}{||y||}\right)\right)\\ &= b(|x|,|y|). \end{aligned}$$

By Theorem 4.2 on page 29 it follows that b is a Riesz bimorphism. Clearly b is continuous. □

Let N be an arbitrary normed Riesz space and let $\phi: L_1\times L_2\to N$ be a continuous Riesz bimorphism. Let $\phi_1: S_1\times S_2\to N$ be the restriction of ϕ to $S_1\times S_2$. Note that ϕ_1 is a bounded map. Let $\psi:\mathrm{FBL}(S_1\times S_2)\to N$ be the unique Riesz homomorphism that satisfies $\phi_1=\psi\circ\iota$ with the property $||\psi||=||\phi_1||=||\phi||$. We will prove that $J\subset\ker(\psi)$. Since $\ker(\psi)$ is a $||\cdot||_F$-norm closed order ideal, we only have to prove that the set that generates J is contained in $\ker(\psi)$. Let $\alpha\in\mathbb{R}\backslash\{0\}, x\in L_1\backslash\{0\}, y\in L_2\backslash\{0\}$. Then

$$\begin{aligned} &\psi\left(||\alpha x||\,||y||\iota\left(\tfrac{\alpha x}{||\alpha x||},\tfrac{y}{||y||}\right)-\alpha||x||\,||y||\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\\ =\;& ||\alpha x||\,||y||\psi\left(\iota\left(\tfrac{\alpha x}{||\alpha x||},\tfrac{y}{||y||}\right)\right)-\alpha||x||\,||y||\psi\left(\iota\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\right)\\ =\;& ||\alpha x||\,||y||\phi_1\left(\tfrac{\alpha x}{||\alpha x||},\tfrac{y}{||y||}\right)-\alpha||x||\,||y||\phi_1\left(\tfrac{x}{||x||},\tfrac{y}{||y||}\right)\\ =\;& \phi(\alpha x,y)-\alpha\phi(x,y)\\ =\;& 0. \end{aligned}$$

Thus $||\alpha x||\,||y||\iota\left(\frac{\alpha x}{||\alpha x||},\frac{y}{||y||}\right)-\alpha||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)\in\ker(\psi)$, for all $x\in L_1\backslash\{0\}, y\in L_2\backslash\{0\}$ and $\alpha\in\mathbb{R}\backslash\{0\}$. Likewise $||x||\,||\alpha y||\iota\left(\frac{x}{||x||},\frac{\alpha y}{||\alpha y||}\right)-\alpha||x||\,||y||\iota\left(\frac{x}{||x||},\frac{y}{||y||}\right)\in\ker(\psi)$, for all $x\in L_1\backslash\{0\}, y\in L_2\backslash\{0\}$ and $\alpha\in\mathbb{R}\backslash\{0\}$.

For all $x, y \in L_1\backslash\{0\}$ and $z \in L_2\backslash\{0\}$ with $x+y \neq 0$ we have

$$\begin{aligned}
& \psi\left(||x+y||\,||z||\iota\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right) - ||x||\,||z||\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right) - ||y||\,||z||\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\
= \ & ||x+y||\,||z||\psi\left(\iota\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right)\right) - ||x||\,||z||\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right)\right) - ||y||\,||z||\psi\left(\iota\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right)\right) \\
= \ & ||x+y||\,||z||\phi_1\left(\tfrac{x+y}{||x+y||}, \tfrac{z}{||z||}\right) - ||x||\,||z||\phi_1\left(\tfrac{x}{||x||}, \tfrac{z}{||z||}\right) - ||y||\,||z||\phi_1\left(\tfrac{y}{||y||}, \tfrac{z}{||z||}\right) \\
= \ & \phi(x+y, z) - \phi(x, z) - \phi(y, z) \\
= \ & 0.
\end{aligned}$$

Thus $||x+y||\,||z||\iota\left(\frac{x+y}{||x+y||}, \frac{z}{||z||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||}, \frac{z}{||z||}\right) - ||y||\,||z||\iota\left(\frac{y}{||y||}, \frac{z}{||z||}\right) \in \ker(\psi)$. Likewise, for all $x \in L_1\backslash\{0\}$ and $y, z \in L_2\backslash\{0\}$ with $y+z \neq 0$, we have that $||x||\,||y+z||\iota\left(\frac{x}{||x||}, \frac{y+z}{||y+z||}\right) - ||x||\,||y||\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right) - ||x||\,||z||\iota\left(\frac{x}{||x||}, \frac{z}{||z||}\right) \in \ker(\psi)$.
For all $x \in L_1\backslash\{0\}$ and $y \in L_2\backslash\{0\}$ we have

$$\begin{aligned}
& \psi\left(||x||\,||y||\left|\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right| - ||x||\,||y||\iota\left(\tfrac{|x|}{||x||}, \tfrac{|y|}{||y||}\right)\right) \\
= \ & ||x||\,||y||\left|\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right)\right| - ||x||\,||y||\psi\left(\iota\left(\tfrac{|x|}{||x||}, \tfrac{|y|}{||y||}\right)\right) \\
= \ & ||x||\,||y||\left|\phi_1\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right| - ||x||\,||y||\phi_1\left(\tfrac{|x|}{||x||}, \tfrac{|y|}{||y||}\right) \\
= \ & |\phi(x, y)| - \phi(|x|, |y|) \\
= \ & 0.
\end{aligned}$$

Thus $||x||\,||y||\left|\iota\left(\frac{x}{||x||}, \frac{y}{||y||}\right)\right| - ||x||\,||y||\iota\left(\frac{|x|}{||x||}, \frac{|y|}{||y||}\right) \in \ker(\psi)$. It follows that $J \subset \ker(\psi)$.
Define $\phi_* : T \to N$ through $\phi_*(q(x)) = \psi(x)$, for $x \in \mathrm{FBL}(S_1 \times S_2)$. Then ϕ_* is a well defined Riesz homomorphism. We have

$$\begin{aligned}
\phi_*(b(x, y)) &= \phi_*\left(q\left(||x||\,||y||\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right)\right) \\
&= ||x||\,||y||\psi\left(\iota\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right)\right) \\
&= ||x||\,||y||\phi_1\left(\tfrac{x}{||x||}, \tfrac{y}{||y||}\right) \\
&= \phi(x, y),
\end{aligned}$$

for all $x \in L_1\backslash\{0\}$ and $y \in L_2\backslash\{0\}$. If $x \in L_1, y \in L_2$ and $x = 0$ or $y = 0$ or both, then $\phi_*(b(x, y)) = \phi_*(q(0)) = \psi(0) = 0 = \phi(x, y)$. So $\phi = \phi_* \circ b$.
Suppose $\chi : T \to N$ is a Riesz homomorphism that satisfies $\phi = \chi \circ b$. In particular $\phi_1 = \chi \circ q \circ \iota$. From the uniqueness statement in the definition of a free normed Riesz space follows that $\chi \circ q : \mathrm{FNRS}(S_1 \times S_2) \to N$ is equal to ψ. Thus $\chi(q(x)) = \psi(x) = \phi_*(q(x))$, for all $x \in \mathrm{FBL}(S_1 \times S_2)$. Hence $\chi = \phi_*$.
Note that $||b(x, y)|| = ||q(\iota(x, y))|| \leq ||\iota(x, y)|| = 1 = ||x||\,||y||$, for all $x \in S_1, y \in S_2$. Thus $||b|| \leq 1$. For all $x \in \mathrm{FBL}(S_1 \times S_2)$ of norm one, we have $||\phi_*(q(x))|| = ||\psi(x)||$ thus $||\phi_*|| = ||\psi|| = ||\phi_1|| = ||\phi||$. This concludes our proof that (T, b) is the normed Riesz space tensor product of L_1 and L_2.
Note that b is non-trivial. Thus $||b|| > 0$. We have already seen that $||b|| \leq 1$. Suppose $||b|| = 1 - \varepsilon < 1$ for some $0 < \varepsilon < 1$. Then for all $x \in S_1, y \in S_2$ we have that $||b(x, y)|| = ||b_*(b(x, y))|| \leq ||b_*||\,||b(x, y)|| \leq (1-\varepsilon)||b_*|| = (1-\varepsilon)||b||$, and that is a contradiction. We conclude that $||b|| = 1$. This concludes the proof of the theorem.

$$\begin{tikzcd}
\mathrm{FNRS}(S_1 \times S_2) \arrow[r, "q"] \arrow[dr, "\psi"] & T \arrow[d, "\phi_*"] \\
S_1 \times S_2 \arrow[u, hook, "\iota"] \arrow[r, "\phi_1"'] & N
\end{tikzcd}$$

□

Remark 11.8. D.H. Fremlin proved also the following fact for Banach lattices [4, Theorem 1E(iii)]: let L, M and N be Banach lattices and let (T, b) be the Banach lattice tensor product of L and M. Then there is a one-to-one norm preserving correspondence between the continuous positive bimorphisms $\phi : L \times M \to N$ and the continuous positive linear maps $\phi_* : T \to N$ such that $\phi = \phi_* \circ b$. Moreover ϕ is a continuous Riesz bimorphism if and only if ϕ_* is a continuous Riesz homomorphism. We could not find a proof for this fact with the new construction.

Theorem 11.9. *Let L', L, M' and M be normed Riesz spaces (Banach lattices) with $L' \subset L$ and $M' \subset M$. Let (T', b') be the normed Riesz space (Banach lattice) tensor product of L' and M' and let (T, b) be the normed Riesz space (Banach lattice) tensor product of L and M. Then there is an injective continuous Riesz homomorphism $\iota : T' \to T$ such that $\iota(b'(x, y)) = b(x, y)$, for all $x \in L'$ and $y \in M'$. In particular ι is bipositive.*

Proof. The proof is similar to the proof of Theorem 6.8 on page 41. Note that $||\iota|| \leq ||b|| \leq 1$. Hence ι is continuous. □

Remark 11.10.
1. For Banach lattices L and M we denote the Banach lattice tensor product T by $L\widehat{\otimes}M$ and the bimorphism b by $\otimes$.
2. From the construction follows clearly that $L\widehat{\otimes}M$ is isomorphic as Banach lattice to $M\widehat{\otimes}L$.

11.3 Fremlin space tensor product and Banach lattice tensor product

In this subsection we study the relation between the Fremlin space tensor product and the Banach lattice tensor product. We will see that the Fremlin space tensor product is in a nice way a subspace of the Banach lattice tensor product. We need the representation due to D.H. Fremlin.
According to D.H. Fremlin we have the following.

Theorem 11.11 (D.H. Fremlin)**.** *Let L and M be Banach lattices and $L\widehat{\otimes}M$ the Banach lattice tensor product of L and M. For any Banach lattice N and for any continuous positive linear bimorphism $\phi : L \times M \to N$ there is a unique continuous positive linear map $\phi_* : L\widehat{\otimes}M \to N$ such that $\phi = \phi_* \circ \otimes$ and $||\phi_*|| = ||\phi||$ [4, Theorem 1E(iii)]. The Riesz subspace $L\bar{\otimes}M \subset L\widehat{\otimes}M$ generated by tensors $x \otimes y, x \in L, y \in M$ is the Riesz space tensor product of L and M [4, Theorem 1E(vi)]. The linear subspace $L \otimes M \subset L\widehat{\otimes}M$ spanned by tensors $x \otimes y, x \in L, y \in M$ is the usual vector space tensor product [3, Theorem 4.2(ii)] and also the positive tensor product with induced ordering [3, Proposition 5.1].*

This has the following nice corollary.

Theorem 11.12. *Let L and M be Banach lattices, then $L \otimes M$ with induced ordering and norm of $L\widehat{\otimes}M$ is the Fremlin space tensor product of L and M.*

Proof. Let F be an arbitrary Fremlin space and $\phi : L \times M \to F$ be a continuous positive bimorphism. By Theorem 1.35 on page 15 we can view F as a subspace of a Banach lattice $\mathbb{F}$. Thus we view ϕ as a map into $\mathbb{F}$. By Theorem 11.11 there is a unique continuous positive linear map $\phi'_* : L\widehat{\otimes}M \to \mathbb{F}$ such that $\phi = \phi_* \circ \otimes$. Note that ϕ_* maps $L \otimes M$ into F. Let $\phi_* : L \otimes M \to F$ be the restriction of ϕ'_*. Thus $\phi = \phi_* \circ \otimes$ and $||\phi_*|| \leq ||\phi'_*|| = ||\phi||$. Since $L \otimes M$ is also the positive tensor product of L and M, we have that ϕ_* is the unique positive linear map ψ with $\phi = \psi \circ \otimes$. We conclude that $L \otimes M$ is the Fremlin space tensor product of L and M. □

Theorem 11.13. *Let F and G be subspaces of Banach lattices $\mathbb{F}$ and $\mathbb{G}$ respectively. If $T = Span\{x \otimes y \in \mathbb{F}\widehat{\otimes}\mathbb{G} : x \in F, y \in G\}$, then $(T, \otimes)$ with induced ordering and norm of $\mathbb{F}\widehat{\otimes}\mathbb{G}$ is the Fremlin space tensor product of F and G.*

Proof. Follows from Theorems 10.6 on page 56 and 11.12. □

Corollary 11.14. *The Fremlin space tensor product is also the positive tensor product.*

12 Open problems

We finish our thesis with some open problems.

1. If E is an arbitrary partially ordered vector space and I an order ideal of E, does there exist a topology τ on E, such that E/I is integrally closed if and only if I is τ-closed? We know that with the ru-topology on $E, E/I$ is Archimedean if and only if I is ru-closed (Theorem 1.23 on page 13). If so, we can probably solve open question 4 and give a fourth construction of the positive tensor product as the integrally closed completion of $(E \otimes F, K_T)$, where K_T is the projective cone in $E \otimes F$.

2. Does every Riesz* bimorphism $b : E \times F \to G$ extend to a Riesz bimorphism between the Riesz completions, for arbitrary pre-Riesz spaces E, F and G? If so, then the integrally closed Riesz* tensor product exists for every pair of integrally closed pre-Riesz spaces E and F.

3. Let E and F be arbitrary pre-Riesz spaces with Riesz completions (E^r, φ_E) and (F^r, φ_F) respectively. Let (T', b') be the Riesz tensor product of E^r and F^r, and T the vector subspace of T' spanned by elements $b'(\varphi_E(x), \varphi_F(y)), x \in E, y \in F$. Let $\iota_T : T \to T'$ be the inclusion map. Is (T', ι_T) the Riesz completion of T? In the integrally closed case this is valid (Theorem 6.13 on page 43).

4. Does the 'integrally closed completion' of a partially ordered vector space exists?

5. Is for all integrally closed pre-Riesz spaces E, F and G and for every Riesz* bimorphism $\phi : E \times F \to G$ the induced (positive) linear map $\phi_* : E \otimes F \to G$ a Riesz* homomorphism? In that case the positive and integrally closed Riesz* tensor product coincide.

6. Does the (integrally closed) Riesz* tensor product exists for all pairs of (integrally closed) pre-Riesz spaces E and F?

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