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# HASSE-WEIL ZETA-FUNCTION IN A SPECIAL CASE

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## 1. Introduction

In number theory, arithmetic geometry and algebraic geometry, the theory of L-functions, which is closely connected to automorphic forms, has become a major point. The Hasse-Weil zeta function of varieties over number fields are conjecturally products of automorphic L-functions. Through the efforts of many people, from Eichler, Shimura, Kuga, Sato and Ihara, who studied $\mathrm{GL}_2$, to Langlands, Rapoport, and Kottwitz, the final conjectural description of the zeta function in terms of automorphic L-functions has been verified, in certain cases. This master thesis, following very closely P. Scholze's preprint ([30]), gives a quick review of determining the Hasse-Weil zeta function in a special case of some moduli schemes of elliptic curves with level-structure, via the method of Langlands (cf.[24]) and Kottwitz (cf. [22]). We leave out some proofs, and include some background materials that are needed. Though the result is weaker than that proved by Carayol, [7], it does not involve too much advanced methods. In [7], Carayol determines the restriction of certain representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, and shows that the $L$-function agree up to shift, which implies the result in this thesis.

Recall from [10] that for a projective smooth variety $X$ of dimension $d$ over $\mathbb{Q}$, we can choose $m > 0$ such that $X$ extends to a smooth scheme over $\mathbb{Z}[\frac{1}{m}]$. If $p$ is prime to $m$, then we may consider the good reduction $X_p$ modulo $p$. In this case, for a projective smooth variety $X_p$, the local factor of the Hasse-Weil zeta function is given by

$$\log \zeta(X_p, s) = \sum_{r=1}^{\infty} |X_p(\mathbb{F}_{p^r})| \frac{p^{-rs}}{r}.$$

It converges when $Re(s) > d + 1$.

The Hasse-Weil zeta-function is then defined as a product over all finite places of $\mathbb{Q}$

$$\zeta(X, s) = \prod_p \zeta(X_p, s).$$

In general, Langlands's method is to start with a cohomological definition of the local factor, via the semi-simple trace of the Frobenius, and nearby cycles plays an important part in determining those factors. Then we express its logarithm as a certain sum of orbital integrals, which involves both counting points and the stabilization of the geometric side of the Arthur-Selberg trace formula. Finally we apply the Arthur-Selberg trace formula and express the sum as a trace of a function on automorphic representations appearing in the discrete part of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (in our case $G = \mathrm{GL}_2$). By comparison, the equalities of trace imply a relation of Hasse-Weil zeta function and the automorphic L-functions. The main result is as follows:

**Theorem.** *Let $m$ be the product of two coprime integers, both at least 3, the Hasse-Weil zeta-function of $\overline{\mathcal{M}}_m$ is given by*

$$\zeta(\overline{\mathcal{M}}_m, s) = \prod_{\pi \in \Pi_{disc}(GL_2(\mathbb{A}), 1)} L(\pi, s - \frac{1}{2})^{\frac{1}{2} m(\pi) \chi(\pi_\infty) \dim \pi_f^{K_m}},$$

*where $K_m = \{g \in GL_2(\hat{\mathbb{Z}}) | g \equiv 1 \mod m\}$, and $\prod_{disc}(GL_2(\mathbb{A}), 1)$ is the set of automorphic representations $\pi = \pi_f \otimes \pi_\infty$ of $GL_2(\mathbb{A})$ that occur discretely in $L^2(GL_2(\mathbb{Q})\mathbb{R}^\times \backslash GL_2(\mathbb{A}))$ such that $\pi_\infty$ has trivial central and infinitesimal character. Here $m(\pi)$ is the multiplicity of $\pi$ inside $L^2(GL_2(\mathbb{Q})\mathbb{R}^\times \backslash GL_2(\mathbb{A}))$, $\chi(\pi_\infty) = 2$ if $\pi_\infty$ is a character and $\chi(\pi_\infty) = -2$ otherwise.*

We gradually recall those concepts and skills in representations theory, global and local harmonic analysis, and so on, mainly on $\mathrm{GL}_2$, and build up those results in our case, to deduce the final result. All materials have their classic origin from many famous lectures.

## 2. Preliminaries

We give a summarizing description of moduli space with level structure in this section. It is the foundation of what this master thesis mainly concerns about. [11] and [19] are good references.

### 2.1. **Elliptic curves.**

In this subsection, we recall some basic facts on elliptic curves without proof.

**Definition 2.1.1.** An elliptic curve is a pair $(E, O)$, where $E$ is a curve of genus 1 and $O \in E$. For a field $K$, the elliptic curve $E$ is defined over $K$, written $E/K$, if $E$ is defined over $K$ as a curve and $O \in E(K)$.

**Proposition 2.1.2.** *There exists a unique operation $\oplus$ on $E$ such that $E$ is an Abelian group.*

*Proof.* cf. [32], chapter III, proposition 2.2. □

**Theorem 2.1.3.** *Let $E$ be an elliptic curve over a field $k$ and let $N$ be a positive integer, denote by $E[N]$ the $N$-torsion subgroup $E[N] = \ker([N])$. Then $E[N] \cong \prod E[p^{e_p}]$ where $N = \prod p^{e_p}$. Also, $E[p^e] \cong (\mathbb{Z}/p^e\mathbb{Z})^2$ if $p \neq \operatorname{char}(k)$. Thus $E[N] \cong (\mathbb{Z}/N\mathbb{Z})^2$ if $\operatorname{char}(k) \nmid N$. On the other hand, if $p = \operatorname{char}(k)$, then $E[p^e] \cong \mathbb{Z}/p^e\mathbb{Z}$ for all $e \geqslant 1$ or $E[p^e] = \{0\}$ for all $e \geqslant 1$. In particular, if $\operatorname{char}(k) = p$ then either $E[p] \cong \mathbb{Z}/p\mathbb{Z}$, in which case $E$ is called ordinary, or $E[p] = \{0\}$ and $E$ is supersingular.*

For details, see [12], theorem 8.1.2.

### 2.2. **Moduli space with level structure in good reduction.**

Together with next subsection, we recall some aspects of the moduli space of elliptic curves with level structure (cf. [11], IV.2.) that we mainly concerns about.

**Definition 2.2.1.** A morphism $p : E \longrightarrow S$ of schemes with a section $e : S \longrightarrow E$ is said to be an elliptic curve over $S$ if $p$ is proper, smooth, and all geometric fibers are elliptic curves (with zero section given by $e$).

We simply say that $E/S$ is an elliptic curve. As is well-known, an elliptic curve is canonically a commutative group scheme over $S$, with $e$ as unit section.

**Definition 2.2.2.** A level-$m$-structure on an elliptic curve $E/S$ is an isomorphism $\alpha$ of group schemes over $S$, from $(\mathbb{Z}/m\mathbb{Z})^2_S$ to $E[m]$, where $E[m]$ is the preimage of (the closed subscheme) $e$ under multiplication by $m : E \longrightarrow E$.

As mentioned above, for an algebraically closed field $k$ of characteristic prime to $m$, and $S = \operatorname{Spec} k$, we have (noncanonically) $E[m] \cong (\mathbb{Z}/m\mathbb{Z})^2$. But if $\operatorname{char}(k)|m$, then there is no level-$m$-structure and it follows that if $(E/S, \alpha)$ is an elliptic curve with level-$m$-structure, then $m$ is invertible on $S$.

Consider the functor $\mathfrak{M}_m : (\text{Schemes}/\mathbb{Z}[m^{-1}]) \longrightarrow (\text{Sets})$ by

$$S \mapsto \left\{ \begin{array}{l} (E/S, \alpha) \text{ elliptic curve E over S with} \\ \text{level-}m\text{-structure } \alpha, \text{up to isomorphism} \end{array} \right\}.$$

We give a theorem from [19] without proof.

**Theorem 2.2.3.** *For $m \geqslant 3$, the functor $\mathfrak{M}_m$ is representable by a smooth affine curve $\mathcal{M}_m$ (we write $\mathcal{M}$ for short later) over $\operatorname{Spec}\mathbb{Z}[\frac{1}{m}]$. There is a projective smooth curve $\overline{\mathcal{M}}$ containing $\mathcal{M}$ as an open dense subset such that the boundary $\partial\mathcal{M} = \overline{\mathcal{M}}\backslash\mathcal{M}$ is étale over $\operatorname{Spec}\mathbb{Z}[\frac{1}{m}]$.*

*Proof.* cf. [19]. □

2.3. **Moduli space with level structure in bad reduction.**

Next we extend the moduli spaces $\mathcal{M}_m$, defined over $\operatorname{Spec}\mathbb{Z}[\frac{1}{m}]$, to the primes where they have bad reduction. To go quickly towards the zeta function, we omit the proofs of the following theorems. For more details, see [19].

For any integer $n \geqslant 0$, and $p$ a prime, with $m \geqslant 3$ prime to $p$, we want to extend the $\mathbb{Z}[\frac{1}{pm}]$ scheme $\mathcal{M}_{p^n m}$ to a scheme over $\operatorname{Spec}\mathbb{Z}[\frac{1}{m}]$.

**Definition 2.3.1.** A Drinfeld-level-$p^n$-structure on an elliptic curve $E/S$ is a pair of sections $P, Q : S \longrightarrow E[p^n]$ such that there is an equality of Cartier divisors

$$\sum_{i,j\in\mathbb{Z}/p^n\mathbb{Z}} [iP + jQ] = E[p^n].$$

A Drinfeld-level-$p^n$-structure coincides with an ordinary level-$p^n$-structure when $p$ is invertible on $S$, since in this case the group scheme $E[p^n]$ is étale over $S$. Hence we have an extension of the functor $\mathfrak{M}_{p^n m}$ to schemes over $\operatorname{Spec}\mathbb{Z}[\frac{1}{m}]$ defined as follows:

$$\mathfrak{M}_{\Gamma(p^n),m} : (\text{Schemes}/\mathbb{Z}[m^{-1}]) \longrightarrow (\text{Sets})$$

$$S \mapsto \left\{ \begin{array}{l} (E/S,(P,Q),\alpha) \text{ elliptic curve E over S with Drinfeld-level-}p^n\text{-} \\ \text{structure } (P,Q) \text{ and level-}m\text{-structure } \alpha,\ \text{up to isomorphism} \end{array} \right\}.$$

Like theorem 2.2.3, we have

**Theorem 2.3.2.** *The functor $\mathfrak{M}_{\Gamma(p^n),m}$ is representable by a regular scheme $\mathcal{M}_{\Gamma(p^n),m}$ which is an affine curve over $\operatorname{Spec}\mathbb{Z}[\frac{1}{m}]$. The canonical map $\pi_n : \mathcal{M}_{\Gamma(p^n),m} \longrightarrow \mathcal{M}_m$ is finite. Over $\operatorname{Spec}\mathbb{Z}[\frac{1}{pm}]$, it is an étale cover with Galois group $GL_2(\mathbb{Z}/p^n\mathbb{Z})$.*

So we have a finite Galois cover $\pi_{n\eta} : \mathcal{M}_{\Gamma(p^n),m}[\frac{1}{p}] \cong \mathcal{M}_{p^n m} \longrightarrow \mathcal{M}_m[\frac{1}{p}]$ with Galois group $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$.

We write $\mathcal{M}_{\Gamma(p^n)}$ for $\mathcal{M}_{\Gamma(p^n),m}$ for short.

Also, there is a compactification.

**Theorem 2.3.3.** *There is a smooth proper curve $\overline{\mathcal{M}}_{\Gamma(p^n)}/\mathbb{Z}[m^{-1}][\zeta_{p^n}]$ with $\mathcal{M}_{\Gamma(p^n)}$ as an open subset such that the complement is étale over $\operatorname{Spec}\mathbb{Z}[m^{-1}][\zeta_{p^n}]$ and has a smooth neighborhood, here $\zeta_{p^n}$ is a primitive $p^n$-th root of unity.*

Now, we need one more result. For any direct summand $H \subset (\mathbb{Z}/p^n\mathbb{Z})^2$ of order $p^n$, write $\mathcal{M}^H_{\Gamma(p^n)}$ for the reduced subscheme of the closed subscheme of $\mathcal{M}_{\Gamma(p^n)}$ where

$$\sum_{(i,j)\in H\subset(\mathbb{Z}/p^n\mathbb{Z})^2} [iP + jQ] = p^n[e].$$

**Theorem 2.3.4.** *For any $H$ as above, $\mathcal{M}^H_{\Gamma(p^n)}$ is a regular divisor on $\mathcal{M}_{\Gamma(p^n)}$ which is supported in $\mathcal{M}_{\Gamma(p^n)} \otimes_{\mathbb{Z}} \mathbb{F}_p$. Any two of them intersect exactly at the supersingular points of $\mathcal{M}_{\Gamma(p^n)} \otimes_{\mathbb{Z}} \mathbb{F}_p$. Also, we have*

$$\mathcal{M}_{\Gamma(p^n)} \otimes_{\mathbb{Z}} \mathbb{F}_p = \bigcup_H \mathcal{M}^H_{\Gamma(p^n)}.$$

## 3. Basics of representations

Theory of representations is a widely used basic tool. In this thesis, we will mainly use the representations of $\mathrm{GL}_2(F)$ where $F$ is a non-Archimedean local field. As $\mathrm{GL}_2(F)$ is both locally profinite and reductive, we recall here the basic knowledge of the representations of such groups.

### 3.1. Representations of $\mathbf{GL_2}$.

All details here are almost contained in [6], or alternatively, one can see [7]. Not to go far away, we merely sketch some concepts and propositions without proof. In the following of this subsection, $G = \mathrm{GL}_2(F)$ unless indicated to be others, but many results still hold for other locally profinite groups.

**Proposition 3.1.1.** *Assume that $G$ is locally profinite. Let $\psi : G \longrightarrow \mathbb{C}^\times$ be a group homomorphism into $\mathbb{C}^\times$, the following are equivalent:*

(i) *$\psi$ is continuous;*
(ii) *the kernel of $\psi$ is open.*

*If $\psi$ satisfies these conditions and $G$ is the union of its compact open subgroups, then the image of $\psi$ is contained in the unit circle $|z| = 1$ in $\mathbb{C}$.*

**Definition 3.1.2.** A character of a locally profinite group $G$ is a continuous homomorphism, and we call it unitary if its image is contained in the unit circle.

**Definition 3.1.3.** Assume that $G$ is locally profinite with a representation $(\pi, V)$, then $V$ is a complex vector space and $\pi$ is a group homomorphism $G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$. The representation $(\pi, V)$ is called smooth if for every $v \in V$, there is a compact open subgroup $K$ of $G$ (depending on $v$) such that $\pi(x)v = v$ for all $x \in K$. This is equivalent to say that, if $V^K$ denotes the space of $\pi(K)$-fixed vectors in $V$, then $V = \bigcup_K V^K$, where $K$ ranges over the compact open subgroups of $G$.

**Definition 3.1.4.** A smooth representation $(\pi, V)$ is called admissible if the space $V^K$ is finite dimensional, for each compact open subgroup $K$ of $G$. $(\pi, V)$ is irreducible if $V$ has no nontrivial $G$-stable subspace.

**Proposition 3.1.5.** *For a representation $(\pi, V)$ of a locally profinite group $G$, the following are equivalent:*

(i) *$V$ is the sum of its irreducible $G$-subspaces;*
(ii) *$V$ is the direct sum of a family of irreducible $G$-spaces;*
(iii) *any $G$-subspace of $V$ has a $G$-stable complement in $V$.*

**Definition 3.1.6.** The representation $(\pi, V)$ is called $G$-semisimple if it satisfies the equivalent conditions above.

Now we introduce the notion of induced representation.

Let $G$ be locally profinite, with $H$ a closed subgroups, then $H$ is also locally profinite. Assume that $(\sigma, W)$ is a smooth representation of $H$. Consider the space $X$ of functions $f : G \longrightarrow W$ satisfying

(i) $f(hg) = \sigma(h)f(g)$, for all $h \in H, g \in G$;

(ii) there is a compact open subgroup $K$ of $G$ (depending on $f$) such that $f(gx) = f(g)$ for $g \in G, x \in K$.

**Definition 3.1.7.** Let $\Sigma : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(X)$, $\Sigma(g)f : x \mapsto f(xg)$, $g, x \in G$. Then $(\Sigma, X)$ provides a smooth representation of $G$, the representation of $G$ smoothly induced by $\sigma$, and is denoted by $(\Sigma, X) = \mathrm{Ind}_H^G \sigma$.

**Proposition 3.1.8.** *The map $\sigma \mapsto Ind_H^G \sigma$ gives a functor $Rep(H) \longrightarrow Rep(G)$ that is additive and exact.*

There is a canonical $H$-homomorphism $\alpha_\sigma : \mathrm{Ind}_H^G \sigma \longrightarrow W$ sending $f$ to $f(1)$.

**Theorem 3.1.9** (Frobenius Reciprocity)**.** *With notions above, for a smooth representation $(\sigma, W)$ of $H$ and a smooth representation $(\pi, V)$ of $G$, the canonical map*

$$\hom_G(\pi, Ind_H^G \sigma) \longrightarrow \hom_H(\pi|_H, \sigma),$$

$$\phi \longrightarrow \alpha_\sigma \circ \phi,$$

*is an isomorphism. So the induction is right adjoint to restriction.*

Now we introduce Schur's lemma.

**Lemma 3.1.10.** *If $(\pi, V)$ is an irreducible smooth representation of $G$, then* $\mathrm{End}_G(V) = \mathbb{C}$.

**Corollary 3.1.11.** *Let $(\pi, V)$ be an irreducible smooth representation of $G$, the center $Z$ of $G$ acts on $V$ via a character $\omega_\pi : Z \longrightarrow \mathbb{C}^\times$ satisfying $\pi(z)v = \omega_\pi(z)v$, for all $v \in V, z \in Z$.*

In the following part of this subsection, let $F$ be a non-archimedean local field, $A = M_2(F)$ and $G = \mathrm{GL}_2(F)$, then $A$ is (as additive group) a product of 4 copies of $F$ and a Haar measure is obtained by taking a (tensor) product of 4 copies of a Haar measure on $F$. Let $\mu$ be a Haar measure on $A$.

We introduce several important closed subgroups of $G$. Let

$$B = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in G\},$$

$$N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in G\},$$

$$T = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in G\}.$$

$B$ is called the standard Borel subgroup of $G$, and $N$ is the unipotent radical of $B$. $T$ is the standard split maximal torus in $G$, satisfying $B = T \ltimes N$.

**Proposition 3.1.12** (Iwasawa decomposition)**.** *Let $K = GL_2(\mathcal{O}_F)$, the unique (up to conjugate) maximal compact subgroup of $G$, then $G = BK$, and hence $B\backslash G$ is compact.*

**Definition 3.1.13.** Let $(\pi, V)$ be a smooth representation of $G$, let $V(N)$ denotes the subspace of $V$ spanned by the vectors $v - \pi(x)v$ for $v \in V, x \in N$. The space $V_N = V/V(N)$ inherits a representation $\pi_N$ of $B/N = T$, which is also smooth. The representation $(\pi_N, V_N)$ is called the Jacquet module of $(\pi, V)$ at $N$. An irreducible smooth representation $(\pi, V)$ of $G$ is called cuspidal if $V_N$ is zero.

**Proposition 3.1.14.** *Every irreducible smooth representation of $G$ is admissible. Every cuspidal representation of $G$ is admissible.*

**Definition 3.1.15.** Let $(\pi, V)$ be an irreducible cuspidal representation of $G$. We say that $\pi$ is unramified if there exists an unramified character $\phi \neq 1$ of $F^\times$ (i.e. $\phi$ is trivial on $U_F$) such that $\phi\pi \cong \pi$. Or equivalently, it has a vector which is invariant under the maximal compact subgroup $\mathrm{GL}_2(\mathcal{O}_F)$.

**Definition 3.1.16.** Let $1_T$ be the trivial character of $T$, the trivial character $1_G$ occurs in $\mathrm{Ind}_B^G 1_T$, since $\mathrm{Ind}_B^G \chi$ has length 2 (cf. [6]), we have $\mathrm{Ind}_B^G 1_T = 1_G \oplus St_G$ for a unique irreducible representation $St_G$, the Steinberg representation:

$$0 \longrightarrow 1_G \longrightarrow \mathrm{Ind}_B^G 1_T \longrightarrow St_G \longrightarrow 0.$$

**Proposition 3.1.17.** *The Steinberg representation of $G$ is square-integrable.*

At the end, we introduce the normalized induced representation. We recall the measure first.

Let $C_c^\infty(G)$ be the space of functions $f : G \longrightarrow \mathbb{C}$ which are locally constant and of compact support. Then $G$ acts on $C_c^\infty(G)$ by left translation $\lambda$ and right translation $\rho$:

$$\lambda_g f : x \mapsto f(g^{-1}x),$$
$$\rho_g f : x \mapsto f(xg).$$

Both of the $G$-representations $(C_c^\infty(G), \lambda)$, $(C_c^\infty(G), \rho)$ are smooth.

**Definition 3.1.18.** A right Haar integral on $G$ is a non-zero linear functional

$$I : C_c^\infty(G) \longrightarrow \mathbb{C}$$

such that

(i) $I(\rho_g f) = I(f)$, $g \in G, f \in C_c^\infty(G)$;
(ii) $I(f) \geqslant 0$ for $f \in C_c^\infty(G), f \geqslant 0$.

A left Haar integral is defined similarly.

**Proposition 3.1.19.** *There exists a right (resp. left) Haar integral $I : C_c^\infty(G) \longrightarrow \mathbb{C}$. And a linear functional $I' : C_c^\infty(G) \longrightarrow \mathbb{C}$ is a right (resp. left) Haar integral if and only if $I' = cI$ for some constant $c > 0$.*

**Proposition 3.1.20.** *Let $\mu$ be a Haar measure on $A$. For $\Phi \in C_c^\infty(G)$, the function $x \mapsto \Phi(x)\|\det x\|^{-2}$ (vanishing on $A \backslash G$) lies in $C_c^\infty(A)$. The functional*

$$\Phi \mapsto \int_A \Phi(x)\|\det x\|^{-2} d\mu(x), \Phi \in C_c^\infty(G),$$

*is a left and right Haar integral on $G$. In particular, $G$ is unimodular, i.e. any left Haar integral on $G$ is a right Haar integral.*

Now let $I$ be a left Haar integral on $G$, and $S \neq \emptyset$ be a compact open subset of $G$ with $\Gamma_S$ be its characteristic function. Define $\mu_G(S) = I(\Gamma_S)$. Then $\mu_G$ is a left Haar measure on $G$. The relation with the integral is expressed via the traditional notation

$$I(f) = \int_G f(g) d\mu_G(g), f \in C_c^\infty(G).$$

For a left Haar measure $\mu_G$ on $G$ and $g \in G$, consider the functional $C_c^\infty(G) \longrightarrow \mathbb{C}$ sending $f$ to $\int_G f(xg)d\mu_G(x)$. This is a left Haar integral on $G$, hence there is a unique $\delta_G(g) \in \mathbb{R}_+^\times$ such that

$$\delta_G(g) \int_G f(xg) d\mu_G(x) = \int_G f(x) d\mu_G(x),$$

for all $f \in C_c^\infty(G)$. $\delta_G$ is a homomorphism $G \longrightarrow \mathbb{R}_+^\times$, it is called the module of $G$.

If $\sigma$ is a smooth representation of $T$, define $\iota_B^G \sigma = \mathrm{Ind}_B^G(\delta_B^{-\frac{1}{2}} \otimes \sigma)$. This provides another exact functor $\mathrm{Rep}(T) \longrightarrow \mathrm{Rep}(G)$, the normalized smooth induction. Here $\mathrm{Rep}(G)$ is the abelian category of smooth representations of $G$.

### 3.2. **Weil group.**

Now we give a quick glance of Weil group, all materials are contained in [6].

Let $F$ be a non-Archimedean local field. Denote by $\mathfrak{o}$ the discrete valuation ring in $F$, and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Choose a separable algebraic closure $\overline{F}$ of $F$.

First we recall some features of the Galois theory of $F$. Let $p$ be the characteristic of the residue class field $k = \mathfrak{o}/\mathfrak{p}$. Put $\Omega_F := \mathrm{Gal}(\overline{F}/F)$, then it is a profinite group: $\Omega_F = \varprojlim \mathrm{Gal}(E/F)$, where $E$ ranges over finite Galois extensions with $E \subset \overline{F}$.

The field $F$ admits a unique unramified extension $F_m/F$ of degree $m$ such that $F_m \subset \overline{F}$. Denote by $F_\infty$ the composite of all these fields, then $F_\infty/F$ is the unique maximal unramified extension of $F$ contained in $\overline{F}$. $\mathrm{Gal}(F_m/F)$ is cyclic and an $F$-automorphism of $F_m$ is determined by its action on the residue field $k_{F_m} \cong \mathbb{F}_{q^m}$. Hence there is one unique element $\phi_m \in \mathrm{Gal}(F_m/F)$ which acts on $k_{F_m}$ as $x \mapsto x^q$. Put $\Phi_m = {\phi_m}^{-1}$. Then $\Phi_m \mapsto 1$ gives a canonical isomorphism $\mathrm{Gal}(F_m/F) \cong \mathbb{Z}/m\mathbb{Z}$. So we have $\mathrm{Gal}(F_\infty/F) \cong \varprojlim_{m \geqslant 1} \mathbb{Z}/m\mathbb{Z}$, and a unique element $\Phi_F \in \mathrm{Gal}(F_\infty/F)$ which acts on $F_m$ as $\Phi_m$.

**Definition 3.2.1.** An element of $\Omega_F$ is called a geometric Frobenius element (over $F$) if its image in $\mathrm{Gal}(F_\infty/F)$ is $\Phi_F$, while $\Phi_F$ is called the geometric Frobenius substitution on $F_\infty$.

Put $\mathcal{I}_F = \mathrm{Gal}(\overline{F}/F_\infty)$, the inertia group of $F$. As $\hat{\mathbb{Z}} \cong \prod_\ell \mathbb{Z}_\ell$, we have an exact sequence

$$1 \longrightarrow \mathcal{I}_F \longrightarrow \Omega_F \longrightarrow \hat{\mathbb{Z}} \longrightarrow 0.$$

Let $\mathcal{W}_F^a$ denote the inverse image in $\Omega_F$ of the cyclic subgroup $\langle \Phi_F \rangle$ of $\mathrm{Gal}(F_\infty/F)$ generated by $\Phi_F$. Thus $\mathcal{W}_F^a$ is the dense subgroup of $\Omega_F$ generated by the Frobenius elements. It is normal in $\Omega_F$ and we have

$$1 \longrightarrow \mathcal{I}_F \longrightarrow \mathcal{W}_F^a \longrightarrow \mathbb{Z} \longrightarrow 0.$$

**Definition 3.2.2.** The Weil group $\mathcal{W}_F$ of $F$ is the topological group, with underlying abstract group $\mathcal{W}_F^a$, satisfying

(i) $\mathcal{I}_F$ is an open subgroup of $\mathcal{W}_F$,
(ii) the topology on $\mathcal{I}_F$, as subspace of $\mathcal{W}_F$, coincides with its profinite topology as $\mathrm{Gal}(\overline{F}/F_\infty) \subset \Omega_F$.

Then $\mathcal{W}_F$ is locally profinite, and the identity map $\iota_F : \mathcal{W}_F \longrightarrow \mathcal{W}_F^a \subset \Omega_F$ is a continuous injection.

**Proposition 3.2.3.** *Let $(\rho, V)$ be an irreducible smooth representation of $\mathcal{W}_F$, then $\rho$ has finite dimension.*

**Proposition 3.2.4.** *Let $\tau$ be an irreducible smooth representation of $\mathcal{W}_F$, then the following are equivalent:*

(i) *the group $\tau(\mathcal{W}_F)$ is finite;*
(ii) *$\tau \cong \rho \circ \iota_F$, for some irreducible smooth representation $\rho$ of $\Omega_F$;*
(iii) *the character* $\det \tau$ *has finite order.*

*For any irreducible smooth representation $\tau$ of $\mathcal{W}_F$, there is an unramified character $\chi$ of $\mathcal{W}_F$ such that $\chi \otimes \tau$ satisfies the conditions above.*

**Proposition 3.2.5.** *Let $(\pi, V)$ be a smooth representation of $\mathcal{W}_F$ of finite dimension, let $\Phi \in \mathcal{W}_F$ be a Frobenius element. The following are equivalent:*

(i) *the representation $\rho$ is semisimple;*
(ii) *the automorphism $\rho(\Phi) \in Aut_{\mathbb{C}}(V)$ is semisimple;*
(iii) *the automorphism $\rho(\Psi) \in Aut_{\mathbb{C}}(V)$ is semisimple, for every element $\Psi \in \mathcal{W}_F$.*

Here we mention the local Langlands conjecture in the case $n = 2$. It asserts that the cuspidal representations of $\mathrm{GL}_2(F)$, where $F$ is a non-Archimedean local field, are in bijection with the irreducible 2-dimensional $\ell$-adic representations of $\mathcal{W}_F$.

### 3.3. **The Bernstein center.**

We now recall some properties of the Bernstein Center built in [8]. It is also summarized in [30].

Let $F$ be a local field, and $G = \mathrm{GL}_n(F)$, then $G$ is unimodular (consider $dg = |\det(g)|^{-n} d_a g$ where $d_a g$ denotes the additive Haar measure on $M_n(F)$). With respect to the convolution $*$, $\mathcal{H}(G) = (C_c^\infty(G), *)$ is an associative algebra of locally constant functions with compact support on $G$, called the Hecke algebra of $G$ (cf. [6]).

Now for a compact open subgroup $K$ of $G$, and one chosen Haar measure $\mu$, let $e_K \in \mathcal{H}(G)$ be the idempotent associated to $K$ defined by

$$e_K(x) = \begin{cases} \mu(K)^{-1} & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases}$$

The space $\mathcal{H}(G, K) := e_K * \mathcal{H}(G) * e_K$ is a sub-algebra of $\mathcal{H}(G)$, with unit element $e_K$. Denote its center by $\mathcal{Z}(G, K)$ and put $\mathcal{Z}(G) = \varprojlim \mathcal{Z}(G, K)$, $\widehat{\mathcal{H}}(G) = \varprojlim \mathcal{H}(G, K)$, which is identified with the space of distributions $T$ of $G$ such that $T * e_K$ is of compact support for all compact open subgroups (cf. [8]). Then $\mathcal{Z}(G)$ is the center of $\widehat{H}(G)$ and consists of the conjugation-invariant distributions in $\widehat{H}(G)$,

Let $\widehat{G}$ be the set of irreducible smooth representations of $G$ over $\mathbb{C}$ module isomorphism. Then by Schur's lemma, we have a map $\phi : \mathcal{Z}(G) \longrightarrow \mathrm{Map}(\widehat{G}, \mathbb{C}^\times)$.

Let $P$ be a standard parabolic subgroup and $L = \prod_{i=1}^{k} \mathrm{GL}_{n_i}$ the corresponding Levi subgroup (cf. [26] or [17] more generally). Concretely, for such a $G = \mathrm{GL}_n(F) = \mathrm{GL}(V)$, where $V$ is an $n$ dimensional $F$ vector space, a flag in $V$ is a strictly increasing sequence of subspaces $W_\bullet = \{W_0 \subset W_1 \subset \cdots \subset W_k = V\}$, and a parabolic subgroup $P$ of $G$ is precisely the subgroup of $\mathrm{GL}(V)$ which stabilizes the flag $W_\bullet$, and the Levi subgroup of $P$ is $L = \prod_{i=0}^{k-1} \mathrm{GL}(W_{i+1}/W_i)$. Let $\sigma$ be a supercuspidal representation of $L$, i.e. every matrix coefficient of $\sigma$ is compactly supported modulo the center of $G$ (cf. [29]). Now denote by $\mathbb{G}_m$ the multiplicative group scheme (cf. [18]), and $D = (\mathbb{G}_m)^k$. Then we have a universal unramified character $\chi$: $L \longrightarrow \Gamma(D, \mathcal{O}_D) \cong \mathbb{C}[T_1^{\pm 1}, \ldots, T_k^{\pm 1}]$ sending $(g_i)_{i=1,\ldots,k}$ to $\prod_{i=1}^{k} T_i^{v_p(\det(g_i))}$. Now we get a corresponding family of representations n-$\mathrm{Ind}_P^G(\sigma\chi)$ (the normalized induction) of $G$ parameterized by the scheme $D$.

Assume $W(L, D)$ is the subgroup of $N_G(L)/L$ consisting of those $n$ such that the set of representations $D$ coincides with its conjugate via $n$.

**Theorem 3.3.1.** *Fix a cuspidal representation $\sigma$ of a Levi subgroup $L$ as above. Suppose $z \in \mathcal{Z}(G)$, then $z$ acts by a scalar on n-$\mathrm{Ind}_P^G(\sigma\chi_0)$ for any character $\chi_0$. The corresponding function on $D$ is a $W(L, D)$-invariant regular function. This induces an isomorphism of $\mathcal{Z}(G)$ with the algebra of regular functions on $\bigcup_{(L,D)} D/W(L, D)$.*

*Proof.* cf. [8], Theorem 2.13. □

## 4. Harmonic analysis

Along with the representation theory, harmonic analysis is another powerful tool in number theory and arithmetic geometry. We need the following knowledge in this thesis. R. E. Kottwitz's lecture in [2] is the resource of the section.

### 4.1. Basics of integration.

For the use of orbital integrals, we recall the basics of integration here.

As mentioned before, $G = \mathrm{GL}_2(F)$ is locally profinite, it admits a left invariant Haar measure $dg$, and $dg$ is unique up to a positive scalar. Hence we obtains the modulus character $\delta_G$ characterized by the property $d(gh^{-1}) = \delta_G(h)dg$. As $G$ is unimodular, we deduce that $d(g^{-1}) = dg$.

For $G$, integration is simple. Fix some compact open subgroup $K_0$, then there is a unique Haar measure $dg$ giving $K_0$ measure 1. For any compact open subgroup $K$ of $G$ the measure of $K$ is $[K : K \cap K_0][K_0 : K \cap K_0]^{-1}$. Moreover for any compact open subset $S$ of $G$, there is a compact open subgroup $K$ that is small enough to assure that, $S$ is a disjoint union of cosets $gK$, Hence the measure of $S$ is the number of such cosets times the measure of $K$.

For a unimodular closed subgroup $H$ of $G$, there exists a Haar measure $dh$. Then there is a quotient measure $dg/dh$ on $H\backslash G$ characterized by the formula

$$\int_G f(g)dg = \int_{H\backslash G}\int_H f(hg)dhdg/dh,$$

for all $f \in C_c^\infty(G)$.

Any function in $C_c^\infty(H\backslash G)$ lies in the image of the linear map $C_c^\infty(G) \longrightarrow C_c^\infty(H\backslash G)$, via $f \mapsto f^\#$ defined by $f^\#(g) = \int_H f(hg)dh$, hence the integration in stages formula characterizes the invariant integral on $H\backslash G$. Indeed, any compact open subset of $H\backslash G$ can be written as a disjoint union of ones of the form $H\backslash HgK$ (for some compact open subgroup $K$ of $G$), and the measure of $H\backslash HgK$ is given by $\mathrm{meas}_{dg}(K)/\mathrm{meas}_{dh}(H \cap gKg^{-1})$, as one sees by applying integration in stages to the characteristic function of $gK$.

Let $F$ be a $p$-adic field and $G$ be a connected reductive group over $F$.

**Definition 4.1.1.** Let $\gamma \in G(F)$, the orbital integral $O_\gamma(f)$ of a function $f \in C_c^\infty(G(F))$ is by definition the integral

$$O_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)d\dot{g}$$

where $d\dot{g}$ is a right $G(F)$-invariant measure on the homogeneous space over which we are integrating.

**Remark 4.1.2.** $O_\gamma$ depends on a choice of measure, but once the choice is made we get a well-defined linear functional on $C_c^\infty(G(F))$.

**Proposition 4.1.3.** *The group $G_\gamma(F)$ is unimodular, hence the measure $d\dot{g}$ exists.*

**Proposition 4.1.4.** *The orbital integral $O_\gamma(f)$ converges.*

For proofs of these two lemmas, see [2], p. 407-408.

### 4.2. Character of representations.

First we recall that for a smooth irreducible representation $\pi$ of $G = \mathrm{GL}_2(F)$ with $F$ a non-Archimedean local field and $f \in C_c^\infty(G)$, there is an operator $\pi(f)$ on the underlying vector space $V$ of $\pi$, defined by

$$\pi(f)(v) := \int_G f(g)\pi(g)(v)dg, v \in V,$$

with $dg$ a fixed Haar measure on $G$.

By proposition 3.1.14, $\pi$ is admissible, hence $\pi(f)$ has finite rank and has a trace. The character $\Theta_\pi$ of $\pi$ is the distribution on $G$ defined by

$$\Theta_\pi(f) = \mathrm{tr}\pi(f)$$

on $C_c^\infty(G)$. By a deep theorem of Harish-Chandra, the distribution $\Theta_\pi$ can be represented by integration against a locally constant function, still denoted $\Theta_\pi$, on the set $G_{rs}$ of regular semisimple elements (the characteristic polynomial has distinct roots) in $G$. For all $f \in C_c^\infty(G)$, there is an equality

$$\Theta_\pi(f) = \int_G f(g)\Theta_\pi(g)dg.$$

The function $\Theta_\pi$ is independent of the choice of Haar measure, and we get formally $\Theta_\pi(g) = \mathrm{tr}\pi(g)$, though the right hand side does not make sense literally when $\pi$ is infinite dimensional.

4.3. **Selberg trace formula.**

We give a rough description of Selberg trace formula. Materials are contained in [2], and [14] is also a good reference.

Let $G$ be a locally compact, unimodular topological group, and $\Gamma$ be a discrete subgroup of $G$. The space $\Gamma\backslash G$ of right cosets has a right $G$-invariant Borel measure. Let $R$ be the unitary representation of $G$ by right translation on the corresponding Hilbert space $L^2(\Gamma\backslash G)$: $(R(y)\phi)(x) = \phi(xy), \phi \in L^2(\Gamma\backslash G), x, y \in G$. We study $R$ by integrating it against a test function $f \in C_c(G)$: define $R(f)\phi(x) = \int_G f(y)\phi(xy)dy$, then the computation shows that

$$R(f)\phi(x) = \int_G f(y)\phi(xy)dy = \int_G f(x^{-1}y)\phi(y)dy = \int_{\Gamma\backslash G} (\sum_{\gamma\in\Gamma} f(x^{-1}\gamma y))\phi(y)dy,$$

for $\phi \in L^2(\Gamma\backslash G), x \in G$.

Then $R(f)$ is an integral operator with kernel $K(x, y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$. The sum here is finite since it may be taken over the intersection of the discrete group $\Gamma$ with the compact subset $x\mathrm{supp}(f)y^{-1}$ of $G$.

In the special case when $\Gamma\backslash G$ is compact, the operator $R(f)$ has two properties. On the one hand, $R$ decomposes discretely into irreducible representations $\pi$, with finite multiplicities $m_\pi$. Since the kernel $K(x, y)$ is a continuous function on the compact space $(\Gamma\backslash G) \times (\Gamma\backslash G)$, hence square integrable, and $R(f)$ is of Hilbert-Schmidt class. Applying the spectral theorem to the compact self adjoint operators attached to functions of the form $f(x) = (g*g^*)(x) = \int_G g(y)\overline{g(x^{-1}y)}dy$ where $g \in C_c(H)$, we obtain a spectral expansion in terms of irreducible unitary representations $\pi$ of $G$. On the other hand, if $H$ is a Lie group, one can require that $f$ be smooth and compactly supported. Thus $R(f)$ is an integral operator with smooth kernel on the compact manifold $\Gamma\backslash G$, and it is of trace class with $\mathrm{tr}R(f) = \int_{\Gamma\backslash G} K(x, x)dx$. Now for a representatives $\Delta$ of conjugacy classes in $\Gamma$, using a subscript $\gamma$ to indicate the centralizer of $\gamma$, we have

$$\begin{aligned}
\operatorname{tr}(R(f)) &= \int_{\Gamma\backslash G} K(x,x)dx \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)dx \\
&= \int_{\Gamma\backslash G} \sum_{\gamma\in\Delta} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)dx \\
&= \sum_{\gamma\in\Delta} \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)dx \\
&= \sum_{\gamma\in\Delta} \int_{G_\gamma\backslash G} \int_{\Gamma_\gamma\backslash G_\gamma} f(x^{-1}u^{-1}\gamma ux)dudx \\
&= \sum_{\gamma\in\Delta} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx.
\end{aligned}$$

This is regarded as a geometric expansion of $\operatorname{tr}(R(f))$ in terms of conjugacy classes $\gamma \in \Gamma$. Thus we have an equality, the Selberg trace formula:

$$\sum_{\gamma} \upsilon_\gamma O_\gamma(f) = \sum_{\pi} m_\pi \operatorname{tr}(\pi(f)),$$

where $\upsilon_\gamma = \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma), \operatorname{tr}(\pi(f)) = \operatorname{tr}(\int_G f(y)\pi(y)dy)$.

We will make advantage of a special case of the Arthur-Selberg trace formula in $\mathrm{GL}_2$ for the trace of Hecke operators on the $L^2$-cohomology of locally symmetric spaces later.

## 5. Advanced tools

In this part, we afford several powerful tools that will be needed later.

### 5.1. **Crystalline cohomology.**

We say a few words on crystalline cohomology in this subsection.

First we recall the Witt Vectors.

Let $p$ be a prime number, $(X_0, \dots, X_n, \dots)$ be a sequence of indeterminates.

The Witt polynomials are defined by

$$\begin{aligned}
W_0 &= X_0, \\
W_1 &= X_0^p + pX_1, \\
&\vdots \\
W_n &= X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n. \\
&\vdots
\end{aligned}$$

Let $(Y_0, \dots, Y_n, \dots)$ be another sequence of indeterminates.

**Lemma 5.1.1.** *For $\Psi \in \mathbb{Z}[X, Y]$, there exists a unique sequence $(\psi_0, \dots, \psi_n, \dots)$ of elements of $\mathbb{Z}[X_0, \dots, X_n, \dots; Y_0, \dots, Y_n, \dots]$ such that*

$$W_n(\psi_0, \dots, \psi_n, \dots) = \Psi(W_n(X_0, \dots), W_n(Y_0, \dots))$$

*for $n = 0, 1, 2, \dots$.*

*Proof.* cf. [31], II.6 Theorem 6. □

Denote by $S_0, \dots, S_n, \dots$ (resp. $P_0, \dots, P_n, \dots$) the polynomials $\psi_0, \dots, \psi_n, \dots$ associated by the lemma with the polynomial $\Psi(X, Y) = X+Y$ (resp. $\Psi(X, Y) = XY$). For a commutative ring $A$, and $\mathfrak{a} = (a_0, \dots, a_n, \dots), \mathfrak{b} = (b_0, \dots, b_n, \dots)$ elements of $A^{\mathbb{N}}$, define

$$\mathfrak{a} + \mathfrak{b} = (S_0(a, b), \dots, S_n(a, b), \dots)$$

$$\mathfrak{a}\mathfrak{b} = (P_0(a, b), \dots, P_n(a, b), \dots).$$

**Theorem 5.1.2.** *The laws of composition defined above make $A^{\mathbb{N}}$ into a commutative unitary ring, the ring of Witt vectors with coefficients in $A$ and denoted $W(A)$, elements of $W(A)$ are called Witt vectors with coefficients in $A$.*

*Proof.* cf. [31], II.6 Theorem 7. □

Now let $k_r = \mathbb{F}_{p^r}$ be a finite field with ring of Witt vectors $W(k_r)$. The fraction field $L_r$ of $W(k_r)$ is an unramified extension of $\mathbb{Q}_p$ and its Galois group is the cyclic group of order $r$ generated by the Frobenius element $\sigma : x \mapsto x^p$. Note that $\sigma$ acts on Witt vectors by $\sigma(a_0, a_1, \dots) = (a_0^p, a_1^p, \dots)$.

For an abelian variety $A$ over $k_r$ of dimension $g$, we have the integral isocrystal associated to $A/k_r$, given by the data $\mathbb{D}(A) = (H^1_{crys}(A/W(k_r)), F, V)$. Here the crystalline cohomology group $H^1_{crys}(A/W(k_r))$ (see [25] for details) is a free $W(k_r)$-module of rank $2g$, equipped with a $\sigma$-linear endomorphism $F$ (Frobenius) and the $\sigma^{-1}$-linear endomorphism $V$ (Verschiebung) which induce bijections on $H^1_{crys}(A/W(k_r)) \otimes_{W(k_r)} L_r$. We also have the identity $FV = VF = p$, hence the inclusions of $W(k_r)$-lattices

$$pH^1_{crys}(A/W(k_r)) \subset FH^1_{crys}(A/W(k_r)) \subset H^1_{crys}(A/W(k_r)),$$

$$pH^1_{crys}(A/W(k_r)) \subset VH^1_{crys}(A/W(k_r)) \subset H^1_{crys}(A/W(k_r)).$$

Let $A[p^n] = \ker(p^n : A \longrightarrow A)$, and $A[p^\infty] := \varinjlim A[p^n]$. The crystalline cohomology of $A/k_r$ is connected to the contravariant Dieudonné module of the $p$-divisible group $A[p^\infty]$ (cf. [5]).

The classical contravariant Dieudonné functor $G \mapsto D(G)$ establishes an exact anti-equivalence between the category

$$\{p\text{-divisible groups } G = \varinjlim G_n \text{ over } k_r\}$$

and the category

$$\{\text{free } W(k_r)\text{-modules } M = \varprojlim M/p^nM, \text{ equipped with operators } F, V\},$$

Here $F$ and $V$ are, $\sigma$ and $\sigma^{-1}$-linear endomorphisms respectively, inducing bijections on $M \otimes_{W(k_r)} L_r$.

The crystalline cohomology of $A/k_r$, together with the operators $F$ and $V$, is the same as the Dieudonné module of the $p$-divisible group $A[p^\infty]$, in the sense that there is a canonical isomorphism $H^1_{crys}(A/W(k_r)) \cong D(A[p^\infty])$ which respects the endomorphisms $F$ and $V$ on both sides. It is a standard fact, cf. [4].

5.2. **Nearby cycles.**

We give a summary of nearby cycles from [2]. It is through nearby cycles to determine the local factors (cf. [27]).

Let $k$ be a finite or algebraically closed field, $X$ be a scheme of finite type over $k$ (The following works as well if $k$ is the fraction of a discrete valuation ring $R$ with finite residue field, and assume that $X$ is finite type over $R$). Denote by $\overline{k}$ an algebraic closure of $k$, and $X_{\overline{k}}$ the base change $X \times_k \overline{k}$. Denote by $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ the 'derived' category of $\overline{\mathbb{Q}}_\ell$-sheaves on $X$, which is not actually the derived category of the category of $\overline{\mathbb{Q}}_\ell$-sheaves in the original sense, but is obtained as a localization of a projective limit of derived categories, under certain finiteness assumption (cf. [21]). The category $D^b_c(X, \overline{\mathbb{Q}}_\ell)$ is a triangulated category which admits the usual functorial formalism, and which can be equipped with a natural $t$-structure having as its core the category of $\overline{\mathbb{Q}}_\ell$-sheaves. If $f : X \longrightarrow Y$ is a morphism of schemes of finite type over $k$, we have the derived functors $f_*, f_! : D^b_c(X, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(Y, \overline{\mathbb{Q}}_\ell)$ and $f^*, f^! : D^b_c(Y, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X, \overline{\mathbb{Q}}_\ell)$. Occasionally we denote these same derived functors by $Rf_*$, etc.

Let $S$ be a spectrum of a complete discrete valuation ring, with special point $s$ and generic point $\eta$. Let $k(s)$ and $k(\eta)$ denote the residue fields of $s$ and $\eta$ respectively. Choose a separable closure $\overline{\eta}$ of $\eta$ and define the Galois group $\Gamma = \mathrm{Gal}(\overline{\eta}/\eta)$ and the inertia subgroup $\Gamma_0 = \ker(\mathrm{Gal}(\overline{\eta}/\eta) \longrightarrow \mathrm{Gal}(\overline{s}/s))$, where $\overline{s}$ is the residue field of the normalization $\overline{S}$ of $S$ in $\overline{\eta}$.

Now let $X$ denote a finite type scheme over $S$. The category $D^b_c(X \times_s \eta, \overline{\mathbb{Q}}_\ell)$ is the category of sheaves $\mathcal{F} \in D^b_c(X_{\overline{s}}, \overline{\mathbb{Q}}_\ell)$ together with a continuous action of $\mathrm{Gal}(\overline{\eta}/\eta)$ which is compatible with the action on $X_{\overline{s}}$.

**Definition 5.2.1.** For $\mathcal{F} \in D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell)$, we define the nearby cycles sheaf to be the object in $D^b_c(X \times_s \eta, \overline{\mathbb{Q}}_\ell)$ given by $R\Psi^X(\mathcal{F}) = \overline{i}^* R\overline{j}_*(\mathcal{F}_{\overline{\eta}})$, where $\overline{i} : X_{\overline{s}} \hookrightarrow X_{\overline{S}}$ and $\overline{j} : X_{\overline{\eta}} \hookrightarrow X_{\overline{S}}$ are the closed and open immersions of the geometric special and generic fibers of $X/S$, and $\mathcal{F}_{\overline{\eta}}$ is the pull-back of $\mathcal{F}$ to $X_{\overline{\eta}}$.

**Theorem 5.2.2.** *The functors* $R\Psi : D^b_c(X_\eta, \overline{\mathbb{Q}}_\ell) \longrightarrow D^b_c(X \times_s \eta, \overline{\mathbb{Q}}_\ell)$ *have the following properties*

(i) *$R\Psi$ commutes with proper-push-forward: if $f : X \longrightarrow Y$ is a proper $S$-morphism, then the canonical base change morphism of functors to $D^b_c(Y \times_s \eta, \overline{\mathbb{Q}}_\ell)$ is an isomorphism: $R\Psi f_* \cong f_* R\Psi$. In particular, if $X \longrightarrow S$ is proper there is a $Gal(\overline{\eta}/\eta)$-equivariant isomorphism $H^i(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) = H^i(X_{\overline{s}}, R\Psi(\overline{\mathbb{Q}}_\ell))$.*

(ii) *Suppose $f : X \longrightarrow S$ is finite type but not proper. Suppose that there is a compactification $j : X \hookrightarrow \overline{X}$ over $S$ such that the boundary $\overline{X} \backslash X$ is a relative normal crossings divisor over $S$. Then there is a $Gal(\overline{\eta}/\eta)$-equivariant isomorphism $H^i_c(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) = H^i_c(X_{\overline{s}}, R\Psi(\overline{\mathbb{Q}}_\ell))$.*

(iii) *$R\Psi$ commutes with smooth pull-back: if $p : X \longrightarrow Y$ is a smooth $S$-morphism, then the base change morphism is an isomorphism: $p^*R\Psi \cong R\Psi p^*$.*

*Proof.* cf. [2], p. 619. □

5.3. **Base change.**

Here we recall certain facts about base change of representations and establish a base change identity which will be used later. [23] is a good reference for base change, and [30] builds up many results here.

Let $\mathbb{Q}_{p^r}$ be an unramified extension of $\mathbb{Q}_p$ of degree $r$, this field carries a unique automorphism $\sigma$ lifting the Frobenius automorphism $x \mapsto x^p$ on its residue field. Furthermore, $\sigma$ is a generator of $\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p)$. We say two elements $x, y \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$ are $\sigma$-conjugate if there exists $h \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$ such that $y = h^{-1}x\sigma(h)$.

**Definition 5.3.1.** For an element $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$, let $\mathrm{N}\delta = \delta\delta^\sigma \cdots \delta^{\sigma^{r-1}}$ be the norm.

Then we have

**Proposition 5.3.2.** *The $GL_2(\mathbb{Q}_{p^r})$-conjugacy class of $N\delta$ contains an element of $GL_2(\mathbb{Q}_p)$.*

*Proof.* Let $y = \mathrm{N}\delta$, and $\overline{\mathbb{Q}}_p$ be the algebraic closure containing $\mathbb{Q}_{p^r}$. Then it is enough check that the set of eigenvalues of $y$, with multiplicities, is invariant under those $\sigma' \in \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ with image $\sigma \in \mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p)$. Acting with $\sigma'$ on the set we get the eigenvalues of $\sigma(y)$. Because $\sigma(y) = \delta^{-1}y\delta$, we deduce the invariance. □

**Proposition 5.3.3.** *If $N\delta$ and $N\delta'$ are conjugate, then $\delta$ and $\delta'$ are $\sigma$-conjugate.*

*Proof.* cf. [23], Lemma 4.2. □

Now for $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p), \delta \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$, define the centralizer $G_\gamma(R) = \{g \in \mathrm{GL}_2(R) | g^{-1}\gamma g = \gamma\}$, and the twisted centralizer $G_{\delta\sigma}(R) = \{h \in \mathrm{GL}_2(R \otimes \mathbb{Q}_{p^r}) | h^{-1}\delta h^\sigma = \delta\}$.

For a function $f \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p))$, define the orbital integral

$$O_\gamma(f) = \int_{G_\gamma(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_p)} f(g^{-1}\gamma g)dg$$

and for $\phi \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^r}))$, define the twisted orbital integral

$$TO_{\delta\sigma}(\phi) = \int_{G_{\delta\sigma}(\mathbb{Q}_p)\backslash \mathrm{GL}_2(\mathbb{Q}_{p^r})} \phi(h^{-1}\delta h^\sigma)dh.$$

**Definition 5.3.4.** The functions $f \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p)), \phi \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^r}))$ have matching (twisted) orbital integrals (or simply 'associated') if the following condition holds: for all semi-simple $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$, the orbital integral $O_\gamma(f)$ vanishes if $\gamma$ is not a norm (i.e. conjugate to $\mathrm{N}\delta$ for some $\delta$), and if $\gamma$ is a norm, then $O_\gamma(f) = \pm TO_{\delta\sigma}(\phi)$, where the sign is $-$ if $\mathrm{N}\delta$ is a central element, but $\delta$ is not $\sigma$-conjugate to a central element, and otherwise is $+$.

**Proposition 5.3.5.** *Assume* $\delta \in GL_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})$*, then*

$$G_{\delta\sigma}(\mathbb{Z}/p^n\mathbb{Z}) = \{h \in GL_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r}) | h^{-1}\delta h^\sigma = \delta\}$$

*and*

$$G_{N\delta}(\mathbb{Z}/p^n\mathbb{Z}) = \{g \in GL_2(\mathbb{Z}/p^n\mathbb{Z}) | g^{-1}N\delta g = N\delta\}$$

*have the same cardinality. And* $\sigma$*-conjugacy classes in* $GL_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})$ *are mapped bijectively via the norm to conjugacy classes in* $GL_2(\mathbb{Z}/p^n\mathbb{Z})$*.*

*Proof.* Fix $\gamma \in \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$. Clearly, the groups $Z_{\gamma,p} = (\mathbb{Z}/p^n\mathbb{Z}[\gamma])^\times$ and $Z_{\gamma,p^r} = (\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r}[\gamma])^\times$ are commutative. With the norm map, we have a homomorphism $N : Z_{\gamma,p^r} \longrightarrow Z_{\gamma,p}$ and define a homomorphism $d : Z_{\gamma,p^r} \longrightarrow Z_{\gamma,p^r}$ via $d(x) = xx^{-\sigma}$. By definition $H^1(\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p), Z_{\gamma,p^r}) = \ker(N)/\mathrm{im}(d)$.

**Lemma 5.3.6.** $H^1(Gal(\mathbb{Q}_{p^r}/\mathbb{Q}_p), Z_{\gamma,p^r}) = 0$*. Thus we have an exact sequence*

$$0 \longrightarrow Z_{\gamma,p} \longrightarrow Z_{\gamma,p^r} \xrightarrow{d} Z_{\gamma,p^r} \xrightarrow{N} Z_{\gamma,p} \longrightarrow 0.$$

*Proof.* It is not hard for the exactness at the first step. By the definition of $d$, it is also clear at the second step.

Let $X_i = \ker(Z_{\gamma,p^r} \longrightarrow \mathrm{GL}_2(\mathbb{Z}_{p^r}/p^i\mathbb{Z}_{p^r}))$ for $i = 0, \dots, n$, this is a $\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p)$-invariant filtration on $Z_{\gamma,p^r}$. We first prove the vanishing of the cohomology for the successive quotients. As $X_i/X_{i+1}$ is a $\mathbb{F}_{p^r}$-subvectorspace of $\ker(\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^{i+1}\mathbb{Z}_{p^r}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}_{p^r}/p^i\mathbb{Z}_{p^r})) \cong \mathbb{F}_{p^r}^4, i \geqslant 1$. As familiar, we have $\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^r}/\mathbb{F}_p)$. By Lang's lemma we have $H^1(\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p), \mathbb{F}_{p^r}) = 0$. For the case $i = 0$, it is also true since the groups considered are connected. In sum, the sequence is exact at the third step. Now by the exact sequence

$$0 \longrightarrow X_0/X_1 \longrightarrow X_0/X_2 \longrightarrow X_1/X_2 \longrightarrow 0,$$

and the long exact sequence for cohomology, we have $H^1(\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p), X_0/X_2) = 0$, and step by step, we finally deduce that $H^1(\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p), X_0/X_n) = 0$, which is as claimed.

The last surjectivity now follows from the other exactness. From above, we now have

$$\begin{aligned} Z_{\gamma,p^r}/\ker(N) &\cong \mathrm{im}(N), \\ Z_{\gamma,p^r}/\ker(d) &\cong \mathrm{im}(d), \\ ker(d) &= Z_{\gamma,p}, \\ \mathrm{im}(d) &= \ker(N). \end{aligned}$$

Hence we conclude that $\mathrm{im}(N) = Z_{\gamma,p}$. □

Now by the lemma, for $\gamma \in \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$, there exists $\delta \in Z_{\gamma,p^r}$ such that $\mathrm{N}\delta = \gamma$. So it suffices to prove that $G_{\delta\sigma}(\mathbb{Z}/p^n\mathbb{Z}) = G_\gamma(\mathbb{Z}/p^n\mathbb{Z})$ as sets.

On the one hand, note that $\delta \in \mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r}[\gamma]$, which implies directly that $G_\gamma(\mathbb{Z}/p^n\mathbb{Z}) \subset G_{\delta\sigma}(\mathbb{Z}/p^n\mathbb{Z})$.

On the other hand, for $x \in G_{\delta\sigma}(\mathbb{Z}/p^n\mathbb{Z})$, we have $x^{-1}\delta x^\sigma = \delta$ by definition, so $x^{-\sigma^i}\delta^{\sigma^i}x^{\sigma^{i+1}} = \delta^{\sigma^i}$ for $i = 0, \dots, r-1$. Thus we get $x^{-1}\mathrm{N}\delta x = \mathrm{N}\delta$, i.e. $\gamma x = x\gamma$. So $x$ commutes with $\delta \in \mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r}[\gamma]$, hence $x^{-1}\delta x = \delta$. Combining the results, we get $x = x^\sigma$, $x \in G_\gamma(\mathbb{Z}/p^n\mathbb{Z})$ and hence $G_{\delta\sigma}(\mathbb{Z}/p^n\mathbb{Z}) \subset G_\gamma(\mathbb{Z}/p^n\mathbb{Z})$

For the second part of the proposition, choose representatives $\gamma_1, \ldots, \gamma_t$ of the conjugacy classes in $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$, then from above there are $\delta_1, \ldots, \delta_t$ satisfying $\mathrm{N}\delta_i = \gamma_i$, which represent different $\sigma$-conjugacy classes by the proposition 5.3.3. By group theory, we know the size of their $\sigma$-conjugacy classes is

$$\frac{|\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})|}{|G_{\delta_i\sigma}(\mathbb{Z}/p^n\mathbb{Z})|} = \frac{|\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})|}{|G_{\gamma_i}(\mathbb{Z}/p^n\mathbb{Z})|},$$

as a result, we deduce that

$$\begin{aligned} \frac{|\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})|}{|\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})|} \sum_{i=1}^{t} \frac{|\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})|}{|G_{\gamma_i}(\mathbb{Z}/p^n\mathbb{Z})|} &= \frac{|\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})|}{|\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})|} |\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})| \\ &= |\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})|. \end{aligned}$$

Hence we conclude that $\delta_1, \ldots, \delta_t$ are representatives of the conjugacy classes in $\mathrm{GL}_2(\mathbb{Z}_{p^r}/p^n\mathbb{Z}_{p^r})$.

□

Define the principal congruence subgroups (cf. [12]) by

$$\Gamma(p^k)_{\mathbb{Q}_p} = \{g \in \mathrm{GL}_2(\mathbb{Z}_p) | g \equiv 1 \mod p^k\},$$
$$\Gamma(p^k)_{\mathbb{Q}_{p^r}} = \{g \in \mathrm{GL}_2(\mathbb{Z}_{p^r}) | g \equiv 1 \mod p^k\}.$$

**Corollary 5.3.7.** *Assume $f$ is conjugation-invariant locally integrable complex valued function on $GL_2(\mathbb{Z}_p)$, then the function $\phi$ on $GL_2(\mathbb{Z}_{p^r})$ given by $\phi(\delta) = f(N\delta)$ is locally integrable, and $(e_{\Gamma(p^k)_{\mathbb{Q}_{p^r}}} * \phi)(\delta) = (e_{\Gamma(p^k)_{\mathbb{Q}_p}} * f)(N\delta)$ for all $\delta \in GL_2(\mathbb{Z}_{p^r})$.*

*Proof.* First assume simply that $f$ is locally constant, conjugation-invariant by $\Gamma(p^k)_{\mathbb{Q}_p}$. Then $\phi$ is $\sigma$-conjugation-invariant by $\Gamma(p^k)_{\mathbb{Q}_{p^r}}$ and locally integrable. Hence the identity follows from the proposition. In general, as $\Gamma(p^k)_{\mathbb{Q}_p}$ (resp. $\Gamma(p^k)_{\mathbb{Q}_{p^r}}$) give a fundamental system of open neighborhoods of 1 in $\mathrm{GL}_2(\mathbb{Z}_p)$ (resp. $\mathrm{GL}_2(\mathbb{Z}_{p^r})$), the corollary follows from approximating $f$ by locally constant functions. □

Now let $\pi$ (resp. $\Pi$) be tempered representation (cf. [29], VII.2.) of $\mathrm{GL}_2(\mathbb{Q}_p)$ (resp. $\mathrm{GL}_2(\mathbb{Q}_{p^r})$.

**Definition 5.3.8.** $\Pi$ is called a base-change lift of $\pi$, if $\Pi$ is invariant under $\mathrm{GL}_2(\mathbb{Q}_{p^r}/\mathbb{Q}_p)$ and for some extension of $\Pi$ to a representation of $\mathrm{GL}_2(\mathbb{Q}_{p^r}) \rtimes \mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p)$, $\mathrm{tr}(\mathrm{N}g|\pi) = \mathrm{tr}((g,\sigma)|\Pi)$ for all $g \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$ such that the conjugacy class of $\mathrm{N}g$ is regular semi-simple.

**Remark 5.3.9.** It is proved that there exists a unique base-change lift (cf. [23], section 2).

**Theorem 5.3.10.** *Suppose $f \in \mathcal{Z}(GL_2(\mathbb{Q}_p))$, $\phi \in \mathcal{Z}(GL_2(\mathbb{Q}_{p^r}))$, and for all tempered irreducible smooth representation $\pi$ of $GL_2(\mathbb{Q}_p)$ with base-change lift $\Pi$, the scalars $c_{f,\pi}$ (resp. $c_{\Phi,\Pi}$) through which $f$ (resp. $\Phi$) acts on $\pi$ (resp. $\Pi$) are the same. Then $f * h$ and $\phi * h'$ have matching (twisted) orbital integrals for every* *associated* $h \in C_c^\infty(GL_2(\mathbb{Q}_p))$ *and* $h' \in C_c^\infty(GL_2(\mathbb{Q}_{p^r}))$. *Moreover,* $e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$ *and* $e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ *are associated.*

*Proof.* As $h$ and $h'$ are associated, while $\Pi$ is a base-change of $\pi$, by the twisted version of Weyl integration formula (cf. [23]), we have $\mathrm{tr}(h|\pi) = \mathrm{tr}((h', \sigma)|\Pi)$. Hence $\mathrm{tr}(f * h|\pi) = c_{f,\pi}\mathrm{tr}(h|\pi) = c_{\Phi,\Pi}\mathrm{tr}((h', \sigma)|\Pi = \mathrm{tr}((\phi * h', \sigma)|\Pi)$. As proved in [23], there exists $f' \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_p))$ having matching (twisted) orbital integrals with $\phi * h'$, we have $\mathrm{tr}((\phi * h', \sigma)|\Pi) = \mathrm{tr}(f'|\pi)$. Hence for every tempered irreducible smooth representation $\pi$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, $\mathrm{tr}((f * h - f')|\pi) = 0$. Kazhdan's density theorem (cf. [20], theorem 1) implies that all regular semi-simple orbital integrals of $f * h - f'$ vanish. By the choice of $f'$, we know $f * h$ and $\phi * h'$ have matching regular semi-simple (twisted) orbital integrals. Then proposition 7.2, [9] tells that while it is true for all regular semi-simple (twisted) orbital integrals, it is true over all semi-simple (twisted) orbital integrals as well.

For the last part, note that by corollary 5.3.7, taking $f$ the character of $\pi$ restricted to $\mathrm{GL}_2(\mathbb{Z}_p)$, $k = n$ and $\delta = 1$, it gives that

$$\mathrm{tr}(e_{\Gamma(p^n)_{\mathbb{Q}_p}}|\pi) = \mathrm{tr}((e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}, \sigma)|\Pi).$$

So the rest is a repetition as above. □

5.4. **Counting points over finite fields.**

In this subsection, we explain the method of Langlands and Kottwitz to count the number of points mod $p$ of modular curve with good reduction. More details are can be found in [30] and [10].

Let $p$ be a prime coprime to $m$, $r$ a positive integer, and $E$ an elliptic curve over $\mathbb{F}_{p^r}$. Define the set $\mathcal{M}(\mathbb{F}_{p^r})(E)$ to be $\{x \in \mathcal{M}(\mathbb{F}_{p^r})|E_x$ is $\mathbb{F}_{p^r}$-isogeneous to $E\}$. Our aim is just to count its cardinality (cf. [10]).

For a prime $\ell \neq p$, we may consider the dual of the $\ell$-adic Tate module, $H^1_{et}(E_{\overline{F}_{p^r}}, \mathbb{Z}_\ell) = H_\ell E$. Let $\mathrm{End}(E)$ be the ring of endomorphisms of $E$ defined over $\mathbb{F}_{p^r}$, and $\mathrm{End}_{\mathbb{Q}}(E) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\Gamma = (\mathrm{End}_{\mathbb{Q}}(E))^\times$. The crystalline cohomology $H^1_{crys}(E/\mathbb{Z}_{p^r})$ is a free $\mathbb{Z}_{p^r}$-module of rank 2, equipped with a $\sigma$-linear endomorphism $F$. Define $H_p = H^1_{crys}(E_0/\mathbb{Z}_{p^r}) \otimes_{\mathbb{Z}_{p^r}} \mathbb{Q}_{p^r}$. Let $\mathbb{A}_f^p$ be the ring of finite adèles of $\mathbb{Q}$ with trivial $p$-component and $\hat{\mathbb{Z}}^p \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$ be the integral elements in $\mathbb{A}_f^p$. Define $H^p = H^1_{et}(E_{\overline{\mathbb{F}}_{p^r}}, \mathbb{A}_f^p) = \prod'_{\ell \neq p} H^1_{et}(E_{\overline{\mathbb{F}}_{p^r}}, \mathbb{Q}_\ell)$.

Take any $x \in \mathcal{M}(\mathbb{F}_{p^r})(E)$, and $f : E \longrightarrow E_x$ an $\mathbb{F}_{p^r}$-isogeny, we have a $G_{\mathbb{F}_{p^r}} = \mathrm{Gal}(\overline{\mathbb{F}}_{p^r}/\mathbb{F}_{p^r})$-invariant $\hat{\mathbb{Z}}^p$-lattice $L = f^*(H^1_{et}(E_{x,\overline{\mathbb{F}}_{p^r}}, \hat{\mathbb{Z}}^p)) \subset H^p$, an $F, pF^{-1}$-invariant $\mathbb{Z}_{p^r}$-lattice $\Lambda = f^*(H^1_{crys}(E_x/\mathbb{Z}_{p^r})) \subset H_p$, and a $G_{\mathbb{F}_{p^r}}$-invariant isomorphism $\phi : (\mathbb{Z}/m\mathbb{Z})^2 \longrightarrow L \otimes \mathbb{Z}/m\mathbb{Z}$, corresponding to the level-$m$-structure. Let $\Lambda^p$ be the set of such $(L, \phi)$ and $\Lambda_p$ be the set of $\Lambda$ as above. Dividing by the choice of $f$, we get a map $\Psi : \mathcal{M}(\mathbb{F}_{p^r})(E) \longrightarrow \Gamma\backslash\Lambda^p \times \Lambda_p$.

**Theorem 5.4.1.** *The map* $\Psi$ *is a bijection.*

*Proof.* cf. [30] or [10].

□

Fix a basis of $H^p$, let $\gamma \in \mathrm{GL}_2(\mathbb{A}_f^p)$ be the endomorphism induced by the geometric Frobenius $\Phi_{p^r}$ on $H^p$ and $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$ be induced by the $p$-linear endomorphism $F$ on $H_p$: for the $p$-linear isomorphism $\sigma$ on $H_p$ preserving the chosen basis, defined through $F = \delta\sigma$. Then $\gamma\Lambda^p = \Lambda^p$ and $p\Lambda_p \subset \delta\sigma\Lambda_p \subset \Lambda_p$. We define the centralizer

$$G_\gamma(\mathbb{A}_f^p) = \{g \in \mathrm{GL}_2(\mathbb{A}_f^p) | g^{-1}\gamma g = \gamma\}$$

and the twisted centralizer

$$G_{\delta\sigma}(\mathbb{Q}_p) = \{h \in \mathrm{GL}_2(\mathbb{Q}_{p^r}) | h^{-1}\delta h^\sigma = \delta\}.$$

Let $K^p = \{g \in \mathrm{GL}_2(\hat{\mathbb{Z}}^p) | g \equiv 1 \mod m\}$ and $K_p = \mathrm{GL}_2(\mathbb{Z}_{p^r})$. Let $f^p$ be the characteristic function of $K^p$ divided by its volume, $\phi_{p,0}$ be the characteristic function of $K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p$ divided by the volume of $K_p$.

For $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}_f^p))$, define the orbital integral

$$O_\gamma(f) = \int_{G_\gamma(\mathbb{A}_f^p)\backslash \mathrm{GL}_2(\mathbb{A}_f^p)} f(g^{-1}\gamma g) dg.$$

We have

**Corollary 5.4.2.** *The cardinality of* $\mathcal{M}(\mathbb{F}_{p^r})(E)$ *is*

$$\mathrm{vol}(\Gamma\backslash G_\gamma(\mathbb{A}_f^p) \times G_{\delta\sigma}(\mathbb{Q}_p)) O_\gamma(f^p) TO_{\delta\sigma}(\phi_{p,0}),$$

*where the Haar measure on* $\Gamma$ *gives points measure* 1.

*Proof.* Identify the set $X^p$ of pairs $(L, \phi)$ as above with $\mathrm{GL}_2(\mathbb{A}_f^p)/K^p$, without the Galois-invariance condition, and identify the set $X_p$ of all lattices $\Lambda$ with $\mathrm{GL}_2(\mathbb{Q}_{p^r})/K_p$. If $gK^p \in X^p$ lies in $\Lambda^p$, then $\gamma g K^p = gK^p$, so $g^{-1}\gamma g \in K^p$. If $hK_p \in X_p$ lies in $\Lambda_p$, then $FhK_p \subset hK_p$ and $VhK_p \subset hK_p$. Since $FV = p$, we get $phK_p \subset FhK_p \subset hK_p$, hence $pK_p \subset h^{-1}\delta h^\sigma K_p \subset K_p$.

We have $v_p(\det \delta) = 1$ because the Weil pairing gives an isomorphism of the second exterior power of $H_p$ with $\mathbb{Q}_{p^r}(-1)$. Hence we deduce that $h^{-1}\delta h^\sigma \in K_p \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right) K_p$. So the cardinality of $\Gamma\backslash\Lambda^p \times \Lambda_p$ equals to

$$\int_{\Gamma\backslash \mathrm{GL}_2(\mathbb{A}_f^p)\times \mathrm{GL}_2(\mathbb{Q}_{p^r})} f^p(g^{-1}\gamma g)\phi_{p,0}(h^{-1}\delta h^\sigma) dg dh.$$

which is easily checked to be $\mathrm{vol}(\Gamma\backslash G_\gamma(\mathbb{A}_f^p) \times G_{\delta\sigma}(\mathbb{Q}_p)) O_\gamma(f^p) TO_{\delta\sigma}(\phi_{p,0})$. Now the claim follows from the bijection of $\Psi$. □

**Remark 5.4.3.** $TO_{\delta\sigma}(\phi_{p,0}) \neq 0$ whenever $\mathcal{M}(\mathbb{F}_{p^r})(E) \neq \varnothing$.

## 6. The semi-simple trace and semi-simple local factor

In this section we introduce the semi-simple trace and the semi-simple local factor, which turn out to play a crucial role in the generalization of Hasse-Weil local factors.

6.1. **Basics of the semi-simple trace and semi-simple local factor.**

The semi-simple trace was introduced by Rapoport in [27].

Let $X$ be a variety over a local field $K$ with residue field $\mathbb{F}_q$. Let $G_K$ be the Galois group $\mathrm{Gal}(\overline{K}/K)$ and $I_K \subset G_K$ be the inertia subgroup satisfying the exact sequence

$$1 \longrightarrow I_K \longrightarrow G_K \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1.$$

Let $\Phi_q$ be a geometric Frobenius element. Suppose $\ell$ is a prime not dividing $q$.

To introduce the cohomological definition, we first recall briefly the étale cohomology with compact support.

**Definition 6.1.1.** For a torsion sheaf $\mathcal{F}$ on a scheme $X$, with $j : X \longrightarrow \overline{X}$ an open immersion into a complete scheme $\overline{X}$, define the étale cohomology with compact support $H_c^*(X, \mathcal{F})$ to be $H_c^q(X, \mathcal{F}) = H^q(\overline{X}, j_!\mathcal{F})$, where $j_!\mathcal{F}$ is the extension by zero to $\overline{X}$.

It is proved that the compactification $\overline{X}$ exists by Nagata, and the definition is independent of the choice of the compactification.

**Definition 6.1.2.** The Hasse-Weil local factor of $X$ is given by

$$\zeta(X, s) = \prod_{i=0}^{2\dim X} \det(1 - \Phi_q q^{-s} | H_c^i(X \otimes_K \overline{K}, \overline{\mathbb{Q}}_\ell)^{I_K})^{(-1)^{i+1}}.$$

**Remark 6.1.3.** The monodromy conjecture for curves is proved in [28], so that this definition, apparently depending on $\ell$, is independent of $\ell$.

**Remark 6.1.4.** In case of good reduction, $I_K$ acts trivially on all cohomology groups, then this is simply Grothendieck's cohomological expression of the initial definition as a power series.

**Remark 6.1.5.** Recall that if $\sigma$ is an endomorphism of a finite dimensional vector space over a field of characteristic zero, then

$$\log \det(1 - T\sigma | V) = -\sum_{j=1}^{\infty} \frac{\mathrm{tr}\sigma^j}{j} T^j,$$

Thus the determination of the local factor is equivalent with that of the alternating trace of $\sigma^{*j}$ on the $I_K$-invariants in the cohomology for $j \geqslant 1$, which is approached through the method of vanishing cycles.

Let $(\pi, V)$ be a continuous, finite dimensional $\ell$-adic representation of $G_K$, where $\ell$ is a prime number prime to the residue characteristic of $K$. Let $H$ be a finite group, which acts on $V$ commutatively with the action of $G_K$. We have

**Definition 6.1.6.** A filtration $\mathcal{F} : 0 = V_0 \subset V_1 \subset \cdots \subset V_k = V$ is called admissible if it is stable under $G_K \times H$ and $I_K$ operates on the associated $\mathrm{gr}_\bullet^{\mathcal{F}}(V) = \bigoplus_{i=1}^k V_i/V_{i-1}$ through a finite quotient.

**Lemma 6.1.7.** *Admissible filtration always exists.*

*Proof.* See [30] for a complete proof. Or we can take the filtration defined by the kernels of the powers of the logarithm of $\pi$ (cf. [13], section 3.1). □

We now define the semi-simple trace as follows.

**Definition 6.1.8.** For $h \in H$, and $\mathcal{F}$ as above, put

$$\mathrm{tr}^{ss}(\Phi_q^r h|V) = \mathrm{tr}(\Phi_q^r h|(\mathrm{gr}_\bullet^{\mathcal{F}}(V))^{I_K}) = \sum_i \mathrm{tr}(\Phi_q^r h|\mathrm{gr}_i^{\mathcal{F}}(V)^{I_K}).$$

**Proposition 6.1.9.** *The semi-simple trace does not depend on the admissible filtration chosen, and in particular, the semi-simple trace is additive in short exact sequences.*

*Proof.* First consider the case that $I_K$ acts on $V$ through a finite quotient. Let $\mathcal{F}'$ be the filtration of $V^{I_K}$ induced by $\mathcal{F}$, since taking invariants under a finite group acting on an $\ell$-adic vector space is an exact functor, we have $\mathrm{gr}_i^{\mathcal{F}'}(V^{I_K}) = \mathrm{gr}_i^{\mathcal{F}}(V)^{I_K}$, hence $\mathrm{tr}(\Phi_q^r h|V^{I_K}) = \sum_i \mathrm{tr}(\Phi_q^r h|\mathrm{gr}_i^{\mathcal{F}}(V)^{I_K})$.

In general, two admissible filtrations admit a common refinement. As noted above, we deduce that the semi-simple trace associated to each of the two filtrations is equal to the semi-simple trace associated to the refinement.

For additivity, it is just the statement that for an endomorphism $\phi$ on a vector space $V$ with $\phi$-invariant subspace $W$, we have $\mathrm{tr}(\phi|V) = \mathrm{tr}(\phi|W) + \mathrm{tr}(\phi|V/W)$. □

This proposition allows us to define the semi-simple trace on the Grothendieck group of $G_K \times H$, or on the derived category of finite dimensional continuous $\ell$-adic representations of $G_K \times H$.

Let $\mathcal{R}$ be the category of continuous, finite dimensional $\overline{\mathbb{Q}}_\ell$-representations of $G_K \times H$. For any object $C$ of the derived category associated to $\mathcal{R}$, let

$$\mathrm{tr}^{ss}(\Phi_q^r h|C) = \sum_i (-1)^i \mathrm{tr}^{ss}(\Phi_q^r h|H^i(C)).$$

From above, it is additive in distinguished triangles (cf. [13]).

For a variety $X$ over $K$, we define

**Definition 6.1.10.** The semi-simple local factor is defined by

$$\log \zeta^{ss}(X, s) = \sum_{r \geqslant 1} \sum_{i=0}^{2\dim X} (-1)^i \mathrm{tr}^{ss}(\Phi_q^r|H_c^i(X \otimes_K \overline{K}, \overline{\mathbb{Q}}_\ell)) \frac{q^{-rs}}{r}.$$

This agrees with the usual local factor if $I_K$ acts through a finite quotient and the semi-simple local factor determines the true local factor, which is proved by Rapoport, [27].

6.2. **Nearby cycles.**

We recall the definition of nearby cycles according to our case. For $\mathcal{O} \subset K$ the ring of integers and for a scheme $X_{\mathcal{O}}/\mathcal{O}$ of finite type, write $X_s, X_{\overline{s}}, X_\eta$ and $X_{\overline{\eta}}$ respectively for its special, geometric special, generic and geometric generic fiber, respectively. Let $X_{\overline{\mathcal{O}}}$ denote the base change to the ring of integers in a fixed algebraic closure of $K$. We have maps $\overline{\imath} : X_{\overline{s}} \longrightarrow X_{\overline{\mathcal{O}}}$ and $\overline{j} : X_{\overline{\eta}} \longrightarrow X_{\overline{\mathcal{O}}}$ .

**Definition 6.2.1.** For a $\overline{\mathbb{Q}}_\ell$-sheaf $\mathcal{F}$ on $X_\eta$, the complex of nearby cycle sheaves is defined to be $R\psi\mathcal{F} = \bar{\iota}^* R\bar{j}_*\mathcal{F}_{\overline{\eta}}$, where $\mathcal{F}_{\overline{\eta}}$ is the pullback of $\mathcal{F}$ to $X_{\overline{\eta}}$. This is an element of the (so-called) derived category of $\overline{\mathbb{Q}}_\ell$-sheaves on $X_{\overline{s}}$ with an action of $G_K$ that is compatible with its action on $X_{\overline{s}}$.

The following two results, connect the semi-simple trace and local factor with nearby cycles.

**Theorem 6.2.2.** *Assume that $X_\mathcal{O}/\mathcal{O}$ is a scheme of finite type such that there exists an open immersion $X_\mathcal{O} \subset \overline{X}_\mathcal{O}$ where $\overline{X}_\mathcal{O}$ is proper over $\mathcal{O}$, with complement $D$ a relative normal crossings divisor (i.e. there is an open neighborhood $U$ of $D$ in $\overline{X}_\mathcal{O}$ which is smooth over $\mathcal{O}$, such that $D$ is a relative normal crossings divisor in $U$). Then there is a canonical $G_K$-equivariant isomorphism*

$$H^i_c(X_{\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \cong H^i_c(X_{\overline{s}}, R\psi\overline{\mathbb{Q}}_\ell),$$

*and*

$$\log \zeta^{ss}(X_\eta, s) = \sum_{r\geqslant 1} \sum_{x\in X_s(\mathbb{F}_{q^r})} tr^{ss}(\Phi_{q^r}|(R\psi\overline{\mathbb{Q}}_\ell)_x)\frac{q^{-rs}}{r}.$$

*Proof.* The first part is from theorem 5.2.2 and the rest is from the discussion in [13], section 3. □

This is not true for $X = \mathcal{M}_{\Gamma(p^n)}$ as its divisor at infinity is not étale over $\operatorname{Spec}\mathbb{Z}[m^{-1}]$. However, we have the following similar theorem.

**Theorem 6.2.3.** *There is a canonical $G_{\mathbb{Q}_p}$-equivariant isomorphism*

$$H^i_c(\mathcal{M}_{\Gamma(p^n),\overline{\eta}}, \overline{\mathbb{Q}}_\ell) \longrightarrow H^i_c(\mathcal{M}_{\Gamma(p^n),\overline{s}}, R\psi\overline{\mathbb{Q}}_\ell).$$

*Also, the formula for the semi-simple local factor from theorem 6.2.2 holds.*

*Proof.* This is the results from theorem 6.2.2 and theorem 2.3.3. For the proof, see [30], theorem 7.11.

□

As nearby cycles relates closely to the semi-simple local factor, we calculate the nearby cycles in the case of interest as follows.

Let $X_\mathcal{O}/\mathcal{O}$ be a scheme of finite type, $X_{\eta^{ur}}$ be the base change of $X_\mathcal{O}$ to the maximal unramified extension $K^{ur}$ of $K$ and let $X_{\mathcal{O}^{ur}}$ be the base change to the ring of integers in $K^{ur}$. We have $\iota : X_{\overline{s}} \longrightarrow X_{\mathcal{O}^{ur}}$ and $j : X_{\eta^{ur}} \longrightarrow X_{\mathcal{O}^{ur}}$.

With a special case of Grothendieck's purity conjecture, in [30], Scholze gives the following results of calculation.

**Theorem 6.2.4.** *Let $X/\mathcal{O}$ be regular and flat of relative dimension 1 and suppose that $X_s$ is globally the union of regular divisors. Let $x \in X_s(\mathbb{F}_q)$ and let $D_1, \dots, D_i$ be the divisors passing through $x$. Let $W_1$ be the $i$-dimensional $\overline{\mathbb{Q}}_\ell$-vector space with basis given by those $D_t$, and $W_2$ be the kernel of the map $W_1 \longrightarrow \overline{\mathbb{Q}}_\ell$ sending all $D_t$ to 1. Then there are canonical isomorphisms*

$$(\iota^* R^k j_* \overline{\mathbb{Q}}_\ell)_x \cong \begin{cases} \overline{\mathbb{Q}}_\ell & k=0 \\ W_1(-1) & k=1 \\ W_2(-2) & k=2 \\ 0 & \textit{else} \ . \end{cases}$$

*Proof.* cf. [30], theorem 8.2.

□

Let $B$ denote the Borel subgroup of $\mathrm{GL}_2$. We have associated an element $\delta \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$ to any point $x \in \mathcal{M}(\mathbb{F}_{p^r})$ by looking at the action of $F$ on the crystalline cohomology. And the covering $\pi_n : \mathcal{M}_{\Gamma(p^n)} \longrightarrow \mathcal{M}$ and the sheaf $\mathcal{F}_n = \pi_{n\eta *}\mathbb{Q}_\ell$ on the generic fibre of $\mathcal{M}_{\Gamma(p^n)}$.

**Corollary 6.2.5.** *Let $x \in \mathcal{M}(\mathbb{F}_{p^r})$ and $g \in GL_2(\mathbb{Z}_p)$.*

(i) *If $x$ corresponds to an ordinary elliptic curve and $a$ is the unique eigenvalue of $N\delta$ with valuation 0, then $tr^{ss}(\Phi_{p^r} g|(R\psi\mathcal{F}_n)_x) = tr(\Phi_{p^r} g|V_n)$, where $V_n$ is a $G_{\mathbb{F}_{p^r}} \times GL_2(\mathbb{Z}/p^n\mathbb{Z})$-representation isomorphic to*

$$\bigoplus_{\chi \in ((\mathbb{Z}/p^n\mathbb{Z})^\times)^\vee} Ind_{B(\mathbb{Z}/p^n\mathbb{Z})}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes \chi$$

*as a $GL_2(\mathbb{Z}/p^n\mathbb{Z})$-representation. Here $\Phi_{p^r}$ acts as the scalar $\chi(a)^{-1}$ on $Ind_{B(\mathbb{Z}/p^n\mathbb{Z})}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes \chi$.*

(ii) *If $x$ corresponds to a supersingular elliptic curve, then*

$$tr^{ss}(\Phi_{p^r} g|(R\psi\mathcal{F}_n)_x) = 1 - tr(g|St)p^r,$$

*where $St = \ker(Ind_{B(\mathbb{Z}/p^n\mathbb{Z})}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes 1 \longrightarrow 1)$ is the Steinberg representation of $GL_2(\mathbb{Z}/p^n\mathbb{Z})$.*

*Proof.* Here we omit the proof in p. 19, [30], which applies theorem 6.2.4 and theorem 2.3.4.

□

For $x \in \mathcal{M}(\mathbb{F}_{p^r})$, put $(R\psi\mathcal{F}_\infty)_x = \varinjlim (R\psi\mathcal{F}_n)_x$, then it carries a natural smooth action of $\mathrm{GL}_2(\mathbb{Z}_p)$ and a commuting continuous action of $G_{\mathbb{Q}_{p^r}}$. Define $\mathrm{tr}^{ss}(\Phi_{p^r} h|(R\psi\mathcal{F}_\infty)_x)$ for $h \in C_c^\infty(\mathrm{GL}_2(\mathbb{Z}_p))$ in the following way: Pick $n$ such that $h$ is $\Gamma(p^n)_{\mathbb{Q}_p}$-biinvariant and take invariants under $\Gamma(p^n)_{\mathbb{Q}_p}$:

$$\mathrm{tr}^{ss}(\Phi_{p^r} h|(R\psi\mathcal{F}_\infty)_x) := \mathrm{tr}^{ss}(\Phi_{p^r} h|(R\psi\mathcal{F}_n)x).$$

For $h \in C_c^\infty(\mathrm{GL}_2(\mathbb{Z}_p))$, the value of $\mathrm{tr}^{ss}(\Phi_{p^r} h|(R\psi\mathcal{F}_\infty)_x)$ depends only on $\gamma = \mathrm{N}\delta$ associated to $x$, which leads to the following definition.

**Definition 6.2.6.** For $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$, $h \in C_c^\infty(\mathrm{GL}_2(\mathbb{Z}_p))$, define $c_r(\gamma, h) = 0$ unless $\upsilon_p(\det\gamma) = r$, $\upsilon_p(\mathrm{tr}\gamma) \geqslant 0$. Now assume that these conditions are fulfilled. Then for $\upsilon_p(\mathrm{tr}\gamma) = 0$, we define

$$c_r(\gamma, h) = \sum_{\chi_0 \in ((\mathbb{Z}/p^n\mathbb{Z})^\times)^\vee} \mathrm{tr}(h|\mathrm{Ind}_{B(\mathbb{Z}/p^n\mathbb{Z})}^{\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes \chi_0)\chi_0(t_2)^{-1}$$

where $t_2$ is the unique eigenvalue of $\gamma$ with $\upsilon_p(t_2) = 0$. For $\upsilon_p(\mathrm{tr}\gamma) \geqslant 1$, we take

$$c_r(\gamma, h) = \mathrm{tr}(h|1) - p^r \mathrm{tr}(h|\mathrm{St}).$$

That $x$ is supersingular is equivalent to $\mathrm{trN}\delta \equiv 0 \mod p$, so we have

$$\mathrm{tr}^{ss}(\Phi_{p^r} h|(R\psi\mathcal{F}_\infty)_x) = c_r(\mathrm{N}\delta, h)$$

whenever $\delta$ is associated to $x \in \mathcal{M}(\mathbb{F}_{p^r})$ as in subsection 5.4.

6.3. **The semi-simple trace of Frobenius as a twisted orbital integral.** As explained in [22], the twisted orbital integral plays an important role in Langlands's method of computing the Hasse-Weil zeta function. In this subsection, we give a quick sketch connecting the semi-simple trace with twisted orbital integral, all are contained in [30].

We first need the function $\phi_p$ which has the correct twisted orbital integrals.

**Lemma 6.3.1.** *There exists a function $\phi_p$ of the Bernstein center of $GL_2(\mathbb{Q}_{p^r})$ such that for all irreducible smooth representations $\Pi$ of $GL_2(\mathbb{Q}_{p^r})$, $\phi_p$ acts by the scalar $p^{\frac{1}{2}} tr^{ss}(\Phi_{p^r}|\sigma_\Pi)$, where $\sigma_\Pi$ is the representation of the Weil group $W_{\mathbb{Q}_{p^r}}$ of $\mathbb{Q}_{p^r}$ with values in $\overline{\mathbb{Q}}_\ell$ associated to $\Pi$ by the local Langlands correspondence.*

For a representation $\sigma$ of $W_{\mathbb{Q}_{p^r}}$, the definition of the semi-simple trace of Frobenius makes sense. We write $\sigma^{ss}$ for the associated semisimplification.

*Proof.* By theorem 3.3.1, it suffices to check that the corresponding function to $\phi_p$ as in the assumption defines a regular function on $D/W(L, D)$ for all $L, D$. As we are taking the semi-simple trace, the scalar agrees for a 1-dimensional representation $\Pi$ and the corresponding twist of the Steinberg representation. This gives a well-defined function on $D/W(L, D)$. For $L$ and $D$ fixed, take $\Pi$ in the corresponding component, then the semi-simplification $\sigma_\Pi^{ss}$ decomposes as $(\sigma_1 \otimes \chi \circ \det) \otimes \cdots \otimes (\sigma_t \otimes \chi_t \circ \det)$ for certain fixed irreducible representations $\sigma_1, \ldots, \sigma_t$ and varying unramified characters $\chi_1, \ldots, \chi_t$ parametrized by $D$. So $\mathrm{tr}^{ss}(\Phi_{p^r}|\sigma_\Pi) = \sum_{i=1}^{t} \mathrm{tr}^{ss}(\Phi_{p^r}|\sigma_i)\chi_i(p)$, it is a regular function on $D$ and necessarily $W(L, D)$-invariant, hence descends to a regular function on $D/W(L, D)$. □

Next we consider the function $\phi_{p,0} = \phi_p * e_{\mathrm{GL}_2(\mathbb{Z}_{p^r})} \in \mathcal{H}(\mathrm{GL}_2(\mathbb{Q}_{p^r}), \mathrm{GL}_2(\mathbb{Z}_{p^r}))$. It is compatible with the previous one, since it is showed in p. 21, [30] that

**Proposition 6.3.2.** *The function $\phi_{p,0}$ is the characteristic function of the set*

$$GL_2(\mathbb{Z}_{p^r}) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_{p^r})$$

*divided by the volume of $GL_2(\mathbb{Z}_{p^r})$.*

The following theorem makes connection with the semi-simple trace and twisted orbital integral.

**Theorem 6.3.3.** *Let $\delta \in GL_2(\mathbb{Q}_{p^r})$ with semisimple norm $\gamma \in GL_2(\mathbb{Q}_p)$. If $h \in C_c^\infty(GL_2(\mathbb{Z}_p))$ and $h' \in C_c^\infty(GL_2(\mathbb{Z}_{p^r}))$ have matching (twisted) orbital integrals. Then $TO_{\delta\sigma}(\phi_p * h') = TO_{\delta\sigma}(\phi_{p,0})c_r(\gamma, h)$.*

*Proof.* Denote by $f_1 = \phi_p * h'$ and $f_2 = \phi_{p,0}$. Let $\Pi$ be the base change lift to $\mathrm{GL}_2(\mathbb{Q}_{p^r})$ of some tempered representation $\pi$ of $\mathrm{GL}_2(\mathbb{Q}_p)$. From the construction of $\mathrm{tr}^{ss}(\Phi_{p^r} h|(R\psi\mathbb{F}_\infty)_x)$, we can take $n$ such that $h, h'$ are $\Gamma(p^n)_{\mathbb{Q}_p}$, $\Gamma(p^n)_{\mathbb{Q}_{p^r}}$-biinvariant, respectively. Then by the assumption, we have

$$\mathrm{tr}((f_1,\sigma)|\Pi) = p^{\frac{1}{2}r}\mathrm{tr}((h',\sigma)|\Pi^{\Gamma(p^n)})\mathrm{tr}^{ss}(\Phi_{p^r}|\sigma_\Pi) = p^{\frac{1}{2}r}\mathrm{tr}(h|\pi^{\Gamma(p^n)})\mathrm{tr}^{ss}(\phi_{p^r}|\sigma_\Pi).$$

Because $e_{\Gamma(1)\mathbb{Q}_p}$ and $e_{\Gamma(1)\mathbb{Q}_{p^r}}$ are associated by theorem 5.3.10, we have

$$\mathrm{tr}((f_2,\sigma)|\Pi) = p^{\frac{1}{2}r}\dim \pi^{\mathrm{GL}_2(\mathbb{Z}_p)}\mathrm{tr}^{ss}(\Phi_{p^r}|\sigma_\Pi).$$

Then the rest is finished by the following two lemmas. For the detailed proof, follow p. 21-24, [30].

**Lemma 6.3.4.** *Assume that* $\delta = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ *with* $Nt_1 \neq Nt_2$. *Then the twisted orbital integrals* $TO_{\delta\sigma}(\phi_p * h') = TO_{\delta\sigma}(\phi_{p,0}) = 0$ *except in the case where, up to exchanging* $t_1, t_2$, *we have* $v_p(t_1) = 1$ *and* $v_p(t_2) = 0$. *In the latter case,*

$$TO_{\delta\sigma}(\phi_p * h') = \mathrm{vol}(T(\mathbb{Z}_p))^{-1} \sum_{\chi_0 \in ((\mathbb{Z}/p^n\mathbb{Z})^\times)^\vee} tr(h|Ind_{B(\mathbb{Z}/p^n\mathbb{Z})}^{GL_2(\mathbb{Z}/p^n\mathbb{Z})} 1 \boxtimes \chi_0)\chi_0(Nt_2)^{-1}$$

*and*

$$TO_{\delta\sigma}(\phi_{p,0}) = \mathrm{vol}(T(\mathbb{Z}_p))^{-1}.$$

Now if $\delta$ is not $\sigma$-conjugate to an element as in the previous lemma, then the eigenvalues of eigenvalues of $\mathrm{N}\delta$ have the same valuation. Let $H_1 = \mathrm{tr}(h|\mathrm{St})$ and $H_2 = \mathrm{tr}(h|1)$ with St and 1 the Steinberg and trivial representation of $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ respectively. Let $f = f_1 + (H_1 p^r - H_2) f_2$.

**Lemma 6.3.5.** *If the eigenvalues of* $N\delta$ *have the same valuation, then the twisted orbital integral* $TO_{\delta\sigma}(f)$ *vanishes.*

□

**Lemma 6.3.6.** *For any* $\delta \in GL_2(\mathbb{Q}_{p^r})$ *associated to an elliptic curve over* $\mathbb{F}_{p^r}$, *the norm* $N\delta$ *is semisimple (i.e. diagonalizable).*

*Proof.* cf. [30], lemma 9.8. □

With this lemma, combining theorem 6.3.3 with corollary 6.2.5, we get the desired result.

**Corollary 6.3.7.** *Let* $x \in \mathcal{M}(\mathbb{F}_{p^r})$ *with associated* $\delta$. *Let* $h \in C_c^\infty(GL_2(\mathbb{Z}_p)), h' \in C_c^\infty(GL_2(\mathbb{Z}_{p^r}))$ *have matching (twisted) orbital integrals. Then*

$$TO_{\delta\sigma}(\phi_p * h') = TO_{\delta\sigma}(\phi_{p,0}) tr^{ss}(\Phi_{p^r} h|(R\psi\mathcal{F}_\infty)_x).$$

By theorem 5.3.10, taking $h$ to be the idempotent $e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ and $h'$ to be the idempotent $e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$. By remark 5.4.3, we immediately get that

**Corollary 6.3.8.** *Let* $x \in \mathcal{M}(\mathbb{F}_{p^r})$ *with associated* $\delta$. *Then*

$$tr^{ss}(\Phi_{p^r}|(R\psi\mathcal{F}_n)_x) = TO_{\delta\sigma}(\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}})(TO_{\delta\sigma}(\phi_{p,0}))^{-1}.$$

**Remark 6.3.9.** The explicit determination of $\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$ is proved in [30] as follows, which we do not need in the rest of the thesis:

For $g \in \mathrm{GL}_2(\mathbb{Q}_{p^r})$, let $k(g)$ be the minimal number $k$ such that $p^k g$ has integral entries. If $\upsilon_p(\det g) \geqslant 1$ and $\upsilon_p(\mathrm{tr} g) = 0$, then $g$ has a unique eigenvalue $x \in \mathbb{Q}_{p^r}$ with $\upsilon_p(x) = 0$; in this case we define $l(g) = \upsilon_p(x-1)$. For $n \geqslant 1$, define a function $\phi_{p,n} : \mathrm{GL}_2(\mathbb{Q}_{p^r}) \longrightarrow \mathbb{C}$ requiring that

- $\phi_{p,n}(g) = 0$ except if $\upsilon_p(\det g) = 1$, $\upsilon_p(\mathrm{tr} g) \geqslant 0$ and $k(g) \leqslant n-1$. Assume now that $g$ has these properties.
- $\phi_{p,n}(g) = -1 - q$ if $\upsilon_p(\mathrm{tr} g) \geqslant 1$,
- $\phi_{p,n}(g) = 1 - q^{2l(g)}$ if $\upsilon_p(\mathrm{tr} g) = 0$ and $l(g) < n - k(g)$,
- $\phi_{p,n}(g) = 1 + q^{2(n-k(g))-1}$ if $\upsilon_p(\mathrm{tr} g) = 0$ and $l(g) \geqslant n - k(g)$.

Take the Haar measure on $\mathrm{GL}_2(\mathbb{Q}_{p^r})$ such that a maximal compact subgroup has measure $p^r - 1$, then we have $\phi_{p,n} = \phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$.

## 6.4. **Lefschetz number.**

In order to compute the semi-simple local factor, we will compute the Lefschetz number $\sum_{x \in \mathcal{M}_{\Gamma(p^n)}(\mathbb{F}_{p^r})} \mathrm{tr}^{ss}(\Phi_{p^r} | R\psi\mathcal{F}_n)_x)$, because by theorem 6.2.3,

$$\log \zeta^{ss}(\mathcal{M}_{\Gamma(p^n)}, \overline{\mathbb{Q}}_\ell) = \sum_{r \geqslant 1} \sum_{x \in \mathcal{M}(\mathbb{F}_{p^r})} \mathrm{tr}^{ss}(\Phi_{p^r} | (R\psi\mathcal{F}_n)_x) \frac{p^{-rs}}{r}.$$

We have just got that

$$\mathrm{tr}^{ss}(\Phi_{p^r} | (R\psi\mathcal{F}_n)_x) = TO_{\delta\sigma}(\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}})(TO_{\delta\sigma}(\phi_{p,0}))^{-1}.$$

Combining this with corollary 5.4.2, we actually have the following

**Corollary 6.4.1.**

$$\sum_{x \in \mathcal{M}(\mathbb{F}_{p^r})(E)} tr^{ss}(\Phi_{p^r} | R\psi\mathcal{F}_n)_x) = \mathrm{vol}(\Gamma \backslash G_\gamma(\mathbb{A}_f^p) \times G_{\delta\sigma}(\mathbb{Q}_p)) O_\gamma(f^p) TO_{\delta\sigma}(\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}).$$

Now we try to eliminate the twisted orbital integral. Let $f_{p,r}$ be the function of the Bernstein center for $\mathrm{GL}_2(\mathbb{Q}_p)$ such that for all irreducible smooth representations $\pi$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, $f_{p,r}$ acts by the scalar $p^{\frac{1}{2}r}\mathrm{tr}^{ss}(\Phi_p^r | \sigma_\pi)$. Here $\sigma_p$ is the $\ell$-adic representation, associated to $\pi$, of the Weil group $W_{\mathbb{Q}_p}$ with $\overline{\mathbb{Q}}_\ell$ coefficients. In the same way as lemma 6.3.1, it can be proved that $f_{p,r}$ exists. From [15], if $\pi$ is tempered and $\Pi$ is a base-change lift of $\pi$, then $\sigma_\Pi$ is the restriction of $\sigma_\pi$. As proved in [30], there is one result

**Lemma 6.4.2.** *For any tempered irreducible smooth representation $\pi$ of $GL_2(\mathbb{Q}_p)$ with base-change lift $\Pi$, we have $tr^{ss}(\Phi_p^r | \sigma_\pi) = tr^{ss}(\Phi_{p^r} | \sigma_\Pi)$.*

By this lemma, we deduce that $f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ and $\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$ satisfy the hypothesis of theorem 5.3.10. Together with lemma 6.3.6, we get from corollary 6.4.1 the following

**Corollary 6.4.3.**

$$\sum_{x\in\mathcal{M}(\mathbb{F}_{p^r})(E)} tr^{ss}(\Phi_{p^r}|R\psi\mathcal{F}_n)_x) = \pm\mathrm{vol}(\Gamma\backslash G_\gamma(\mathbb{A}_f^p)\times G_{\delta\sigma}(\mathbb{Q}_p))O_\gamma(f^p)O_{N\delta}(f_{p,r}*e_{\Gamma(p^n)_{\mathbb{Q}_p}}).$$

The Honda-Tate theory allows us a further simplification of the above result. We first recall certain facts here.

**Theorem 6.4.4.** *Fix a finite field $\mathbb{F}_q$ of characteristic $p$.*

(i) *For any elliptic curve $E/\mathbb{F}_q$, the action of Frobenius on $H^1_{et}(E,\mathbb{Q}_\ell)$ is semisimple with characteristic polynomial $p_E\in\mathbb{Z}[T]$ independent of $\ell$. Additionally, if $F$ acts as $\delta\sigma$ on $H^1_{crys}(E/\mathbb{Z}_q)\otimes\mathbb{Q}_q$, then $N\delta$ is semisimple with characteristic polynomial $p_E$.*
*Let $\gamma_E\in GL_2(\mathbb{Q})$ be semisimple with characteristic polynomial $p_E$. Then*

(ii) *The map $E\mapsto\gamma_E$ gives a bijection between $\mathbb{F}_q$-isogeny classes of elliptic curves over $\mathbb{F}_q$ and conjugacy classes of semisimple elements $\gamma\in GL_2(\mathbb{Q})$ with $\det\gamma=q$, $tr\gamma\in\mathbb{Z}$ which are elliptic in $GL_2(\mathbb{R})$, and there exists $\delta\in GL_2(\mathbb{Q}_p)$ such that $\gamma$ is conjugate to $N(\delta)$.*

(iii) *Let $G_{\gamma_E}$ be the centralizer of $\gamma_E$. Then $\mathrm{End}(E)^\times$ is an inner form of $G_{\gamma_E}$. We have*

$$(\mathrm{End}(E)\otimes\mathbb{Q}_\ell)^\times\cong G_{\gamma_E}\otimes\mathbb{Q}_\ell,\ \textit{for } \ell\neq p$$
$$(\mathrm{End}(E)\otimes\mathbb{Q}_p)^\times\cong G_{\delta\sigma}.$$

*Furthermore, the algebraic group $(\mathrm{End}(E)\otimes\mathbb{R})^\times$ is anisotropic modulo center.*

*Proof.* This is a combination of the fixed point formulas in étale and crystalline cohomology, the Weil conjectures for elliptic curves and the main theorems of [33], [16]. Refer to [5] for (i), refer to [16] for (ii), and refer to [33], [5] for (iii). □

For one isogeny class, the right-hand side of lemma 6.4.3 equals

$$\pm\mathrm{vol}(\Gamma\backslash(\mathrm{End}(E)\otimes\mathbb{A}_f)^\times)O_\gamma(f^p)O_\gamma(f_{p,r}*e_{\Gamma(p^n)_{\mathbb{Q}_p}}),$$

Write $\gamma=\gamma_E\in\mathrm{GL}_2(\mathbb{Q})$, which is compatible with previous use of $\gamma$ by part (i), and write $f$ for the function $f^p(f_{p,r}*e_{\Gamma(p^n)_{\mathbb{Q}_p}})\in C_c^\infty(\mathrm{GL}_2(\mathbb{A}_f))$.

Now we rewrite the expression above as

$$\pm\mathrm{vol}((\mathrm{End}(E)\otimes\mathbb{Q})^\times\backslash(\mathrm{End}(E)\otimes\mathbb{A}_f)^\times)\int_{G_\gamma(\mathbb{A}_f)\backslash\mathrm{GL}_2(\mathbb{A}_f)} f(g^{-1}\gamma g)dg.$$

**Theorem 6.4.5.** *The Lefschetz number $\sum_{x\in\mathcal{M}_{\Gamma(p^n)}(\mathbb{F}_{p^r})} tr^{ss}(\Phi_{p^r}|R\psi\mathcal{F}_n)_x)$ equals*

$$-\sum_{\gamma\in\mathbb{Z}(\mathbb{Q})}\mathrm{vol}(\overline{GL_2}(\mathbb{Q})\backslash\overline{GL_2}(\mathbb{A}_f))f(\gamma)+{\sum}'\mathrm{vol}(\overline{G}_\gamma(\mathbb{Q})\backslash\overline{G}_\gamma(\mathbb{A}_f))\int_{G_\gamma(\mathbb{A}_f)\backslash GL_2(\mathbb{A}_f)} f(g^{-1}\gamma g)dg.$$

*where the second sum $\sum'$ is taken over $\gamma\in GL_2(\mathbb{Q})\backslash Z(\mathbb{Q})$ semisimple conjugate class with $\gamma_\infty$ elliptic.*

*Proof.* It is enough to check that the contributions of $\gamma$ vanish where $\det\gamma \neq p^r$ or $\mathrm{tr}\gamma \notin \mathbb{Z}$.

Assume that $\det\gamma \neq p^r$. The orbital integrals of $f^p$ vanish except if $\det\gamma$ is a unit away from $p$, hence $\det\gamma$ is up to sign a power of $p$. The orbital integrals of $f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ vanish except if $v_p(\det\gamma) = r$, hence $\det\gamma = \pm p^r$. But $\gamma$ is hyperbolic at $\infty$ if $\det\gamma = -p^r < 0$, so $\det\gamma = p^r$, a contradiction.

Now assume that $\mathrm{tr}\gamma \notin \mathbb{Z}$. If a prime $\ell \neq p$ is in the denominator of $\mathrm{tr}\gamma$, then the orbital integrals of $f^p$ vanish. The orbital integrals of $f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ match with the twisted orbital integrals of $\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$, which were computed in theorem 6.3.3. Consequently, they are nonzero only if $v_p(\mathrm{tr}\gamma) \geqslant 0$, which implies that $\mathrm{tr}\gamma$ is integral, a contradiction again. □

## 7. The Hasse-Weil zeta-function

With the above results, now we are ready to conclude the main result. This section is summarized from [30].

### 7.1. **Contributions from the boundary.**

Recall from section 2, let $j : \mathcal{M}_{p^n m} \longrightarrow \overline{\mathcal{M}}_{p^n m}$ be a smooth projective compactification with boundary $\partial\mathcal{M}_{p^n m}$. We use a subscript $\overline{\mathbb{Q}}$ to denote base change to $\overline{\mathbb{Q}}$.

Let $H^*(\overline{\mathcal{M}}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) = \sum_{i=0}^{2}(-1)^i H^i(\overline{\mathcal{M}}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$ in the Grothendieck group of representations of $G_{\mathbb{Q}} \times \mathrm{GL}_2(\mathbb{Z}/p^n m\mathbb{Z})$. we are interested in the semi-simple trace of Frobenius on the cohomology groups, which will turn out to be related to the Authur-Selberg trace formula. Similarly, put

$$H_c^*(\mathcal{M}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) = \sum_{i=0}^{2}(-1)^i H_c^i(\mathcal{M}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell).$$

The long exact cohomology sequence for the short exact sequence

$$0 \longrightarrow j_!\overline{\mathbb{Q}}_\ell \longrightarrow \overline{\mathbb{Q}}_\ell \longrightarrow \bigoplus_{x\in\partial\mathcal{M}_{p^n m\overline{\mathbb{Q}}}} \overline{\mathbb{Q}}_{l,x} \longrightarrow 0$$

implies that $H^*(\overline{\mathcal{M}}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) = H_c^*(\mathcal{M}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell) + H^0(\partial\mathcal{M}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$.

The discussions above have actually imply the semi-simple trace of $\Phi_{p^r}$ on $H_c^*(\mathcal{M}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$ by theorem 6.4.5. We focus on the boundary here.

**Lemma 7.1.1.** *There is a $G_{\mathbb{Q}} \times GL_2(\mathbb{Z}/p^n m\mathbb{Z})$-equivariant bijection*

$$\partial\mathcal{M}_{p^n m\overline{\mathbb{Q}}} \cong \{\pm \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}\backslash GL_2(\mathbb{Z}/p^n m\mathbb{Z}),$$

*where $GL_2(\mathbb{Z}/p^n m\mathbb{Z})$ acts on the right hand side by multiplication from the right, and $G_{\mathbb{Q}}$ acts on the right hand side by multiplication from the left through the map*

$$G_{\mathbb{Q}} \longrightarrow Gal(\mathbb{Q}(\zeta_{p^n m})/\mathbb{Q}) \cong (\mathbb{Z}/p^n m\mathbb{Z})^\times \longrightarrow GL_2(\mathbb{Z}/p^n m\mathbb{Z}),$$

*the last map being given by* $x \mapsto \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}$.

*Proof.* cf. [30]. □

P. Scholze then has proved the following

**Corollary 7.1.2.** *The semi-simple trace of the Frobenius $\Phi_{p^r}$ on $H^0(\partial\mathcal{M}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$ is given by $\frac{1}{2}\int_{GL_2(\hat{\mathbb{Z}})}\int_{\mathbb{A}_f} f(k^{-1}\left(\begin{smallmatrix}1&0\\0&p^r\end{smallmatrix}\right)\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)k)dudk$. Here, for all $p'$ we use the Haar measure on $\mathbb{Q}_{p'}$ that gives $\mathbb{Z}_{p'}$ measure 1, hence the subgroup $\hat{\mathbb{Z}}$ of $\mathbb{A}_f$ gets measure 1.*

*Proof.* If $f^p(k^{-1}\left(\begin{smallmatrix}1&u\\0&p^r\end{smallmatrix}\right)k)\neq 0$, then $p^r\equiv 1 \mod m$, so if $p^r\not\equiv 1 \mod m$, then the integral is identically zero. In this situation, $\Phi_{p^r}$ has no fixed points on $\partial\mathcal{M}_{p^n m\overline{\mathbb{Q}}}$. Thus assuming now that $p^r\equiv 1 \mod m$, then the inertia subgroup at $p$ groups the points of $\partial\mathcal{M}_{p^n m\overline{\mathbb{Q}}}$ into packets of size $p^{n-1}(p-1)$ on which $\Phi_{p^r}$ acts trivially. Therefore the semi-simple trace of $\Phi_{p^r}$ is

$$\frac{1}{p^{n-1}(p-1)}\#(\{\pm\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\}\backslash \mathrm{GL}_2(\mathbb{Z}/p^n m\mathbb{Z})).$$

As $f^p(k^{-1}\left(\begin{smallmatrix}1&0\\0&p^r\end{smallmatrix}\right)\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)k) = \#\mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})\mathrm{vol}(\mathrm{GL}_2(\hat{\mathbb{Z}}^p))^{-1}$ if $u\equiv 0 \mod m$ and is 0 otherwise, we deduce that

$$\int_{\mathrm{GL}_2(\hat{\mathbb{Z}}^p)}\int_{\mathbb{A}_f^p} f^p(k^{-1}\left(\begin{smallmatrix}1&0\\0&p^r\end{smallmatrix}\right)\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)k)dudk = \#(\{\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\}\backslash \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})).$$

So we have to prove

$$\int_{\mathrm{GL}_2(\mathbb{Z}_p)}\int_{\mathbb{Q}_p}(f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}})(k^{-1}\left(\begin{smallmatrix}1&u\\0&p^r\end{smallmatrix}\right)k)dudk = p^{2n}-p^{2n-2}.$$

Generally for $\gamma_1\neq\gamma_2$ and $h\in C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_p))$,

$$\begin{aligned}
&\int_{\mathrm{GL}_2(\mathbb{Z}_p)}\int_{\mathbb{Q}_p} h(k^{-1}\left(\begin{smallmatrix}\gamma_1&0\\0&\gamma_2\end{smallmatrix}\right)\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)k)dudk\\
=\ & |1-\frac{\gamma_2}{\gamma_1}|_p^{-1}\int_{\mathrm{GL}_2(\mathbb{Z}_p)}\int_{\mathbb{Q}_p} h(k^{-1}\left(\begin{smallmatrix}1&-u\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}\gamma_1&0\\0&\gamma_2\end{smallmatrix}\right)\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)k)dudk\\
=\ & |1-\frac{\gamma_2}{\gamma_1}|_p^{-1}\mathrm{vol}(T(\mathbb{Z}_p))\int_{T(\mathbb{Q}_p)\backslash\mathrm{GL}_2(\mathbb{Q}_p)} h(g^{-1}\left(\begin{smallmatrix}\gamma_1&0\\0&\gamma_2\end{smallmatrix}\right)g)dg,
\end{aligned}$$

as $\mathrm{GL}_2(\mathbb{Q}_p) = B(\mathbb{Q}_p)\mathrm{GL}_2(Z_p)$, so the left hand side of the desired equality is the orbital integral of $f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ for $\gamma = \left(\begin{smallmatrix}1&0\\0&p^r\end{smallmatrix}\right)$.

In our case, $|1-\frac{\gamma_2}{\gamma_1}|_p^{-1} = |1-p^r|_p^{-1} = 1$. Then it follows from the twisted orbital integral of $\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$ which equals the orbital integral of $f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}}$ and was calculated in lemma 6.3.4. □

## 7.2. **The Arthur-Selberg Trace formula.**

For the final comparison in Langlands' method, we give the special case of the Arthur-Selberg trace formula for $\mathrm{GL}_2$ that will be needed. For a fully detailed explanation, refer to [1].

Let $H^i_{(2)} = \varinjlim H^i_{(2)}(\mathcal{M}_m(\mathbb{C}), \mathbb{C})$ be the direct limit of the $L^2$-cohomology of the spaces $\mathcal{M}_m(\mathbb{C})$, which is a smooth, admissible representation of $\mathrm{GL}_2(\mathbb{A}_f)$. Define $H^*_{(2)} = \sum_{i=0}^{2}(-1)^i H^i_{(2)}$ and for $h \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}_f))$, let $\mathcal{L}(h) = \mathrm{tr}(h|H^*_{(2)})$.

Let $Z \subset T \subset B \subset \mathrm{GL}_2$ be the center, the diagonal torus and the upper triangular matrices. For $\gamma \in \mathrm{GL}_2$, let $G_\gamma$ be the centralizer. Put $T(\mathbb{Q})' = \{\gamma = \left(\begin{smallmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{smallmatrix}\right) | \gamma_1\gamma_2 > 0, |\gamma_1| < |\gamma_2|\}$, where we adopt the real absolute value.

**Theorem 7.2.1.** *For any $h \in C_c^\infty(GL_2(\mathbb{A}_f))$, we have*

$$\begin{aligned}
\frac{1}{2}\mathcal{L}(h) &= -\sum_{\gamma\in\mathbb{Z}(\mathbb{Q})} \mathrm{vol}(\overline{GL_2}(\mathbb{Q}))\backslash\overline{GL_2}(\mathbb{A}_f))h(\gamma) \\
&+ \sum{}' \mathrm{vol}(\overline{G}_\gamma(\mathbb{Q})\backslash\overline{G}_\gamma(\mathbb{A}_f)) \int_{G_\gamma(\mathbb{A}_f)\backslash GL_2(\mathbb{A}_f)} h(g^{-1}\gamma g)dg \\
&+ \frac{1}{2}\sum_{\gamma\in T(\mathbb{Q})'} \int_{GL_2(\hat{\mathbb{Z}})}\int_{\mathbb{A}_f} h(k\gamma \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) k^{-1})dudk \\
&+ \frac{1}{4}\sum_{\gamma\in Z(\mathbb{Q})} \int_{GL_2(\hat{\mathbb{Z}})}\int_{\mathbb{A}_f} h(k\gamma \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) k^{-1})dudk.
\end{aligned}$$

*where the sum $\sum'$ is taken over the same set as in theorem 6.4.5.*

*Proof.* cf. [30] for a specialized proof. This is a special case of Theorem 6.1 of [1]. □

Let $\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}), 1)$ denote the set of irreducible automorphic representations $\pi = \bigotimes_{p\leqslant\infty} \pi_p$ of $\mathrm{GL}_2(\mathbb{A})$ with $\pi_\infty$ having trivial central and infinitesimal character, that occur discretely in $L^2(\mathrm{GL}_2(\mathbb{Q})\mathbb{R}_{>0}\backslash\mathrm{GL}_2(\mathbb{A}))$. For $\pi \in \Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}), 1)$, let $m(\pi)$ be the multiplicity of $\pi$ in $L^2(\mathrm{GL}_2(\mathbb{Q})\mathbb{R}_{>0}\backslash\mathrm{GL}_2(\mathbb{A}))$. To deduce the spectral expansion of $\mathcal{L}(h)$, we need

**Lemma 7.2.2.** *For any $i = 0, 1, 2$, there is a canonical $GL_2(\mathbb{A}_f)$-equivariant isomorphism*

$$H^i_{(2)} \cong \bigoplus_{\pi\in\Pi_{disc}(GL_2(\mathbb{A}),1)} m(\pi)H^i(\mathfrak{gl}_2, SO_2(\mathbb{R}), \pi_\infty)\pi_f.$$

*There are the following possibilities for the representation $\pi_\infty$, which has trivial central and infinitesimal character:*

(i) *$\pi_\infty$ is the trivial representation or $\pi_\infty = sgn\,\mathrm{det}$. Then*

$$H^i(\mathfrak{gl}_2, SO_2(\mathbb{R}), \pi_\infty)) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i = 1 \\ \mathbb{C} & i = 2; \end{cases}$$

(ii) *$\pi_\infty$ is a single discrete series representation, then*

$$H^i(\mathfrak{gl}_2, SO_2(\mathbb{R}), \pi_\infty)) = \begin{cases} 0 & i = 0 \\ \mathbb{C}\oplus\mathbb{C} & i = 1 \\ 0 & i = 2. \end{cases}$$

*Proof.* It is a combination of section 2, [1] and the content in [8]. □

Denote $\chi(\pi_\infty) = \sum_{i=0}^{2}(-1)^i \dim H^i(\mathfrak{gl}_2, SO_2(\mathbb{R}), \pi_\infty)$.

**Corollary 7.2.3.** *For* $h \in C_c^\infty(GL_2(\mathbb{A}_f))$, $\mathcal{L}(h) = \sum_{\pi\in\Pi_{disc}(GL_2(\mathbb{A}),1)} m(\pi)\chi(\pi_\infty)tr(h|\pi_f)$.

### 7.3. Hasse-Weil zeta-function from comparison.

Fix an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, and recall $f = f^p(f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}}) \in C_c^\infty(\mathrm{GL}_2(\mathbb{A}_f))$, where $f^p$ is the characteristic function of $K^p$ divided by its volume, as in subsection 5.4, and $f_{p,r}$ is defined in subsection 6.4.

**Theorem 7.3.1.** $2tr^{ss}(\Phi_{p^r}|H^*(\overline{\mathcal{M}}_{p^n m\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)) = \mathcal{L}(f)$.

*Proof.* Combining theorem 6.4.5, corollary 7.1.2 and theorem 7.2.1, the rest is to show that for $\gamma = \left(\begin{smallmatrix}\gamma_1 & 0\\ 0 & \gamma_2\end{smallmatrix}\right) \in T(\mathbb{Q})$ with $\gamma_1\gamma_2 > 0$ and $|\gamma_1| \leqslant |\gamma_2|$, we have

$$\int_{\mathrm{GL}_2(\hat{\mathbb{Z}})}\int_{\mathbb{A}_f} f(k^{-1}\gamma \left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right) k)dudk = 0$$

except for $\gamma_1 = 1$, $\gamma_2 = p^r$.

As the integral is a product of local integrals, where the local one for a prime $\ell \neq p$ is only nonzero if $\gamma \in \mathrm{GL}_2(\mathbb{Z}_\ell)$, we deduce that $\gamma_1$ and $\gamma_2$ are up to sign a power of $p$. We now prove that

$$\int_{\mathrm{GL}_2(\mathbb{Z}_p)}\int_{\mathbb{Q}_p} (f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}})(k^{-1}\gamma \left(\begin{smallmatrix}1 & u\\ 0 & 1\end{smallmatrix}\right) k)dudk \neq 0$$

only if $v_p(\gamma_1) = 0$ and $v_p(\gamma_2) = r$. Similar to the proof of corollary 7.1.2, as long as $\gamma_1 \neq \gamma_2$, it is up to a constant an orbital integral of $f_{p,r} * e_{\Gamma(p^n)_{\mathbb{Q}_p}}$, which we have computed by the twisted orbital integrals of the matching function $\phi_p * e_{\Gamma(p^n)_{\mathbb{Q}_{p^r}}}$. Then the continuity of the integrals implies the case $\gamma_1 = \gamma_2$. We have proved that $\gamma_1$ and $\gamma_2$ are up to sign a power of $p$, but the case $\gamma_1 = -1, \gamma_2 = -p^r$ also gives 0, since no conjugate of $\gamma$ will be $\equiv 1 \mod m$. Hence the only possibility is $\gamma_1 = 1, \gamma_2 = p^r$. □

Now we are ready to compute the zeta-function of the varieties $\overline{\mathcal{M}}_m$, where $m$ is an integer which is the product of two coprime integers, both at least 3, and we do not consider any distinguished prime.

**Theorem 7.3.2.** *The Hasse-Weil zeta-function of* $\overline{\mathcal{M}}_m$ *is given by*

$$\zeta(\overline{\mathcal{M}}_m, s) = \prod_{\pi\in\Pi_{disc}(GL_2(\mathbb{A}),1)} L(\pi, s-\frac{1}{2})^{\frac{1}{2}m(\pi)\chi(\pi_\infty)\dim \pi_f^{K_m}},$$

*where* $K_m = \{g \in GL_2(\hat{\mathbb{Z}}) | g \equiv 1 \mod m\}$. $\chi(\pi_\infty) = 2$ *if* $\pi_\infty$ *is a character and* $\chi(\pi_\infty) = -2$ *otherwise.*

We give the proof of P. Scholze in [30], which shows how the Langlands's method works by comparison of Lefschetz and Arthur-Selberg trace formula.

*Proof.* By definition, we want to compute the semi-simple local factors at all primes $p$. Write $m = p^n m'$, where $m'$ is not divisible by $p$, then $m' \geqslant 3$. From theorem 7.3.1 and corollary 7.2.3, we have

$$\sum_{i=0}^{2}(-1)^i \mathrm{tr}^{ss}(\Phi_{p^r}|H^i(\overline{\mathcal{M}}_{m,\overline{\mathbb{Q}}_p},\overline{\mathbb{Q}}_\ell)) = \frac{1}{2}p^{\frac{1}{2}r} \sum_{\pi\in\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}),1)} m(\pi)\chi(\pi_\infty)\mathrm{tr}^{ss}(\Phi_{p^r}|\sigma_{\pi_p}) \dim \pi_f^{K_m}.$$

We also deduce that

$$\sum_{i\in\{0,2\}}(-1)^i \mathrm{tr}^{ss}(\Phi_{p^r}|H^i(\overline{\mathcal{M}}_{m,\overline{\mathbb{Q}}_p},\overline{\mathbb{Q}}_\ell)) = \frac{1}{2}p^{\frac{1}{2}r} \sum_{\substack{\pi\in\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}),1)\\ \dim\pi_\infty=1}} m(\pi)\chi(\pi_\infty)\mathrm{tr}^{ss}(\Phi_{p^r}|\sigma_{\pi_p}) \dim \pi_f^{K_m}.$$

Indeed, the sum on the right hand side gives nonzero terms only for 1-dimensional representations $\pi$ which are trivial on $K_m$. Using $\chi(\pi_\infty) = 2$, $\dim \pi_f^{K_m} = 1$ and $m(\pi) = 1$, the statement then reduces to class field theory, as the geometric connected components of $\overline{\mathcal{M}}_m$ are parameterized by the primitive $m$-th roots of unity. Notice that we may replace the semi-simple trace by the usual trace on the $I_{\mathbb{Q}_p}$-invariants everywhere, we then have

$$\prod_{i\in\{0,2\}} \det(1-\Phi_p p^{-s}|H^i(\overline{\mathcal{M}}_{m,\overline{\mathbb{Q}}_p},\overline{\mathbb{Q}}_\ell)^{I_{\mathbb{Q}_p}}) = \prod_{\substack{\pi\in\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}),1)\\ \dim\pi_\infty=1}} L(\pi_p, s-\frac{1}{2})^{\frac{1}{2}m(\pi)\chi(\pi_\infty)\dim\pi_f^{K_m}}.$$

Hence by subtracting, we deduce that

$$-\mathrm{tr}^{ss}(\Phi_{p^r}|H^1(\overline{\mathcal{M}}_{m,\overline{\mathbb{Q}}_p},\overline{\mathbb{Q}}_\ell)) = \frac{1}{2}p^{\frac{1}{2}r} \sum_{\substack{\pi\in\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}),1)\\ \dim\pi_\infty>1}} m(\pi)\chi(\pi_\infty)\mathrm{tr}^{ss}(\Phi_{p^r}|\sigma_{\pi_p}) \dim \pi_f^{K_m},$$

or equivalently

$$\det{}^{ss}(1-\Phi_p p^{-s}|H^1(\overline{\mathcal{M}}_{m,\overline{\mathbb{Q}}_p},\overline{\mathbb{Q}}_\ell))^{-1} = \prod_{\substack{\pi\in\Pi_{\mathrm{disc}}(\mathrm{GL}_2(\mathbb{A}),1)\\ \dim\pi_\infty>1}} L(\sigma_{\pi_p}^{ss}, s-\frac{1}{2})^{\frac{1}{2}m(\pi)\chi(\pi_\infty)\dim\pi_f^{K_m}},$$

with the obvious definition for the semi-simple determinant. All zeroes of the left hand side have imaginary part $0, \frac{1}{2}$ or 1: Indeed, if $\overline{\mathcal{M}}_{m,\mathbb{Q}_p}$ had good reduction, Weil conjectures would imply that all zeroes have imaginary part $\frac{1}{2}$. In general, the semistable reduction theorem for curves together with the Rapoport-Zink spectral sequence imply that all zeroes have imaginary part $0, \frac{1}{2}$ or 1. Changing the semi-simple determinant to the usual determinant on the invariants under $I_{\mathbb{Q}_p}$ exactly eliminates the zeroes of imaginary part 1, by the monodromy conjecture, proven in dimension 1 in [28]. We also see that all zeroes of the right hand side have imaginary part $0, \frac{1}{2}$ or 1. Assume $\pi$ gives a nontrivial contribution to the right hand side. Then $\pi_p$ cannot be 1-dimensional, because otherwise $\pi$ and hence $\pi_\infty$ would be 1-dimensional. Hence $\pi_p$ is generic. Being also unitary, the $L$-factor $L(\pi_p, s-\frac{1}{2})$ consists again in removing all zeroes of imaginary part 1. We find that

$$\det(1-\Phi_p p^{-s}|H^1(\overline{\mathcal{M}}_{m,\overline{\mathbb{Q}}_p},\overline{\mathbb{Q}}_\ell)^{I_{\mathbb{Q}_p}})^{-1} = \prod_{\substack{\pi\in\Pi_{\rm disc}(\mathrm{GL}_2(\mathbb{A}),1)\\ \dim\pi_\infty>1}} L(\pi_p, s-\frac{1}{2})^{\frac{1}{2}m(\pi)\chi(\pi_\infty)\dim\pi_f^{K_m}}.$$

Combining the results above yields the result. □

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