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The Order Bicommutant

A study of analogues of the von Neumann Bicommutant Theorem, reflexivity results and Schur's Lemma for operator algebras on Dedekind complete Riesz spaces

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Abstract

It this thesis we investigate whether an analogue of the von Neumann Bicommutant Theorem and related results are valid for Riesz spaces. Let $\mathcal{H}$ be a Hilbert space and $\mathscr{D} \subset \mathcal{L}_b(\mathcal{H})$ a $*$-invariant subset. The bicommutant $\mathscr{D}''$ equals $\mathcal{P}(\mathscr{D}')'$, where $\mathcal{P}(\mathscr{D}')$ denotes the set of projections in $\mathscr{D}'$. Since the sets $\mathscr{D}''$ and $\mathcal{P}(\mathscr{D}')'$ agree, there are multiple possibilities to define an analogue of bicommutant for Riesz spaces. Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. Since the band generated by the projections in $\mathcal{L}_n(E)$ is given by $\mathrm{Orth}(E)$ and order projections in the commutant correspond bijectively to reducing bands, our approach is to define the bicommutant of $\mathscr{A}$ on E by $\mathscr{U} := (\mathscr{A}' \cap \mathrm{Orth}(E))'$.

Our first result is that the bicommutant $\mathscr{U}$ equals $\{T \in \mathcal{L}_n(E) : T$ is reduced by every $\mathscr{A}$-reducing band$\}$. Hence $\mathscr{U}$ is fully characterized by its reducing bands. This is the analogue of the fact that each von Neumann algebra in $\mathcal{L}_b(\mathcal{H})$ is reflexive. This result is based on the following two observations. Firstly, in Riesz spaces there is a one-to-one correspondence between bands and order projections, instead of a one-to-one correspondence between closed subspaces and projections. Secondly, every $*$-invariant subset of $\mathcal{L}_b(\mathcal{H})$ is reduced by each invariant subspace. Therefore, "invariant closed subspaces" is replaced by "reducing bands" in the reflexivity result. Similarly, we obtain Schur's Lemma with "invariant subspaces" replaced by "reducing bands". There is no natural counterpart of the adjoint for Riesz spaces. However, we may define $\mathscr{A} \subset \mathcal{L}_n(E)$ to have the $*$-property, if every $\mathscr{A}$-invariant band is reducing. If $\mathscr{A}$ has the $*$-property, we obtain our classical reflexivity result and Schur's Lemma as known for Hilbert spaces. An instance in which $\mathscr{A}$ has the $*$-property is a subgroup $\mathscr{A}$ of the Riesz automorphisms on E.

Furthermore, we obtain that the bicommutant $\mathscr{U}$ is a unital band algebra. Conversely, if $\mathscr{A}$ is a unital band algebra with the $*$-property and $x \in E$, then, for every operator $T \in \mathscr{U}$, the element Tx is approached in order by a net from $\mathscr{A}x$. If E is atomic, we get approximation in order of each $T \in \mathscr{U}$ by a net of operators from $\mathscr{A}$. Therefore, if E is an atomic Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ has the $*$-property, then $\mathscr{A}$ equals its bicommutant $(\mathscr{A}' \cap \mathrm{Orth}(E))'$ if and only if $\mathscr{A}$ is a unital band algebra. So we retrieve an analogue of the von Neumann Bicommutant Theorem for atomic Riesz spaces. A direct consequence of the von Neumann Bicommutant Theorem is that each von Neumann algebra is the commutant of a group of unitaries. Similarly, the order bicommutant $(\mathscr{A}' \cap \mathrm{Orth}(E))'$ is the commutant of some group of invertible orthomorphisms for every $\mathscr{A} \subset \mathcal{L}_n(E)$. Combining these facts gives that, if E is atomic, then each $\mathscr{A} \subset \mathcal{L}_n(E)$ with the $*$-property is a unital band algebra if and only if $\mathscr{A}$ is the commutant of some group of invertible orthomorphisms.

To obtain the above results, we study operator algebras on Riesz spaces, a subject which is hardly treated in literature at the moment. Moreover, we deal with invariance questions under a set of operators on a Riesz space.

Contents

1 Introduction

1.1 Motivation

A unital strongly closed sub-C^*-algebra of the bounded operators $\mathcal{L}_b(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ was considered by von Neumann considered in [NE]. Later, such an algebra came to be known as a von Neumann algebra. The fundamental theorem of the paper states that a von Neumann algebra is equal to its own bicommutant. Moreover, it states the bicommutant of $*$-invariant subsets $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ is always a von Neumann algebra. Nowadays this theorem is known as the von Neumann Bicommutant Theorem. It is the fundamental result in the study of von Neumann algebras.

Von Neumann's motivation for studying this subject was plurifold. One of the main motivations for his study was the connection with representation theory. Since then representation theory on Hilbert spaces has been well-developed and von Neumann algebras play a key role in this theory. However, positive group representations can be naturally generalized to Banach lattices. This procedure is done in [WO]. The question then arises if there is an analogue of a von Neumann algebra for Banach lattices. Our approach is to address this question for the more general class of Riesz spaces and zoom in on subclasses, if neccesary. A von Neumann algebra occurs as the bicommutant of set of operators. Since the concept of a bicommutant can be defined on Riesz spaces, we can study the existence of analogues of the von Neumann Bicommutant Theorem and related results for Riesz spaces as a first step. This will be the main subject of this thesis.

The key ingredients of the von Neumann Bicommutant Theorem and related results are the orthogonal structure of a Hilbert space, the strong and weak operator topologies and the spectral theorem for normal operators. Fortunately, it is possible to define an orthogonality concept for Riesz spaces comparable to the one on Hilbert spaces. The notion of orthogonal elements on Riesz spaces also leads to projections, which play an important role in the proof of the von Neumann Bicommutant Theorem. Furthermore, we can define order convergence for nets on Riesz spaces with properties similar to those of convergence in the strong operator topology. Finally, with the Freudenthal Spectral Theorem we are able to consider the building blocks of operators on Riesz spaces, as can be done with the spectral theorem for normal operators on Hilbert spaces. This all motivates our study of the bicommutant for operators on Riesz spaces in order to retrieve a similar theory as known for von Neumann algebras on Hilbert spaces.

1.2 Related work

The question whether an analogue of the von Neumann Bicommutant Theorem holds for a set of operators $\mathscr{A}$ on a Banach space X, is a well-known problem. Does the bicommutant of $\mathscr{A}$ coincide with the closure of the algebra generated by $\mathscr{A}$ (and the identity operator) in the strong (or weak) topology? Here we give a summary of work already done in this direction. A complete discussion and a comparison with our results is presented in section 11.

Especially progress has been made, when X is a function space and $\mathscr{A}$ consists of multiplication operators. In [PR] de Pagter and Ricker showed that, if X is an L^p-spaces with $1 \leq p < \infty$, such a bicommutant theorem is valid for any algebra of multiplication operators. In [KI] Kitover investigated the situation $X = C(K)$, where K is a metrizable compact space. In this article necessary and sufficient conditions on K are presented for a bicommutant theorem to be valid. However, there are examples when such a bicommutant theorem does not hold for a set of multiplication operators $\mathscr{A}$. Such an example can be found in [DI].

Another direction in which research on an analogue of the von Neumann Bicommutant Theorem has evolved, is the case where X is a reflexive Banach space. In [DA] Daws proved that under certain conditions on X the weak closure of the range $\mathscr{A}$ of a bounded homomorphism, from a unital Banach algebra into $\mathcal{L}_b(X)$, equals its bicommutant. Furthermore, given a unital Banach algebra $\mathscr{U}$, there exists a reflexive Banach space E and an isometric homomorphism $\mathscr{U} \to \mathcal{L}_b(E)$ such that the range $\mathscr{A}$ equals its own bicommutant.

The projections $\mathcal{P}(X)$ on a Banach space X can be ordered by range inclusion. A last case, when $\mathscr{A} \subset \mathcal{P}(X)$ is a Boolean algebra of projections, is studied by de Pagter and Ricker in [PI]. There are conditions on $\mathscr{A}$ ensuring a bicommutant theorem holds true.

1.3 Questions

To make a distinction between the bicommutant taken in the operators on a Hilbert space or on a Riesz space, we talk about the von Neumann bicommutant, respectively the order bicommutant.

When wondering if a theory on the order bicommutant is fruitful, it is natural to ask if the basic results about the von Neumann bicommutant hold true for the order bicommutant as well. This will be our main subject of study. We restrict ourself to the following questions derived from fundamental results known for the von Neumann bicommutant.

Q1: Description of the bicommutant

The von Neumann bicommutant is a strongly closed unital algebra. Can we derive such a description for the order bicommutant?

Q2: Reflexivity

Using the spectral theorem for normal operators on a Hilbert space, we obtain that the von Neumann bicommutant of a $*$-invariant subset of $\mathcal{L}_b(\mathcal{H})$ is reflexive. This means the von Neumann bicommutant is completely determined by its invariant closed subspaces. Reflexivity can also be defined for sets of operators on Riesz spaces. Is the order bicommutant reflexive? There is no natural counterpart of the adjoint for operators on Riesz spaces. How can this obstruction be solved? Is it a necessary ingredient for proving reflexivity?

Q3: Schur's Lemma

Schur's Lemma is a standard result in representation theory. It states that the commutant of a $*$-closed subset $\mathscr{A}$ of $\mathcal{L}_b(\mathcal{H})$ consists of multiples of the identity if and only if $\mathscr{A}$ leaves only the trivial subspaces invariant. Does Schur's Lemma have an analogue for Riesz spaces? What to do with the $*$-invariance?

Q4: Approximation results

Let $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a unital strongly closed $*$-invariant algebra. By the von Neumann Bicommutant Theorem $\mathscr{A}$ equals its own von Neumann bicommutant. In the proof of the theorem we approximate an operator in the von Neumann bicommutant of $\mathscr{A}$ by a net of operators in $\mathscr{A}$. First this approximation is obtained pointwise. From that we derive global approximation in the strong operator topology by a diagonalization process. Is pointwise and global approximation in order possible for the order bicommutant? Furthermore, does the diagonalization process still work?

Q5: Bicommutant theorem

Using the approximation results for the von Neumann bicommutant we obtain the von Neumann Bicommutant Theorem. Is there a counterpart of this theorem for Riesz spaces? A direct consequence of this theorem is that von Neumann algebras are the commutant of a group of unitaries. If a bicommutant theorem proves to be true, is there an analogue of this consequence?

If $\mathscr{A}$ is a $*$-closed subset of $\mathcal{L}_b(\mathcal{H})$ and $\mathscr{A}'$ denotes its commutant, the von Neumann bicommutant of $\mathscr{A}$ equals the commutant of the projections in $\mathscr{A}'$. Since the notion of projections is also known for Riesz spaces, there are different possibilities for defining the order bicommutant in a Riesz space. It may be possible that those definitions do not coincide when considering operators on Riesz spaces. So, besides the questions mentioned above, we also investigate what is the 'right' analogous definition for the order bicommutant such that most of the structure of the von Neumann bicommutant is preserved.

1.4 Outline and prerequisites

The five questions formulated above are inspired by fundamental results known for the von Neumann bicommutant. Therefore, in section 2 we first treat the five questions for the von Neumann bicommutant such that we can refer to the methods and techniques used here. The reader is presumed to be familiar with operator theory on Hilbert spaces. In particular, we assume familiarity with the strong operator topology and some basic C^*-algebra theory.

In section 3 we give a short overview of the theory of Riesz spaces necessary for understanding the proofs. This overview is intended for the reader unfamiliar with ordered vector spaces and Riesz spaces. Furthermore, we introduce the main examples that will illustrate our results. Operator theory for Riesz spaces is treated in section 4. We start with some basic material, which is present in most of the literature on the subject. From paragraph 4.2 onward we will focus on operator algebras on Riesz spaces, a subject which is hardly treated in the literature at the moment.

In section 5 we give a short overview of the theory of orthomorphisms and in particular projections. We give some necessary results about atomic Riesz spaces in section 6 and about the Freudenthal Spectral Theorem in section 7. In section 8 we consider the commutant. We derive some results, which prepare us for answering the five questions formulated above. However, they should also be considered as interesting in their own right. Finally, in section 9 we treat the five questions for the order bicommutant and we obtain our main results. A summary of the results gathered here, can be found in section 10. As already mentioned, a complete discussion about the literature on the subject and a comparison with our results is presented in section 11.

2 Von Neumann Bicommutant Theorems

2.1 Definition. Let $\mathscr{A}$ be a subset of the bounded operators $\mathcal{L}_b(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. The *commutant* of $\mathscr{A}$ is defined by

$$\mathscr{A}' = \{S \in \mathcal{L}_b(\mathcal{H}) : ST = TS \text{ for all } T \in \mathscr{A}\}.$$

The *von Neumann bicommutant* is given by the set $\mathscr{A}'' := (\mathscr{A}')'$.

The answers to the five questions, as formulated in the introduction, are well-known in the case of the von Neumann bicommutant. In our route of answering these questions for the order bicommutant, we will use a similar approach. Therefore, we fully treat the five questions for the von Neumann bicommutant in this section, in order to refer to the methods and techniques when dealing with the order bicommutant. A more complete discussion on the von Neumann bicommutant, as well as most of the proofs given here, can be found in [CO]. Before we start our discussion, the following property of the von Neumann bicommutant is worthwhile noticing.

2.2 Proposition. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$. Take $\mathscr{U} = \mathscr{A}''$. The commutant $\mathscr{U}'$ coincides with $\mathscr{A}'$. Moreover, $\mathscr{U}$ equals its own von Neumann bicommutant $\mathscr{U}''$.*

Proof. Since $\mathscr{A}$ is obviously contained in $\mathscr{U}$, it follows $\mathscr{U}'$ is contained in $\mathscr{A}'$. Conversely, any operator $S \in \mathscr{A}'$ commutes with all operators in $\mathscr{U}$ and is therefore contained in $\mathscr{U}'$. We conclude $\mathscr{A}' = \mathscr{U}'$. Taking the commutant once again, the second claim $\mathscr{U} = \mathscr{U}''$ follows. $\square$

2.1 Q1: a description of the von Neumann bicommutant

To describe the von Neumann bicommutant, we first consider the commutant.

2.3 Proposition. *Let $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a subset. The commutant $\mathscr{A}'$ is a strongly closed full[1] algebra containing the identity operator I. Furthermore, if $\mathscr{A}$ is closed under taking adjoints, then $\mathscr{A}'$ is also $*$-closed.*

Proof. It is immediate $\mathscr{A}'$ is an algebra containing I. Suppose $S \in \mathscr{A}'$ is invertible. Each $T \in \mathscr{A}$ satisfies $ST = TS$. Applying S^{-1} on both sides of the previous identity yields $TS^{-1} = S^{-1}T$. Therefore, S^{-1} is in $\mathscr{A}'$. Hence $\mathscr{A}'$ is a full algebra. We show $\mathscr{A}'$ is strongly closed. Let $\{S_\lambda\}_\lambda$ be a net in $\mathscr{A}'$ strongly convergent to some $S \in \mathcal{L}_b(\mathcal{H})$. Take $T \in \mathscr{A}$. For each $x \in \mathcal{H}$ we have

$$\|(ST - TS)x\| \leq \|(S - S_\lambda)Tx\| + \|T\|\|(S - S_\lambda)x\| \to 0$$

by strong convergence of $\{S_\lambda\}_\lambda$ to S. It follows T commutes with S for each $T \in \mathscr{A}$ and therefore we have $S \in \mathscr{A}'$. We conclude that $\mathscr{A}'$ is a strongly closed full algebra containing I. Now suppose $\mathscr{A}$ is $*$-closed. Let $S \in \mathscr{A}'$. For all $T \in \mathscr{A}$ the identity $S^*T = (T^*S)^* = (ST^*)^* = TS^*$ holds, since T^* is in $\mathscr{A}$ by assumption. Hence S^* commutes with all $T \in \mathscr{A}$ and therefore S^* is an element of $\mathscr{A}'$. We conclude that $\mathscr{A}'$ is closed under taking adjoints. $\square$

2.4 Corollary. *Let $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a subset. The von Neumann bicommutant $\mathscr{A}''$ is a strongly closed algebra containing the identity operator I.*

[1] That is, if $S \in \mathscr{A}'$ is invertible in $\mathcal{L}_b(\mathcal{H})$, then S^{-1} is contained in $\mathscr{A}'$.

2.2 Q2: reflexivity

2.5 Definition. Let V be a vector space and $A \subset V$ a subset. An operator T on V *leaves A invariant* if $TA \subset A$ holds. In this case A is called *T-invariant.* Furthermore, a subset $\mathscr{A}$ of the linear operators $\mathcal{L}(V)$ on V *leaves A invariant*, if T leaves A invariant for each $T \in \mathscr{A}$. Similarly, A is called *$\mathscr{A}$-invariant* in that case.

2.6 Definition. Let $\mathcal{H}$ be a Hilbert space and $B \subset \mathcal{H}$ a closed subspace. The subspace B *reduces* an operator T on $\mathcal{H}$, if $TB \subset B$ and $TB^{\perp} \subset B^{\perp}$ holds. In this case B is called *T-reducing.* Similarly, B *reduces* $\mathscr{A} \subset \mathcal{L}(\mathcal{H})$, if B reduces T for each $T \in \mathscr{A}$. In that case B is called *$\mathscr{A}$-reducing.*

Reflexive operator algebras are characterized by their invariant subspaces.

2.7 Definition. Let $\mathcal{H}$ be an Hilbert space. A subset $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ is *reflexive*, if it is equal to the algebra of bounded operators which leave invariant each closed subspace, left invariant by $\mathscr{A}$.

We first need some auxiliary statements (which appear to have analogues in the case of the order bicommutant) to obtain the reflexivity result of the von Neumann bicommutant. The notions of invariant and reducing subspaces coincide for subsets of the bounded operators closed under taking adjoints.

2.8 Lemma. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset, then a subspace $B \subset \mathcal{H}$ reduces $\mathscr{A}$ if and only if B is $\mathscr{A}$-invariant.*

Proof. If $B \subset \mathcal{H}$ reduces $\mathscr{A}$, then B is $\mathscr{A}$-invariant. Conversely, suppose $B \subset \mathcal{H}$ is invariant under $\mathscr{A}$. Let $x \in B^{\perp}$ and $T \in \mathscr{A}$. For all $y \in B$ we have $\langle y, Tx\rangle = \langle T^*y, x\rangle = 0$, because T^*y is an element of B, using that $\mathscr{A}$ is $*$-closed and leaves B invariant. It follows $Tx \in B^{\perp}$ for all $x \in B^{\perp}$ and $T \in \mathscr{A}$. So $B^{\perp}$ is also invariant under $\mathscr{A}$. We conclude that B reduces $\mathscr{A}$. $\square$

With the previous we derive an important lemma, which is on the core of most of the von Neumann Bicommutant Theorems.

2.9 Lemma (Projection Lemma)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. A projection $P : \mathcal{H} \to \mathcal{H}$ is in $\mathscr{A}'$ if and only if the closed subspace* $\operatorname{ran}(P)$ *is invariant under $\mathscr{A}$.*

Proof. Suppose P is in the commutant $\mathscr{A}'$. Put $B = \operatorname{ran}(P)$. For all $T \in \mathscr{A}$ we have

$$TB = TPB = PTB \subset B$$

and therefore $\mathscr{A}$ leaves B invariant. Conversely, suppose $B = \operatorname{ran}(P)$ is invariant under $\mathscr{A}$. By 2.8 it follows that $B^{\perp}$ is also invariant under $\mathscr{A}$. For $x \in \mathcal{H}$ we have

$$PTx = PTPx + PT(I - P)x = TPx,$$

since TPx is an element of B and $T(I - P)x$ is in $B^{\perp}$. It follows $P \in \mathscr{A}'$. $\square$

The last lemma, that we need for the reflexivity result, is a consequence of the spectral theorem for normal operators on a Hilbert space.

2.10 Lemma. *Let $\mathcal{H}$ be a Hilbert space. If $\mathscr{A}$ is a strongly closed $*$-invariant algebra of $\mathcal{L}_b(\mathcal{H})$, then $\mathscr{A}$ is the norm closure of the linear span of the set $\mathcal{P}(\mathscr{A})$ of projections in $\mathscr{A}$.*

Proof. [CN, Proposition IX.4.8] □

Our reflexivity result will be a consequence of the following proposition.

2.11 Proposition. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. The von Neumann bicommutant $\mathscr{A}''$ equals $\mathcal{P}(\mathscr{A}')'$.*

Proof. The inclusion $\mathscr{A}'' \subset \mathcal{P}(\mathscr{A}')'$ is trivial, because $\mathcal{P}(\mathscr{A}')$ is contained in $\mathscr{A}'$. For the other inclusion take $R \in \mathcal{P}(\mathscr{A}')'$. By 2.3 $\mathscr{A}'$ is a strongly closed $*$-invariant algebra. Hence $\mathscr{A}' = \overline{\text{span}(\mathcal{P}(\mathscr{A}'))}$ holds by 2.10. Since R commutes with all projections in $\mathscr{A}'$, it is clear R commutes with all linear combinations from $\mathcal{P}(\mathscr{A}')$. So R is in the commutant of $\text{span}(\mathcal{P}(\mathscr{A}'))$. Finally, let $S \in \mathscr{A}'$, then there exists a sequence $S_n \in \text{span}(\mathcal{P}(\mathscr{A}'))$ such that S_n converges to S in norm. We derive

$$\|SR - RS\| \leq \|S - S_n\|\|R\| + \|R\|\|S - S_n\| \to 0.$$

We conclude that R commutes with all $S \in \mathscr{A}'$ and therefore R is in $\mathscr{A}''$. This shows the other inclusion $\mathcal{P}(\mathscr{A}')' \subset \mathscr{A}''$. □

Now, using the projection Lemma 2.9, we are able to obtain the reflexivity result for the von Neumann bicommutant.

2.12 Theorem. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. The von Neumann bicommutant $\mathscr{A}''$ is equal to*

$$\mathscr{A}^{inv} := \{T \in \mathcal{L}_b(\mathcal{H}) : T \textit{ leaves every } \mathscr{A}\textit{-invariant closed subspace invariant}\}.$$

Proof. Suppose $T \in \mathscr{A}''$. Let B be an $\mathscr{A}$-invariant closed subspace and denote by P the projection on B. By 2.9 P is in $\mathscr{A}'$. So P commutes with T. For $x \in B$ we obtain

$$Tx = TPx = PTx \in B.$$

Therefore, B is T-invariant. This yields $T \in \mathscr{A}^{inv}$ and thus $\mathscr{A}''$ is a subset of $\mathscr{A}^{inv}$.

Conversely, let $T \in \mathscr{A}^{inv}$. Let $P \in \mathcal{P}(\mathscr{A}')$ be a projection in $\mathscr{A}'$ and let the closed subspace B be the range of P. Clearly, the projection $I - P$ is an element of $\mathscr{A}'$ by 2.3. Hence $B = \text{ran}(P)$ and $B^{\perp} = \text{ran}(I - P)$ are $\mathscr{A}$-invariant by 2.9. So B and $B^{\perp}$ are also T-invariant. For $x \in \mathcal{H}$ we obtain

$$PTx = PTPx + PT(I - P)x = TPx.$$

Therefore, T commutes with P. We conclude that T commutes with all projections in $\mathscr{A}'$. So T is in $\mathcal{P}(\mathscr{A}')'$. Proposition 2.11 finally yields $T \in \mathscr{A}''$. So $\mathscr{A}^{inv}$ is also a subset of $\mathscr{A}''$ and we conclude $\mathscr{A}^{inv} = A''$. □

2.13 Corollary. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. The von Neumann bicommutant $\mathscr{A}''$ is reflexive.*

Proof. Let $\mathscr{U} = \mathscr{A}''$. By applying 2.3 twice $\mathscr{U}$ is closed under taking adjoints. We have $\mathscr{U} = \mathscr{U}'' = \mathscr{U}^{inv}$ by combining 2.2 and 2.12. So $\mathscr{U}$ is reflexive. □

Combining the last two results, we obtain that all reflexive $*$-invariant subsets of $\mathcal{L}_b(\mathcal{H})$ are von Neumann algebras.

2.14 Corollary. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. Then $\mathscr{A}$ is reflexive if and only if $\mathscr{A}$ equals its von Neumann bicommutant $\mathscr{A}''$.*

2.3 Q3: Schur's Lemma

Schur's Lemma is a classical result in representation theory for Hilbert spaces. It can immediately be derived from the results obtained in the previous paragraph.

2.15 Theorem (Schur's Lemma)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. The following statements are equivalent.*

i. The only closed invariant subspaces for $\mathscr{A}$ are the trivial ones: $\{0\}$ and $\mathcal{H}$.

ii. The commutant $\mathscr{A}'$ consists of multiples of the identity operator $I \in \mathcal{L}_b(\mathcal{H})$.

Proof. Suppose (i) holds. Let $P \in \mathcal{P}(\mathscr{A}')$ be a projection in $\mathscr{A}'$ and $B = \text{ran}(P)$ its range. By the projection Lemma 2.9 B is $\mathscr{A}$-invariant. By assumption B is trivial and hence P is either 0 or I. Now applying 2.10 we conclude that $\mathscr{A}'$ consists of multiples of the identity operator.

Conversely, assume (ii). Take $B \subset \mathcal{H}$ an $\mathscr{A}$-invariant closed subspace. By 2.9 the projection P on B is in $\mathscr{A}'$. Our assumption yields $\mathcal{P}(\mathscr{A}') = \{0, I\}$ and thus P is either 0 or I. We conclude that B is a trivial subspace. So the only closed invariant subspaces for $\mathscr{A}$ are the trivial ones. □

2.4 Q4: approximation results

We approximate operators in the von Neumann bicommutant $\mathscr{A}''$ with operators from $\mathscr{A}$. Since $\mathscr{A}''$ is a unital algebra by 2.4 it is natural to require that $\mathscr{A}$ is also a unital algebra to obtain some approximation results. We take $\mathscr{A}$ closed under taking adjoints to be able to use the projection Lemma 2.9. First we obtain a pointwise approximation result.

2.16 Proposition (Pointwise approximation)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{H}$ be a $*$-closed algebra with $I \in \mathscr{A}$. For all $T \in \mathscr{A}''$ and $x \in \mathcal{H}$ there exists a sequence $\{S_n\}_n$ in $\mathscr{A}$ such that $\{S_n x\}_n$ converges to Tx.*

Proof. Take $T \in \mathscr{A}''$ and $x \in \mathcal{H}$. Since $\mathscr{A}$ is an algebra, $\mathscr{A}x = \{Sx : S \in \mathscr{A}\}$ is a subspace invariant under $\mathscr{A}$. Define the closed subspace $B = \overline{\mathscr{A}x}$. We claim B is still $\mathscr{A}$-invariant. Indeed, let $y \in B$ and let $y_n \in \mathscr{A}x$ be a sequence in $\mathscr{A}x$ converging to y. Take $S \in \mathscr{A}$. Then for all $n \in \mathbb{N}$ we have $Sy_n \in \mathscr{A}x \subset B$, because $\mathscr{A}$ leaves $\mathscr{A}x$ invariant. Since $\mathscr{A}$ is contained in the bounded operators, we have $Sy_n \to Sy$ if $n \to \infty$. It follows $Sy \in B$ by the fact that B is closed. So B is $\mathscr{A}$-invariant.

Let P be the projection on the closed subspace B. By 2.9 P is contained in $\mathscr{A}'$. Therefore, T commutes with P. Using I is contained in $\mathscr{A}$, we derive $x \in \mathscr{A}x \subset B$. Therefore, we derive

$$Tx = TPx = PTx \in B.$$

We conclude that there exists some sequence $\{S_n\}_n$ in $\mathscr{A}$ such that $S_n x \to Tx$. $\square$

By looking at the product of n copies of our Hilbert space $\mathcal{H}$, we obtain global approximation in the strong operator topology.

2.17 Lemma. *Let $n \in \mathbb{N}$ and $\mathcal{H}$ a Hilbert space. Let $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$. Consider the product Hilbert space $\mathcal{H}^n = \{(y_1, \ldots, y_n) : y_i \in \mathcal{H} \text{ for } i = 1, \ldots, n\}$. For $R \in \mathcal{L}_b(\mathcal{H})$ define $R^n \in \mathcal{L}_b(\mathcal{H}^n)$ given by $R^n(y_1, \ldots, y_n) = (Ry_1, \ldots, Ry_n)$. For $\mathscr{U} \subset \mathcal{L}_b(\mathcal{H})$ define $\mathscr{U}_n = \{S^n \in \mathcal{L}_b(\mathcal{H}) : S \in \mathscr{A}\}$. We have the inclusion $(\mathscr{A}'')_n \subset (\mathscr{A}_n)''$.*

Proof. Let $A^n \in (\mathscr{A}'')_n$ for some $A \in \mathscr{A}''$ and $B = [B_{ij}] \in (\mathscr{A}_n)'$. Let $C \in \mathscr{A}$, then B commutes with $C^n \in \mathscr{A}^n$. We obtain the identity

$$[B_{ij}C] = BC^n = C^n B = [CB_{ij}].$$

It follows $B_{ij} \in \mathscr{A}'$ for each i, j and thus A commutes with B_{ij}. We conclude

$$A^n B = [AB_{ij}] = [B_{ij}A] = BA^n.$$

Therefore, A^n is an element of $(\mathscr{A}_n)''$. We have shown $(\mathscr{A}'')_n \subset (\mathscr{A}_n)''$. $\square$

2.18 Proposition (Global approximation)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{H}$ be a $*$-closed algebra with $I \in \mathscr{A}$. For all $T \in \mathscr{A}''$ there exists a net $\{S_\alpha\}_\alpha$ in $\mathscr{A}$ strongly convergent to T.*

Proof. Let $T \in \mathscr{A}''$. Let $\mathscr{U} \subset \mathcal{H}$ be a strongly open neighborhood of T. We show $\mathscr{A} \cap \mathscr{U} \neq \emptyset$. By the properties of the strong operator topology there exists $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{H}$ and $\epsilon_1, \ldots, \epsilon_n > 0$ such that

$$T \in \bigcap_{i=1}^{n} \{S \in \mathcal{L}_b(\mathcal{H}) : \|Sx_i - Tx_i\| < \epsilon_i\} \subset \mathscr{U}.$$

Now consider the diagonal set $\mathscr{A}_n \subset \mathcal{L}_b(\mathcal{H}^n)$ as in 2.17. Let $x = (x_1, \ldots, x_n) \in \mathcal{H}^n$. Observe $\mathscr{A}_n$ is a unital $*$-closed algebra in $\mathcal{L}_b(\mathcal{H}^n)$. By 2.17 the operator T^n is in $(\mathscr{A}'')_n \subset (\mathscr{A}_n)''$. Now, using 2.16, there exists a sequence $\{S_m^n\}$ in $\mathscr{A}_n$ such that $S_m^n x \to T^n x$. This implies $S_m x_i \to Tx_i$ for $i = 1, \ldots, n$. Taking m large enough, we see S_m is in $\mathscr{U}$ by the above identity. It follows $\mathscr{A} \cap \mathscr{U} \neq \emptyset$ for all strongly open neighborhoods $\mathscr{U}$ of T. So T is contained in the strong operator topology closure of $\mathscr{A}$ and therefore there exists a net $\{S_\alpha\}_\alpha$ in $\mathscr{A}$ strongly convergent to T. $\square$

2.5 Q5: the von Neumann Bicommutant Theorem

We are now able to prove the von Neumann Bicommutant Theorem, concerning operator algebras that are equal to their own bicommutant.

2.19 Theorem (von Neumann Bicommutant Theorem)**.** *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a subset closed under taking the adjoint. We have $\mathscr{A}'' = \mathscr{A}$ if and only if $\mathscr{A}$ is a strongly closed algebra containing the identity operator I.*

Proof. Suppose $\mathscr{A} = \mathscr{A}''$. By 2.4 $\mathscr{A}$ is a unital strongly closed algebra. Conversely, suppose $\mathscr{A}$ is a strongly closed algebra containing the identity operator I. The inclusion $\mathscr{A} \subset \mathscr{A}''$ is trivial. Conversely, by 2.18 every $T \in \mathscr{A}''$ is in the strong closure of $\mathscr{A}$. This yields the other inclusion $\mathscr{A}'' \subset \mathscr{A}$. It follows that $\mathscr{A}$ equals $\mathscr{A}''$. $\square$

Combining the above result with 2.14, we obtain the following corollary.

2.20 Corollary. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-invariant subset. Then $\mathscr{A}$ is reflexive if and only if $\mathscr{A}$ is a unital strongly closed algebra.*

The above theorem yields yet another description of the von Neumann bicommutant, as follows.

2.21 Corollary. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. Then $\mathscr{A}''$ equals the strong closure of the algebra* $\mathrm{alg}(\mathscr{A} \cup \{I\})$ *generated by $\mathscr{A} \cup \{I\}$.*

Proof. Define $\mathscr{U}$ to be the strong closure of the unital algebra $\mathscr{D} := \mathrm{alg}(\mathscr{A} \cup \{I\})$. We show $\mathscr{D}$ is $*$-closed. Since I is self-adjoint, we obtain that $\mathscr{A} \cup \{I\}$ is $*$-closed. The algebra $\mathscr{D}$ consists of polynomials in elements of $\mathscr{A} \cup \{I\}$. The adjoint of a monomial $T_1 \dots T_n$ with $T_1, \dots, T_n \in \mathscr{A} \cup \{I\}$ is $T_n^* \dots T_1^*$. This is again a monomial of elements $T_1^*, \dots, T_n^* \in \mathscr{A} \cup \{I\}$. We conclude that $\mathscr{D}$ is also closed under taking adjoints.

Now, take $T \in \mathscr{A}'' \subset \mathscr{D}''$. By 2.18 T is in the strong closure $\mathscr{U}$ of $\mathscr{D}$. This shows one inclusion. For the other inclusion, observe that $\mathscr{A}$ is contained in $\mathscr{A}''$, I is an element of $\mathscr{A}''$ and by 2.4 $\mathscr{A}''$ is an algebra. Therefore, $\mathrm{alg}(\mathscr{A} \cup \{I\})$ is contained in $\mathscr{A}''$. Furthermore, we know by 2.4 that $\mathscr{A}''$ is strongly closed. We conclude that $\mathscr{U}$ is contained in $\mathscr{A}''$. This shows the other inclusion. $\square$

If $\mathscr{A}$ is a $*$-invariant subset of $\mathcal{L}_b(\mathcal{H})$, then by 2.3 the commutant $\mathscr{A}'$ is a full algebra. Therefore, the set of unitaries $U(\mathscr{A}')$ in $\mathscr{A}'$ forms a group. A consequence of the von Neumann Bicommutant Theorem 2.19 is that every von Neumann algebra arises as the commutant of a unitary group.

2.22 Theorem. *Let $\mathcal{H}$ be a Hilbert space and $\mathscr{A} \subset \mathcal{L}_b(\mathcal{H})$ be a $*$-closed subset. Then $\mathscr{A}''$ equals $U(\mathscr{A}')'$. Moreover, $\mathscr{A}$ is a unital strongly closed $*$-invariant algebra if and only if $\mathscr{A}$ is the commutant of some group of unitaries.*

Proof. Since $U(\mathscr{A}')$ is contained in $\mathscr{A}'$, the bicommutant $\mathscr{A}''$ is a subset of $U(\mathscr{A}')'$. Conversely, let $T \in U(\mathscr{A}')'$ and $S \in \mathscr{A}'$. Since $\mathscr{A}'$ is a unital sub-C^*-algebra of $\mathcal{L}_b(\mathcal{H})$ by 2.3, S is a linear combination of unitaries from $\mathscr{A}'$. Therefore, T commutes with S. So T is an element of $\mathscr{A}''$. This yields the other inclusion $\mathscr{U}(\mathscr{A}')' \subset \mathscr{A}''$.

For the second claim, suppose $\mathscr{A}$ is a unital strongly closed $*$-invariant algebra. It follows $\mathscr{A} = \mathscr{A}'' = U(\mathscr{A}')'$ by 2.19. This shows $\mathscr{A}$ is the commutant of the unitary group $U(\mathscr{A}')$. Conversely, suppose $\mathscr{A} = G'$ is the commutant of a group G of unitaries on $\mathcal{H}$. Since G is closed under taking adjoints, 2.3 yields $\mathscr{A} = G'$ is a unital strongly closed $*$-invariant algebra. $\square$

3 Preliminaries about Riesz spaces

Here we present a short overview of the theory of ordered vector spaces and Riesz spaces in particular. This is not meant as a complete discussion about the subject, but should be seen as a treatment of the necessary concepts of the theory to understand the proofs in this thesis. A more comprehensive treatment can be found in [ZA]. In this thesis we assume from now on all vector spaces are real.

3.1 Riesz spaces

3.1 Definition. A vector space E equipped with a partial ordering $\geq$ is said to be an *ordered vector space* if the following properties hold for all $x, y \in E$

i. $x \geq y$ implies $x + z \geq y + z$ for all $z \in E$.

ii. $x \geq y$ implies $\alpha x \geq \alpha y$ for all $\alpha \in \mathbb{R}_{\geq 0}$.

The two properties in the previous definition link the order structure to the algebraic operations on the vector space. Naturally, we write $x \leq y$ as an alternative notation for $y \geq x$. Furthermore, we adopt the interval notation and denote $[x, y] = \{z \in E : x \leq z \leq y\}$. We have the following notions of boundedness.

3.2 Definition. A subset A of an ordered vector space E is *bounded above* if there exists some $x \in E$ with $y \leq x$ for all $y \in A$. Similarly, A is *bounded below* if there exists some $x \in E$ satisfying $y \geq x$ for all $y \in A$. Finally, A is *bounded* if there exists $x, y \in E$ such that A is contained in the interval $[x, y]$.

3.3 Definition. An element x in an ordered vector space E is called *positive* if $x \geq 0$. The *positive cone* E^+ denotes the set of all positive elements in E.

By knowing which elements are positive one can obtain the ordering on an ordered vector space E. Indeed, $x \geq y$ holds if and only if $x - y$ is positive by the first property stated in Definition 3.1. When the order structure on an ordered vector space ensures the existence of suprema and infima of finite subsets we are dealing with Riesz spaces.

3.4 Definition. A *Riesz space* is an ordered vector space E such that for each pair $x, y \in E$ the supremum and infimum of the set $\{x, y\}$ exists. We denote

$$x \vee y := \sup\{x, y\}, \quad x \wedge y := \inf\{x, y\}.$$

Even stronger is the concept of Dedekind completeness, which is the generalization of the well-known supremum property of the real numbers.

3.5 Definition. A Riesz space is *Dedekind complete* whenever every non-empty bounded above subset has a supremum.

Obviously, requiring that every non-empty bounded below subset has an infimum, is equivalent with Dedekind completeness. The utility of Dedekind completeness will become clear later, when considering operators between Riesz spaces. An important class of Dedekind complete Riesz spaces are the L^p-spaces. We introduce some of the main examples.

3.6 Example. Let c_0 be the space of all real-valued sequences converging to 0. Under the ordering $x \leq y$ if $x_i \leq y_i$ for all $i \in \mathbb{N}$, the space c_0 becomes an ordered vector space. Observe c_0 is a Riesz space. Indeed, its lattice operations satisfy $x \vee y = \{x_i \vee y_i\}_{i\in\mathbb{N}}$ and $x \wedge y = \{x_i \wedge y_i\}_{i\in\mathbb{N}}$. One verifies easily that every bounded above subset $A \subset c_0$ has a supremum $x \in c_0$ defined by $x_i = \sup_{y\in A} y_i$. It follows c_0 is a Dedekind complete Riesz space. ■

3.7 Example. Let X be a non-empty set and E a Dedekind complete Riesz space (for example one could take $E = \mathbb{R}$). The space E^X of functions $f : X \to E$ is an ordered vector space under the ordering $f \geq g$ if $f(x) \geq g(x)$ in E for all $x \in X$. Given $f, g \in E^X$ one checks the supremum $f \vee g$ of $\{f, g\}$ is given by $(f \vee g)(x) = f(x) \vee g(x)$ and the infimum $f \wedge g$ is given by $(f \wedge g)(x) = f(x) \wedge g(x)$. This shows E^X is a Riesz space. Moreover, let $A \subset E^X$ be non-empty and bounded above by some function $g \in E^X$. All $f \in A$ satisfy $f(x) \leq g(x)$ for $x \in X$. Since E is Dedekind complete and $\{f(x) : f \in A\} \subset E$ is bounded above by $g(x)$ for all $x \in X$, the function $h \in E^X$ given by $h(x) = \sup_{f\in A} f(x)$ is well-defined and satisfies $h = \sup A$. It follows E^X is also Dedekind complete. ■

3.8 Example. Let c be the space of all convergent real-valued sequences. Under the ordering $x \leq y$ if $x_i \leq y_i$ for all $i \in \mathbb{N}$, the space c becomes an ordered vector space. Observe c is a Riesz space. Indeed, its lattice operations satisfy $x \vee y = \{x_i \vee y_i\}_{i\in\mathbb{N}}$ and $x \wedge y = \{x_i \wedge y_i\}_{i\in\mathbb{N}}$. However, c is not Dedekind complete. To see this, let $e_n \in c$ be the positive sequence whose n-th component is one and every other is zero. Indeed, the set

$$A = \{\sum_{i=1}^{n} (-1)^i e_i : n \in \mathbb{N}\} \subset c,$$

bounded above by the constant sequence whose coordinates are one, has no supremum in c. ■

3.9 Example. Let $C^\infty(\mathbb{R})$ the space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the ordering $f \geq g$ if $f(x) \geq g(x)$ for all $x \in \mathbb{R}$. Let $f, g \in C^\infty(\mathbb{R})$ be given by $f(x) = x$ and $g(x) = -x$. Observe that $\{f, g\}$ has no supremum in $C^\infty(\mathbb{R})$. Therefore, $C^\infty(\mathbb{R})$ is an ordered vector space, but not a Riesz space. ■

Since we are able to take the supremum of two elements in a Riesz space, one can obtain a decomposition in a positive and negative part. This allows us also to define an absolute value. With these notions we retrieve lattice identities that are well-known for the real numbers.

3.10 Definition. Let x be an element of a Riesz space E. Define

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0, \quad |x| := x \vee (-x).$$

x^+ is called the *positive part*, x^- the *negative part* and $|x|$ the *absolute value* of x.

3.11 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. All $f \in E^X$ satisfy $|f|(x) = |f(x)|$ for all $x \in X$. ■

3.12 Proposition. *Let x be an element of a Riesz space E, then we have*

i. $x = x^+ - x^-$*;*

ii. $|x| = x^+ + x^-$*;*

iii. $x^+ \wedge x^- = 0$*;*

iv. $|x| = 0$ *if and only if* $x = 0$*;*

v. $|\lambda x| = |\lambda||x|$ *for* $\lambda \in \mathbb{R}$*.*

Moreover, for $x, y \in E$ we have

vi. $x \vee y = \frac{1}{2}(x + y + |x - y|)$;

vii. $x \wedge y = \frac{1}{2}(x + y - |x - y|)$.

Proof. [AL, Theorems 1.5 and 1.7] and [ZA, Theorems 11.4 and 11.7] □

Furthermore, we retrieve the triangle inequality, which will be used extensively in estimations.

3.13 Proposition. *For elements x and y in a Riesz space, we have*

$$|x + y| \leq |x| + |y|.$$

Proof. [AL, Theorem 1.9] □

Often, when considering mathematical objects, one is interested in structure preserving maps. For Riesz spaces we have the notion of Riesz homomorphisms.

3.14 Definition. Let E and F Riesz spaces. A linear map $T : E \to F$ is a *Riesz homomorphism* if

$$T(x \vee y) = Tx \vee Ty$$

holds for all $x, y \in E$.

The fact that T respects the lattice operation $\vee$ implies that T respects all other lattices operations.

3.15 Proposition. *Let E and F be Riesz spaces and $T : E \to F$ a Riesz homomorphism. For all $x, y \in E$ we have*

$$T(x \wedge y) = Tx \wedge Ty, \quad |Tx| = T|x|, \quad T(x^+) = T(x)^+, \quad T(x^-) = T(x)^-.$$

Proof. [AL, Theorem 2.14] □

3.16 Definition. Let E and F Riesz spaces. A map $T : E \to F$ is a *Riesz isomorphism*, if T is a bijective Riesz homomorphism. If such a map T exists the spaces E and F are *order isomorphic*. Finally, if E equals F, we call T a *Riesz automorphism*.

It is straightforward to check, that if $T : E \to F$ is a Riesz isomorphism, then $T^{-1} : F \to E$ is also a Riesz isomorphism. Therefore, the above definition is symmetric.

3.17 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. The linear maps $T_h : E^X \to E^X$ defined by $T_h f = f \circ h$ with $h : X \to X$ a function are Riesz homomorphisms. If h is a bijection, then the map T_h is a Riesz automorphism with inverse $T_h^{-1} = T_{h^{-1}}$. ■

3.2 Order convergence

For Riesz spaces there is a natural concept of convergence induced by the order structure.

3.18 Definition. A net $\{x_\alpha\}_\alpha$ in a Riesz space E is *decreasing* to an element $x \in E$ if $\alpha \succeq \beta$ implies that $x_\alpha \leq x_\beta$ and $\inf_\alpha x_\alpha = x$ both hold. We write $x_\alpha \downarrow x$. Similarly, $\{x_\alpha\}_\alpha$ is *increasing*, if $\alpha \succeq \beta$ implies that $x_\alpha \geq x_\beta$ and $\sup_\beta x_\beta = x$ both hold. We write $x_\alpha \uparrow x$.

One observes immediately that $x_\alpha \uparrow x$ implies $x - x_\alpha \downarrow 0$. Using the notions of increasing and decreasing nets, we can define the concept of order convergence.

3.19 Definition. A net $\{x_\alpha\}_\alpha$ in a Riesz space is *order convergent* to $x \in E$, if there exists a net $\{y_\alpha\}_\alpha$ with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \leq y_\alpha$ for all α. The element $x \in E$ is called the *order limit* of $\{x_\alpha\}_\alpha$. We write $x_\alpha \xrightarrow{o} x$. A subset $A \subset E$ is *order closed*, if order convergence of a net $\{x_\alpha\}_\alpha$ in A to x implies $x \in A$. Finally, a subset $A \subset E$ is *σ-order closed*, if order convergence of a sequence $\{x_n\}_n$ in A to x implies $x \in A$.

3.20 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. A net $\{f_\alpha\}_\alpha$ in E^X converges in order to an element $f \in E^X$ if and only if $f_\alpha(x) \xrightarrow{o} f(x)$ for each $x \in X$. ■

Clearly order limits, when they exists, are unique. Further, observe that a net converges in order to x, if the net decreases or increases to x. We remark that order convergence can be identified with convergence of nets in a certain topology: the order topology. Since we only need the notion of order convergence, we will not go into detail about the order topology. The interested reader is referred to [ZA].

The following propositions shows that order convergence is compatible with the lattice structure.

3.21 Proposition. *For two nets $\{x_\alpha\}_\alpha$ and $\{y_\beta\}_\beta$ in a Riesz space, satisfying $x_\alpha \xrightarrow{o} x$ and $y_\beta \xrightarrow{o} y$, we have*

i. $\lambda x_\alpha + \mu y_\beta \xrightarrow{o}_{\alpha,\beta} \lambda x + \mu y$ *for all* $\lambda, \mu \in \mathbb{R}$*;*

ii. $|x_\alpha| \xrightarrow{o} |x|$*;*

iii. $x_\alpha \vee y_\beta \xrightarrow{o}_{\alpha,\beta} x \vee y$*;*

iv. $x_\alpha \wedge y_\beta \xrightarrow{o}_{\alpha,\beta} x \wedge y$*.*

Proof. [AB, Theorem 1.6] □

For the real numbers the Archimedean property states that for $x \in \mathbb{R}_{\neq 0}$ the sequence $\{nx\}_{n\in\mathbb{N}}$ is unbounded in $\mathbb{R}$. For Riesz spaces we also have such a notion.

3.22 Definition. A Riesz space E is called *Archimedean*, if for each $x \in E^+$ the sequence $\{\frac{1}{n}x\}_{n\in\mathbb{N}}$ decreases to 0.

3.23 Example. Consider $\mathbb{R}^2$ endowed with the lexicographical ordering. That is $(x_1, y_1) \leq (x_2, y_2)$ if either $x_1 < x_2$ or else $x_1 = x_2$ and $y_1 \leq y_2$. With this ordering $\mathbb{R}^2$ is a Riesz space. For each $n \in \mathbb{N}$ we have $(0, 1) \leq \frac{1}{n}(1, 0)$. Therefore, $\mathbb{R}^2$ is not Archimedean. ■

Most spaces we deal with, such as the function spaces and the L_p-spaces, are Archimedean. If a Riesz space is Dedekind complete, we are ensured that it is Archimedean.

3.24 Proposition. *If a Riesz space E is Dedekind complete, then it is Archimedean.*

Proof. [AO, Lemma 8.4] □

3.25 Example. Consider the space c of convergent sequences from Example 3.8, which is not Dedekind complete. For $x \in c^+$ we have $\frac{1}{n}x_i \downarrow 0$ if $n \to \infty$. This implies $\frac{1}{n}x \downarrow 0$ in c. So c is Archimedean, but not Dedekind complete. ■

The definition of Archimedean is independent of choice of the decreasing sequence.

3.26 Proposition. *A Riesz space E is Archimedean if and only if the sequence $\{\epsilon_n x\}_n$ decreases to 0 for each $x \in E^+$ and every sequence $\{\epsilon_n\}_n$ of real numbers satisfying $\epsilon_n \downarrow 0$.*

Proof. [ZA, Theorem 22.2] □

When proving Schur's Lemma for Riesz spaces, we need a corollary of the above result, which is not a standard result present in the literature. Therefore, we provide a proof.

3.27 Corollary. *Let E be an Archimedean Riesz space and $x \in E$. The linear span of x is σ-order closed.*

Proof. If $x = 0$ the claim is trivial. Therefore, we assume $x \neq 0$. Suppose $y \in E$ is the order limit of some sequence $\{\lambda_n x\}_n$ with $\lambda_n \in \mathbb{R}$ in the linear span of x. By 3.21 we have $|\lambda_n x - \lambda_m x| \xrightarrow{o}_{m,n} |y - y| = 0$. Now suppose $\{\lambda_n\}_n$ is not a Cauchy sequence in $\mathbb{R}$. Then there exists $\epsilon > 0$ and a subsequence $\{\lambda_{n_k}\}_k$ such that $|\lambda_{n_k} - \lambda_{n_l}| \geq \epsilon$ for all $k, l \in \mathbb{N}$. Combining the previous two lines, we obtain

$$0 \leq \epsilon|x| \leq |\lambda_{n_k} - \lambda_{n_l}||x| = |\lambda_{n_k}x - \lambda_{n_l}x| \xrightarrow{o}_{k,l} 0$$

with the aid of 3.12. This implies $\epsilon|x| = 0$. A contradiction with the fact that $x \neq 0$ using 3.12. So $\{\lambda_n\}_n$ is a Cauchy sequence and hence convergent to some $\lambda \in \mathbb{R}$. By 3.12, 3.13, 3.21 and 3.26 it holds

$$0 \leq |\lambda x - y| \leq |\lambda_n - \lambda||x| + |\lambda_n x - y| \xrightarrow{o} 0$$

and therefore y equals λx. □

3.3 Orthogonality in Riesz spaces

Using the absolute value we can introduce an orthogonality concept, which will be of critical importance in obtaining results for the order bicommutant.

3.28 Definition. Two elements x and y in a Riesz space E are *orthogonal* if

$$|x| \wedge |y| = 0.$$

We write $x \perp y$. Furthermore, if $A \subset E$ is non-empty, the set

$$A^{\perp} = \{x \in E : x \perp y \text{ for all } y \in A\}$$

is called the *orthogonal complement* of A.[2]

3.29 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. Then we have $f \perp g$ if and only if $f(x) \perp g(x)$ for all $x \in X$.

Now taking $E = \mathbb{R}$, we derive $f \perp g$ if and only if f and g have disjoint support. If $A \subset \mathbb{R}^X$ is a subset and $Y = \{x \in X : f(x) = 0 \text{ for all } f \in A\}$, then

$$A^{\perp} = \{g \in \mathbb{R}^X : g(x) = 0 \text{ for all } x \in X \setminus Y\}$$

is the orthogonal complement of A. ■

For positive elements the sum is always larger or equal to the maximum. When the involved elements are orthogonal, they coincide.

3.30 Proposition. *For positive orthogonal elements x, y of a Riesz space we have*

$$x + y = x \vee y.$$

Proof. This follows directly from the last two identities of 3.12, see [ZA, Theorem 14.4]. □

Every positive element in an Archimedean Riesz space can be approached from below by a maximal orthogonal system.

3.31 Definition. Let E be a Riesz space. A subset $S \subset E^+$ is an orthogonal system, if $0 \notin S$ and $u \perp v$ for all $u, v \in S$ with $u \neq v$.

Using Zorn's Lemma one derives that every Riesz space has a maximal orthogonal system. We state our approximation result.

3.32 Proposition. *Let E be an Archimedean Riesz space and $S \subset E^+$ a maximal orthogonal system. Let $x \in E$ be positive and define*

$$x_{n,H} = \sum_{u \in H} x \wedge nu$$

for $n \in \mathbb{N}$ and $H \subset S$ finite. We have $x_{n,H} \uparrow_{n,H} x$.

Proof. [SC, Proposition II.1.9] □

[2] Some authors call two orthogonal elements 'disjoint' and talk about the 'disjoint complement' instead of the 'orthogonal complement'. In order to stress the resemblance with the orthogonality concept on Hilbert spaces we decided to use 'orthogonal'.

3.4 Riesz subspaces, ideals and bands

3.33 Definition. A linear subspace A contained in a Riesz space E is a *Riesz subspace*, if A is closed under the lattice operations. That is $x \vee y, x \wedge y \in A$ for all $x, y \in A$.

Riesz subspaces are the natural subsets, closed under the lattice operations, to consider. However, to obtain a rich theory, which has a good interplay with the orthogonality concept, it turns out that one needs ideals and bands.

3.34 Definition. A linear subspace A contained in a Riesz space E is an *ideal*, if $|x| \le |y|$ and $y \in A$ implies $x \in A$. An order closed ideal is said to be a *band*.

3.35 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7 with $E = \mathbb{R}$ and X an infinite set. For an element $f \in \mathbb{R}^X$ we have $|f|(x) = |f(x)|$ for $x \in X$. So, if $f, g \in \mathbb{R}^X$ satisfy $|f| \le |g|$, then $|f(x)| \le |g(x)|$ holds for each $x \in X$. Now, let $p \in [1, \infty)$ and consider the subspace

$$\ell^p(X) = \{f \in \mathbb{R}^X : \sum_{x \in X} |f(x)|^p \text{ exists and is finite}\}$$

of $\mathbb{R}^X$ of p-summable functions. Note that the existence of $\sum_{x \in X} |f(x)|^p$ for $f \in \mathbb{R}^X$ a priori requires that the set $\{x \in X : f(x) \neq 0\}$ is countable. The space $\ell^p(X)$ is a Dedekind complete Riesz space. We show $\ell^p(X)$ is an ideal in $\mathbb{R}^X$, but not a band.

Let $g \in \ell^p(X)$ and $f \in \mathbb{R}^X$ such that $|f| \le |g|$. This immediately implies f is p-summable with $\sum_{x \in X} |f(x)|^p \le \sum_{x \in X} |g(x)|^p < \infty$. It follows $f \in \ell^p(X)$ and therefore $\ell^p(X)$ is an ideal of $\mathbb{R}^X$. On the other hand, for each finite subset $J \subset X$ the element f_J, given by $f_J(x) = 1$ if $x \in J$ and $f_J(x) = 0$ for $x \in X \setminus J$, is an element of $\ell^p(X)$. Let $g \in \mathbb{R}^X$ be the constant function one. Clearly g is not in $\ell^p(X)$. However, $\{f_J\}_J$ increases to g in $\mathbb{R}^X$. This shows $\ell^p(X)$ is not a band in $\mathbb{R}^X$. ■

One can give a complete description of the bands in the space E^X.

3.36 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. Let $A \subset E^X$ be a band. Fix $x \in X$. We show the set $B_x = \{f(x) : f \in A\}$ is a band in E. Define for $z \in X$ and $w \in E$ the function $g_{z,w} \in E^X$ by $g_{z,w}(u) = 0$ for $u \neq z$ and $g_{z,w}(z) = w$. Suppose $|y| \le |f(x)| = |f|(x)$ holds for some $y \in E$ and $f \in A$. We have $|g_{x,y}| \le |f|$ and therefore $g_{x,y}$ is in A. We conclude $g_{x,y}(x) = y \in B_x$. Hence B_x is an ideal.

Furthermore, suppose $f_\alpha(x) \xrightarrow{o} y$ holds for some $y \in E$ and a net $\{f_\alpha\}_\alpha$ in A. Define $y_\alpha := f_\alpha(x)$. We have $g_{x,y_\alpha} \xrightarrow{o} g_{x,y}$ by Example 3.20, since this convergence holds pointwise. Moreover, we have $|g_{x,y_\alpha}| \le |f_\alpha|$ and therefore $g_{x,y_\alpha} \in A$ for every α. Since A is a band, $g_{x,y}$ is an element of A and therefore $g_{x,y}(x) = y$ is an element of B_x. We conclude that B_x is band.

Hence for each $x \in X$ there exists bands $B_x \subset E$ such that

$$A = \{f \in E^X : f(x) \in B_x \text{ for all } x \in X\}.$$

Conversely, it is a routine check that, if B_x is a band in E for each $x \in X$, then $A = \{f \in E^X : f(x) \in B_x$ for all $x \in X\}$ is a band. So all bands are of the above form. Moreover, we have $f \perp g$ if and only if $f(x) \perp g(x)$ for all $x \in X$ by 3.29. Therefore, it is a straightforward check that

$$A^\perp = \{f \in E^X : f(x) \in B_x^\perp \text{ for all } x \in X\}.$$

Finally, note in $\mathbb{R}$ there are only two bands: $\{0\}$ and $\mathbb{R}$. It follows every band in $\mathbb{R}^X$ is of the form $B_Y = \{g \in \mathbb{R}^X : g(x) = 0$ for all $x \in Y\}$ for some $Y \subset \mathbb{R}$. It is a similar check that also in $\ell^p(X) \subset \mathbb{R}^X$ (see Example 3.35) all bands are of the form B_Y for some $Y \subset \mathbb{R}$. ■

Note that ideals (and therefore bands) are closed under the lattice operations $\vee$ and $\wedge$. So ideals are Riesz subspaces. Moreover, an intersection of ideals is again an ideal. The same holds for bands. In this thesis we often consider ideals and bands generated by a certain subset S of a Riesz space.

3.37 Definition. Let E be a Riesz space and $A \subset E$. The *ideal $\mathcal{E}(A)$ generated* by A is the smallest ideal with respect to the inclusion that contains A. Similarly, the *band $\mathcal{B}(A)$ generated* by A is the smallest band with respect to the inclusion that contains A. If A consists of one element $x \in E$, we write $\mathcal{E}(x)$ and $\mathcal{B}(x)$ for the ideal respectively the band generated by A.

3.38 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7 with $E = \mathbb{R}$. Let $f \in \mathbb{R}^X$. Let $Y = \{x \in X : f(x) = 0\}$. The band $\mathcal{B}(f)$ generated by f is given by $B_Y = \{g \in \mathbb{R}^X : g(x) = 0$ for all $x \in Y\}$. ■

A moment's thought reveals $\mathcal{E}(A)$ is the intersection of all ideals containing A. Similarly, $\mathcal{B}(A)$ is the intersection of all bands containing A. Furthermore, we have $\mathcal{B}(A) = \mathcal{B}(\mathcal{E}(A))$. There are however more convenient descriptions of $\mathcal{E}(A)$ and $\mathcal{B}(A)$ in terms of A.

3.39 Proposition. *Let E be a Riesz space and $A \subset E$. The ideal generated by A is given by*

$$\mathcal{E}(A) = \{x \in E : \textit{there exists } x_1, \dots, x_n \in A \textit{ and } \lambda \in \mathbb{R}_{\geq 0} \textit{ with } |x| \leq \lambda \sum_{i=1}^{n} |x_i|\}.$$

Moreover, if A is an ideal in E, the band generated by A is given by

$$\mathcal{B}(A) = \{x \in E : \textit{there exists a net } \{x_\alpha\}_\alpha \textit{ in } A \textit{ with } 0 \leq x_\alpha \uparrow |x|\}.$$

Proof. [AL, Theorem 1.38] □

For a non-empty subset A contained in a Riesz space E, $A^\perp$ is always a band. Moreover, this gives yet another important description of a band generated by a set.

3.40 Proposition. *Let E be an Archimedean Riesz space and $A \subset E$. The band $\mathcal{B}(A)$ generated by A is precisely $A^{\perp\perp} := (A^\perp)^\perp$.*

Proof. [AL, Theorem 1.39] □

Proposition 3.40 gives rise to an important decomposition of a Riesz space.

3.41 Proposition. *If B is a band in a Dedekind complete Riesz space E, then $E = B \oplus B^\perp$ holds. Moreover, for a non-empty subset A of E we have $E = A^\perp \oplus A^{\perp\perp}$.*

Proof. [AL, Theorem 1.42] □

3.42 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7 with $E = \mathbb{R}$. Let $\chi_Y \in \mathbb{R}^X$ be the characteristic function of a subset $Y \subset X$. All bands in $\mathbb{R}^X$ are of the form B_Y for some $Y \subset X$ following Example 3.36. Fix $Y \subset X$. Observe we have $\chi_{X\setminus Y} f \in B_Y$ and $\chi_Y f \in B_{X\setminus Y} = B_Y^{\perp}$ for $f \in E$. Therefore, each function $f \in \mathbb{R}^X$ can be uniquely decomposed as $f = \chi_{X\setminus Y} f + \chi_Y f$ with $\chi_{X\setminus Y} f \in B_Y$ and $\chi_Y f \in B_Y^{\perp}$. ■

Finally, we make the following definition for later purposes.

3.43 Definition. A subset G of a Riesz space E is called *absolutely self-majorizing* if for each $x \in G$ there exists $y \in G$ such that $|x| \leq y$.

Subsets G that are closed under taking the absolute value are absolutely self-majorizing. Hence in particular, Riesz subspaces, bands and ideals are absolutely self-majorizing.

4 Operators on Riesz spaces

In this section we treat the operator theory needed in this thesis. We start with some basic material, which is present in most of the literature on the subject. From paragraph 4.2 onward we will focus on operator algebras on Riesz spaces, a subject which is hardly treated in literature at the moment.

4.1 Basic operator theory for Riesz spaces

Here we treat some of the basics about operator theory. These results are well-known and a more thorough discussion can be found in [AL]. Let E and F be vector spaces. With $\mathcal{L}(E,F)$ we denote the vector space of operators from E to F. With $\mathcal{L}(E)$ we denote the vector space of operators on E. If E is a Riesz space, then an operator $T : E \to F$ is determined by its action on E^+, because for all $x \in E$ we have $Tx = Tx^+ - Tx^-$ by 3.12. When E and F are ordered vector spaces, it is possible to define an ordering on $\mathcal{L}(E,F)$.

4.1 Definition. Let E and F be ordered vector spaces. An operator $T : E \to F$ is *positive*, if $Tx \geq 0$ holds for all $x \in E^+$.

4.2 Example. Let E and F Riesz spaces and $T : E \to F$ a Riesz homomorphism. For $x \in E^+$ we have $Tx = T[x^+] = [Tx]^+ \geq 0$ by 3.15. Therefore, Riesz homomorphisms are positive operators.

4.3 Proposition. *Let E and F be ordered vector spaces. For $S, T \in \mathcal{L}(E,F)$ define $S \leq T$ if $T - S$ is positive. With this partial ordering $\mathcal{L}(E,F)$ is an ordered vector space.*

For general Riesz spaces E and F the space $\mathcal{L}(E,F)$ need not be a Riesz space. To achieve a Riesz space we consider the subspace of order bounded operators and take F Dedekind complete.

4.4 Definition. Let E and F be Riesz spaces. An operator $T : E \to F$ is *order bounded*, if it maps bounded subsets of E to bounded subsets of F. The vector space of all order bounded operators from E to F is denoted by $\mathcal{L}_b(E,F)$.

For positive operators T between Riesz spaces E and F, we have $T[x,y] \subset [Tx, Ty]$ for $x, y \in E$ with $x \leq y$. Thus, every positive operator T is order bounded. In 4.8 we consider an example of an operator that is not order bounded. We are now able to describe the Riesz space $\mathcal{L}_b(E,F)$ of order bounded operators.

4.5 Theorem. *Let E and F be Riesz spaces with F Dedekind complete. Then $\mathcal{L}_b(E,F)$ is a Dedekind complete Riesz space. Moreover, its lattice operations satisfy for all $S, T \in \mathcal{L}_b(E,F)$ and $x \in E^+$:*

$$\begin{aligned}
|T|x &= \sup\{|Ty| : |y| \leq x\};\\
(S \vee T)x &= \sup\{Sy + Tz : y, z \in E^+ \text{ and } y + z = x\};\\
(S \wedge T)x &= \inf\{Sy + Tz : y, z \in E^+ \text{ and } y + z = x\}.
\end{aligned}$$

Proof. [AL, Theorem 1.18] □

In Example 4.8 we show $\mathcal{L}(E, F)$ need not be a Riesz space for F Dedekind complete and in Example 4.7 we show $\mathcal{L}_b(E, F)$ need not be a Riesz space, if F is not Dedekind complete. This justifies our assumptions in Theorem 4.5. The following inequality is of great importance in approximations involving operators.

4.6 Proposition. *Let E and F be Riesz spaces. For an operator $T : E \to F$ for which $|T|$ exists one has*

$$|Tx| \leq |T||x|$$

for all $x \in E$.

Proof. By definition we have $\pm T \leq |T|$. Let $x \in X$, we obtain by 3.12

$$\pm Tx = \pm Tx^+ \mp Tx^- \leq |T|x^+ + |T|x^- = |T||x|.$$

We conclude $|Tx| \leq |T||x|$. □

Using this estimation result we are now able to give two counterexamples justifying the assumption of order boundedness and Dedekind completeness in 4.5.

4.7 Example. This example is based on [AL, Example 1.17]. Let c be the Riesz space of convergent sequences from Example 3.8. We show $\mathcal{L}_b(c)$ is not a Riesz space. Consider the positive operators $S, T : c \to c$ defined by

$$Sx = (x_2, x_1, x_4, x_3, \ldots), \quad Tx = (x_1, x_1, x_3, x_3, \ldots).$$

Take $R = S - T$, then R is order bounded as difference of two positive operators. We show $|R|$ does not exist. For $n \in \mathbb{N}$ define the positive operator $P_n : c \to c$ by

$$P_n x = (x_1, \ldots, x_{n-1}, 0, x_{n+1}, \ldots).$$

Observe that every sequence in the range of $R \in \mathcal{L}_b(c)$ has its even coordinates zero. Therefore, $P_{2n}R = R$ holds for each $n \in \mathbb{N}$. For positive $x \in c$ we have $\pm Rx \leq |R|x$ by definition. Combining the last two lines we derive

$$\pm Rx = \pm P_{2n}Rx \leq P_{2n}|R|x \leq |R|x,$$

using P_{2n} is a positive operator satisfying $P_{2n}y \leq y$ for each $y \in c^+$. We conclude $\pm R \leq P_{2n}|R| \leq |R|$ and thus $|R| = P_{2n}|R|$ holds for each $n \in \mathbb{N}$. We infer each sequence in the range of $|R|$ has its even coordinates zero.

For $n \in \mathbb{N}$, let $e_n \in c$ be the sequence whose n-th coordinate is one and every other zero. Then the n-th coordinate from $-Re_n$ is one. Consider the constant sequence $e \in c$ with ones on all entries. We have $e_n \leq e$. From the inequalities $-Re_n \leq |R|e_n \leq |R|e$ for each $n \in \mathbb{N}$, we derive that the odd coordinates of $|R|e$ are greater or equal to one. Hence, it is impossible for $|R|e$ to converge, noting all even coordinates are zero. We derive $|R|$ can not exist. So $\mathcal{L}_b(c)$ is not a Riesz space. This has to do with the fact that c is not Dedekind complete. ■

4.8 Example. This example is based on [AL, Example 4.73]. Let $C[0, 1]$ the space of continuous functions $f : [0, 1] \to \mathbb{R}$ with the ordering $f \geq g$ if $f(x) \geq g(x)$ for all $x \in [0, 1]$. Since the functions $x \mapsto f(x) \wedge g(x)$ and $x \mapsto f(x) \vee g(x)$ are continuous for $f, g \in E$, it follows that E is a Riesz space.

Consider also the Dedekind complete Riesz space c_0 from Example 3.6. Define the operator $T: C[0,1] \to c_0$ by
$$Tf = \{f(1/i) - f(0)\}_{i\in\mathbb{N}}.$$
The codomain of T is well-defined, because $\lim_{i\to\infty} f(1/i) = f(0)$ holds by continuity of f. We show T is not order bounded. Assume by way of contradiction that T is order bounded. Let $\mathbf{1} \in C[0,1]$ be the constant function one. The interval $[0,\mathbf{1}]$ is bounded, hence there exists $u \in c_0$ such that $|Tf| \le u$ for $f \in [0,\mathbf{1}]$. For each $n \in \mathbb{N}$ choose continuous functions f_n in $[0,\mathbf{1}]$ with $f_n(0) = 0$ and $f_n(x) = 1$ for $x \in [1/n, 1]$. We have $|Tf_i|_i = |f_i(1/i) - f_i(0)| = 1 \le u_i$ for each $i \in \mathbb{N}$. A contradiction with the fact that u converges to 0. It follows T is not order bounded.

Now suppose $|T|$ exists, then for all $f \in [0,\mathbf{1}]$ the inequalities $|Tf| \le |T|f \le |T|\mathbf{1}$ hold by 4.6. Therefore, $\{Tf : f \in [0,\mathbf{1}]\}$ is bounded, which is in contradiction with what we showed before. It follows $|T|$ does not exists. Therefore, $\mathcal{L}(C[0,1], c_0)$ is not a Riesz space, even despite the fact that c_0 is Dedekind complete. ∎

We shift our scope to order convergence in the Riesz space $\mathcal{L}_b(E,F)$. The following example shows that pointwise order convergence of operators does not imply order convergence of the operators itself.

4.9 Example. Consider the Dedekind complete Riesz space c_0 from Example 3.6. For each $n \in \mathbb{N}$ consider the sequence of order bounded positive functionals $\phi_n : c_0 \to \mathbb{R}$ given by $\phi_n(x) = x_n$. For $x \in c_0$ we have by definition $\phi_n(x) = x_n \to 0$ if $n \to \infty$. Now suppose $\phi_n \xrightarrow{o} 0$ holds in $\mathcal{L}_b(c_0, \mathbb{R})$. By definition there exists a sequence $\{\psi_n\}_n$ in $\mathcal{L}_b(c_0,\mathbb{R})$ such that $\phi_n \le \psi_n \downarrow 0$ if $n \to \infty$. This implies $\psi_1 \ge \phi_n$ for all $n \in \mathbb{N}$ and thus $\{\phi_n : n \in \mathbb{N}\}$ is bounded above by ψ_1. We show $\{\phi_n : n \in \mathbb{N}\}$ is not bounded above in $\mathcal{L}_b(c_0,\mathbb{R})$ and therefore $\{\phi_n\}_n$ does not converge in order to 0.

Arguing by contradiction, suppose $\{\phi_n : n \in \mathbb{N}\}$ is bounded above by some $\psi \in \mathcal{L}_b(c_0,\mathbb{R})$. Let $e_n \in c_0$ be the positive sequence whose n-th component is one and every other is zero. So $\psi(e_n) \ge \phi_n(e_n) = 1$ holds for all $n \in \mathbb{N}$. Finally, define $x_n = \sum_{i=1}^{n^3} e_n \in c_0$, then we have $\psi(x_n) \ge n^3$. With the supremum norm $\|x\| = \sup_{i\in\mathbb{N}} x_i$, the space c_0 is a Banach space. Since $\|x_n\| = 1$ holds, $x = \sum_{i=1}^{\infty} \frac{x_n}{n^2}$ exists in c_0. Since all the x_n are positive, we have $0 \le \frac{x_n}{n^2} \le x$ and thus $\psi(x) \ge \psi(\frac{x_n}{n^2}) \ge n$ for all $n \in \mathbb{N}$. Therefore such a ψ does not exist. We have derived a contradiction and shown that $\{\phi_n : n \in \mathbb{N}\}$ is not bounded above in $\mathcal{L}_b(c_0,\mathbb{R})$. ∎

However, when a net of operators decreases pointwise to an operator T, we know the net decreases to T in $\mathcal{L}_b(E,F)$.

4.10 Proposition. *Let E and F be Riesz spaces with F Dedekind complete. A net $\{T_\alpha\}_\alpha$ in $\mathcal{L}_b(E,F)$ decreases to an operator $T \in \mathcal{L}_b(E,F)$ if and only if $T_\alpha x \downarrow Tx$ for all $x \in E^+$. Similarly, $\{T_\alpha\}_\alpha$ increases to T if and only if $T_\alpha x \uparrow Tx$ for all $x \in E^+$.*

Proof. [AL, Theorem 1.18] □

Naturally, one considers operators that preserve convergence properties. Therefore, we make the following definition.

4.11 Definition. Let E and F be Riesz spaces. An operator $T: E \to F$ is *order continuous* if order convergence of a net $\{x_\alpha\}_\alpha$ in E to x implies $Tx_\alpha \xrightarrow{o} Tx$. The vector space of order continuous operators from E to F will be denoted by $\mathcal{L}_n(E,F)$.

4.12 Proposition. *If $T : E \to F$ is a positive operator between Riesz spaces E and F, then T is order continuous if and only if $x_\alpha \uparrow x$ in E implies $Tx_\alpha \uparrow Tx$ in F. Similarly, T is order continuous if and only if $x_\alpha \downarrow x$ in E, implies $Tx_\alpha \downarrow Tx$ in F.*

Proof. See the remark in the text below [AB, Definition 3.6]. □

4.13 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. For a function $h : X \to X$ consider the operator $T_h : E^X \to E^X, T_h g = g \circ h$ as in Example 3.17. Suppose $\{f_\alpha\}_\alpha$ is a net in E^X converging in order to f. Then $f_\alpha(x) \to f(x)$ holds for all $x \in X$ by 3.20. We have

$$(T_h f_\alpha)(x) = f_\alpha(h(x)) \to f(h(x)) = (T_h f)(x)$$

for all $x \in X$. Therefore, $T_h f_\alpha$ converges in order to $T_h f$. It follows T is order continuous. ■

4.14 Proposition. *Let E and F be Riesz spaces. Every order continuous operator $T : E \to F$ is order bounded. If F is Dedekind complete, then $\mathcal{L}_n(E, F)$ is a band in $\mathcal{L}_b(E, F)$.*

Proof. [AL, Lemma 1.54 and Theorem 1.57] □

The facts that a band is order closed and that $\mathcal{L}_b(E, F)$ is Dedekind complete, imply the following corollary of 4.14.

4.15 Corollary. *Let E and F be Riesz spaces and F Dedekind complete, then $\mathcal{L}_n(E, F)$ is a Dedekind complete Riesz space.*

The inclusion of the order continuous operators in the order bounded operators can be proper.

4.16 Example. This example is based on [AL, Example 1.15]. Let $\mathcal{L}^1[0, 1]$ be the ordered vector space of Lesbesgue integrable functions $f : [0, 1] \to \mathbb{R}$ with $f \geq g$ if $f(x) \geq g(x)$ for all $x \in [0, 1]$. Since the functions $x \mapsto f(x) \wedge g(x)$ and $x \mapsto f(x) \vee g(x)$ are Lesbesgue integrable for $f, g \in \mathcal{L}^1[0, 1]$, it follows that $\mathcal{L}^1[0, 1]$ is a Riesz space. Consider the positive operator $T : \mathcal{L}^1[0, 1] \to \mathbb{R}$ given by

$$Tf = \int_0^1 f dx.$$

Let $\mathcal{F}$ be the collection of all finite subsets of $[0, 1]$ and consider the net $\{\chi_\alpha : \alpha \in \mathcal{F}\}$, where χ_α is the characteristic function on α. Clearly the net satisfies $\chi_\alpha \uparrow \mathbf{1}$. On the other hand, we have $T(\chi_\alpha) = 0$ for all $\alpha \in \mathcal{F}$ and $T(\mathbf{1}) = 1$. This shows T is order bounded (since it is positive), but not order continuous. ■

4.2 Multiplying operators on Riesz spaces

From now on we will focus on the algebra $\mathcal{L}(E)$ and, in particular, its subalgebras $\mathcal{L}_b(E)$ and $\mathcal{L}_n(E)$ for a Riesz space E.[3] In this paragraph we investigate the compatibility between the algebraic structure and the order structure on $\mathcal{L}(E)$. We begin with a basic but useful identity.

[3]Note that a product of order bounded or order continuous operators is again order bounded, respectively order continuous.

4.17 Proposition. *Let E be a Riesz space and S, T, R and U operators on E. Suppose $S \leq T$ and $R \leq U$. If either S and U are positive or T and R are positive, we have $SR \leq TU$.*

Proof. In the case S and U are positive we have $SRx \leq SUx \leq TUx$ for all $x \in E^+$. In the case T and R are positive we deduce $SRx \leq TRx \leq TUx$ for all $x \in E^+$. In both cases we conclude $SR \leq TU$. □

The following proposition is of great importance in estimates involving operators.

4.18 Proposition. *Let $S, T \in \mathcal{L}_b(E)$ with E Dedekind complete. We have*

$$|TS| \leq |T||S|.$$

Proof. For positive operators $S, T \in \mathcal{L}_b(E)$ we clearly have $TS \geq 0$ and hence $|TS| = TS$. Furthermore, we may write $T = T^+ - T^-$, $|T| = T^+ + T^-$, $S = S^+ - S^-$ and $|S| = S^+ + S^-$ by 4.5. Now, by combining the previous observations with the triangle inequality 3.13, we have for $S, T \in \mathcal{L}_b(E)$

$$|TS| = |T^+T^- - T^+S^- - T^-S^+ + T^-S^-| \leq T^+T^- + T^+S^- + T^-S^+ + T^-S^- = |T||S|. \qquad \square$$

Finally, order convergence of operators is compatible with multiplication, when we consider a Dedekind complete Riesz space.

4.19 Proposition. *Let $T_\alpha \downarrow T$ in $\mathcal{L}_b(E)$ with E Dedekind complete. For $S \in \mathcal{L}_b(E)$ positive we have $T_\alpha S \downarrow TS$. Furthermore, if $S \in \mathcal{L}_n(E)$ is positive, we also have $ST_\alpha \downarrow ST$. The same results hold when $\downarrow$ is replaced by $\uparrow$.*

Proof. Let $S \in \mathcal{L}_b(E)$ positive. For every $x \in E^+$ we have $T_\alpha Sx \downarrow TSx$ by 4.10. This implies $T_\alpha S \downarrow TS$ by 4.10 again. Further, if $S \in \mathcal{L}_n(E)$ is positive, we have for $x \in E^+$ that $T_\alpha x \downarrow Tx$ by 4.10. By 4.12 we derive $ST_\alpha x \downarrow STx$ for all $x \in E^+$, implying $ST_\alpha \downarrow ST$ by 4.10 again. □

4.20 Proposition. *Let $T_\alpha \xrightarrow{o} T$ in $\mathcal{L}_b(E)$ with E Dedekind complete. For $S \in \mathcal{L}_b(E)$ we have $T_\alpha S \xrightarrow{o} TS$. Furthermore, if $S \in \mathcal{L}_n(E)$, we even have $ST_\alpha \xrightarrow{o} ST$.*

Proof. From $T_\alpha \xrightarrow{o} T$ we know there exists some net $\{R_\alpha\}_\alpha$ such that $|T_\alpha - T| \leq R_\alpha \downarrow 0$. Using 4.18, 4.17 and 4.19, respectively, we deduce

$$|T_\alpha S - TS| \leq |T_\alpha - T||S| \leq R_\alpha|S| \downarrow 0 \text{ for } S \in \mathcal{L}_b(E);$$
$$|ST_\alpha - ST| \leq |S||T_\alpha - T| \leq |S|R_\alpha \downarrow 0 \text{ for } S \in \mathcal{L}_n(E).$$

Hence it follows $T_\alpha S \xrightarrow{o} TS$ if $S \in \mathcal{L}_b(E)$ and $ST_\alpha \xrightarrow{o} ST$ if $S \in \mathcal{L}_n(E)$. □

The question wether multiplication is compatible with order convergence in both variables, is more delicate. Boundedness of the involved nets plays a role. However, we will not study this question here, since an eleborate discussion is not needed to obtain the main results of this thesis. In the proof of Proposition 4.22 below one can find an example of the usage of boundedness to deal with the multiplication of two order convergent nets.

4.3 Algebras of operators

Our first goal, as formulated in the introduction, is to analyze the structure of the order bicommutant. In paragraph 9.1 we shall see it is a band algebra[4]. For that reason we consider ideals and bands generated by subalgebras of $\mathcal{L}_b(E)$ and $\mathcal{L}_n(E)$ for Dedekind complete Riesz spaces E. To avoid confusion we stress that we do not consider algebraic ideals. All ideals mentioned in this thesis are order ideals, as defined in 3.34.

4.21 Proposition. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_b(E)$ absolutely self-majorizing and closed under multiplication. Then the ideal $\mathcal{E}(\mathscr{A})$ generated by $\mathscr{A}$ is an algebra.*

Proof. By 3.39 we have the following description

$$\mathcal{E}(\mathscr{A}) = \{T \in \mathcal{L}_b(E) : \exists\ A_1, \ldots, A_n \in \mathscr{A} \text{ and } \lambda > 0 \text{ with } |T| \leq \lambda \sum_{i=1}^{n} |A_i|\}.$$

For each $A \in \mathscr{A}$ there exists an operator $A' \in \mathscr{A}$ such that $|A| \leq A'$, since $\mathscr{A}$ is absolutely self-majorizing. Therefore, we can rewrite our expression for $\mathcal{E}(\mathscr{A})$ in the following way

$$\mathcal{E}(\mathscr{A}) = \{T \in \mathcal{L}_b(E) : \exists\ A_1, \ldots, A_n \in \mathscr{A} \text{ and } \lambda > 0 \text{ with } |T| \leq \lambda \sum_{i=1}^{n} A_i\}.$$

By definition $\mathcal{E}(\mathscr{A})$ is a linear subspace of $\mathcal{L}_b(E)$. So we only have to show $\mathcal{E}(\mathscr{A})$ is closed under multiplication. Let $S, T \in \mathcal{E}(\mathscr{A})$. Then there exist $\lambda, \mu > 0$ and $A_1, \ldots, A_n, B_1, \ldots, B_m \in \mathscr{A}$ such that $|S| \leq \lambda \sum_{i=1}^{n} A_i$ and $|T| \leq \mu \sum_{j=1}^{m} B_j$. Using 4.17 and 4.18 we deduce

$$|ST| \leq |S||T| \leq \lambda\mu \sum_{i=1}^{n} \sum_{j=1}^{m} A_i B_j.$$

Clearly it holds $\lambda\mu > 0$ and $A_i B_j \in \mathscr{A}$, since $\mathscr{A}$ is closed under multiplication. We conclude $ST \in \mathcal{E}(\mathscr{A})$, which was to be shown. $\square$

4.22 Proposition. *Let E be a Dedekind complete Riesz space. If $A \subset \mathcal{L}_n(E)$ is an ideal and an algebra, then the band $\mathcal{B}(\mathscr{A}) \subset \mathcal{L}_n(E)$ generated by A is an algebra.*

Proof. By 3.39 we have the following description

$$\mathcal{B}(\mathscr{A}) = \{T \in \mathcal{L}_n(E) : \exists\ \{T_\alpha\}_\alpha \subset \mathscr{A}^+ \text{ with } 0 \leq T_\alpha \uparrow |T|\}.$$

By definition $\mathcal{B}(\mathscr{A})$ is a linear subspace. So we only have to show $\mathcal{B}(\mathscr{A})$ is closed under multiplication. Let $S, T \in \mathcal{B}(\mathscr{A})$, then there exist nets $\{S_\alpha\}_\alpha, \{T_\beta\}_\beta$ in $\mathscr{A}^+$ such that $0 \leq S_\alpha \uparrow |S|$ and $0 \leq T_\beta \uparrow |T|$. Clearly this yields $|S| - S_\alpha \downarrow 0$ and $|T| - T_\beta \downarrow 0$. Now observe by 4.18 and the triangle inequality 3.13 we have

$$|S_\alpha T_\beta - |S||T|| \leq |S_\alpha(T_\beta - |T|)| + |(S_\alpha - |S|)|T|| \leq S_\alpha(|T| - T_\beta) + (|S| - S_\alpha)|T| = (*).$$

Now, using Propositions 4.17 and 4.19, we deduce

$$(*) \leq |S|(|T| - T_\beta) + (|S| - S_\alpha)|T| \downarrow_{\alpha,\beta} 0.$$

We conclude $S_\alpha T_\beta \xrightarrow{o}_{\alpha,\beta} |S||T|$ with $S_\alpha T_\beta \in \mathscr{A}$, since $\mathscr{A}$ is an algebra. So $|S||T|$ is an element of $\mathcal{B}(\mathscr{A})$ using $\mathcal{B}(\mathscr{A})$ is order closed. Now by 4.18 we have $|ST| \leq |S||T|$. Using $\mathcal{B}(\mathscr{A})$ is also an ideal, we have $ST \in \mathcal{B}(\mathscr{A})$. Therefore, $\mathcal{B}(\mathscr{A})$ is an algebra. $\square$

[4] That is, a band and an algebra.

We already announced the order bicommutant is a band algebra. For later purposes we make the following definition.

4.23 Definition. Let E a Riesz space and $\mathscr{A} \subset \mathcal{L}_b(E)$ a subset. The *band algebra* $\operatorname{bandalg}(\mathscr{A})$ *generated by* $\mathscr{A}$ is the smallest band algebra containing $\mathscr{A}$ with respect to the inclusion.

Since $\mathcal{L}_b(E)$ is a band algebra itself, the band algebra in $\mathcal{L}_b(E)$ generated by $\mathscr{A}$ always exists. A moment's thought reveals $\operatorname{bandalg}(\mathscr{A})$ is the intersection of all band algebras in $\mathcal{L}_b(E)$ containing $\mathscr{A}$. Moreover, if we have $\mathscr{A} \subset \mathcal{L}_n(E)$, then $\operatorname{bandalg}(\mathscr{A})$ is contained in $\mathcal{L}_n(E)$, since $\mathcal{L}_n(E)$ is a band algebra itself. Combining the above two proposition we come to our next generation result.

4.24 Corollary. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ absolutely self-majorizing and closed under multiplication, then the band $\mathcal{B}(\mathscr{A}) \subset \mathcal{L}_n(E)$ generated by $\mathscr{A}$ is a band algebra.*

The above result combined with the following lemma suggests a nice characterization of $\operatorname{bandalg}(\mathscr{A})$.

4.25 Lemma. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_b(E)$ absolutely self-majorizing. Then* $\operatorname{alg}(\mathscr{A})$ *is absolutely self-majorizing.*

Proof. Let $S \in \operatorname{alg}(\mathscr{A})$. There exists $n \in \mathbb{N}$ and $m_i \in \mathbb{N}, \lambda_i \in \mathbb{R}, i = 1, \ldots, n$ such that

$$S = \sum_{i=1}^{n} \lambda_n \prod_{j=1}^{m_i} S_{i,j}$$

is a polynomial in elements $S_{i,j} \in \mathscr{A}$. Fix i and j. There exists $T_{i,j} \in \mathscr{A}$ such that $|S_{i,j}| \leq T_{i,j}$, since $\mathscr{A}$ is absolutely self-majorizing. By 3.13, 3.12, 4.18 respectively 4.17 we derive

$$|S| \leq \sum_{i=1}^{n} |\lambda_n| \prod_{j=1}^{m_i} |S_{i,j}| \leq \sum_{i=1}^{n} |\lambda_n| \prod_{j=1}^{m_i} T_{i,j} \in \operatorname{alg}(\mathscr{A}).$$

Therefore, $\operatorname{alg}(\mathscr{A})$ is absolutely self-majorizing. ☐

4.26 Proposition. *Let E a be Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ absolutely self-majorizing. We have*

$$\operatorname{bandalg}(\mathscr{A}) = \mathcal{B}(\operatorname{alg}(\mathscr{A})).$$

Proof. Since $\operatorname{alg}(\mathscr{A})$ is multiplicatively closed and absolutely self-majorizing by Lemma 4.25, the band $\mathcal{B}(\operatorname{alg}(\mathscr{A}))$ is an algebra containing $\mathscr{A}$ by 4.24. Therefore, we obtain $\operatorname{bandalg}(\mathscr{A}) \subset \mathcal{B}(\operatorname{alg}(\mathscr{A}))$. Conversely, $\operatorname{bandalg}(\mathscr{A})$ is a band containing $\operatorname{alg}(\mathscr{A})$, therefore $\mathcal{B}(\operatorname{alg}(\mathscr{A}))$ is contained in $\operatorname{bandalg}(\mathscr{A})$. Now our conclusion follows. ☐

4.4 Invariant and reducing bands

Our second goal, as formulated in the introduction, is to retrieve a reflexivity result for the order bicommutant. Therefore, we make the following analogue of Definition 2.6 concerning reducing bands.

4.27 Definition. Let E be a Riesz space and $B \subset E$ a band. The band B *reduces* an operator T on E, if $TB \subset B$ and $TB^{\perp} \subset B^{\perp}$ holds. In this case B is called T*-reducing*. Similarly, B *reduces* $\mathscr{A} \subset \mathcal{L}(E)$, if B reduces T for each $T \in \mathscr{A}$. Then B is called $\mathscr{A}$*-reducing*.

4.28 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. Let $\#X \geq 3$ and $E \neq \{0\}$. Take $h : X \to X$ to be a transposition of elements $x, y \in X, x \neq y$. Let the operator $T_h : E^X \to E^X, T_h f = f \circ h$ as in 3.17. The band

$$B = \{f \in E^X : f(x) = 0 = f(y)\}$$

is T_h-reducing. Take $g : X \to X$ to be the constant map $g(z) = x$ for all $z \in X$. The band B is T_g-invariant, but does not reduce T_g. Finally, let $w \in X$ with $w \neq x, y$ and $j : X \to X$ the transposition of the elements y and w. Clearly, T_j does not leave B invariant. ■

Invariance under a subset $\mathscr{A}$ of operators implies immediately invariance under the whole band generated by $\mathscr{A}$.

4.29 Proposition. *Let E be a Dedekind complete Riesz space, $\mathscr{A} \subset \mathcal{L}_b(E)$ be a subset and $B \subset E$ be a band. The band B is $\mathscr{A}$-invariant if and only if B is $\mathcal{B}(\mathscr{A})$-invariant.*

Proof. Since $\mathscr{A} \subset \mathcal{B}(\mathscr{A})$ it is clear B is $\mathscr{A}$-invariant, if it is $\mathcal{B}(\mathscr{A})$-invariant. So suppose B is $\mathscr{A}$-invariant. Take $S \in \mathscr{A}$ and $x \in B$. Define $D = \{|Sy| : |y| \leq |x|\}$. Suppose $y \in E$ satisfies $|y| \leq |x|$, then we have $y \in B$ and therefore Sy is contained in B using B is $\mathscr{A}$-invariant. This implies $|Sy| \in B$ and hence D is contained in B. By 4.5 we have $|S||x| = \sup D$. Hence $|S||x|$ is an element of B, since B is order closed.

Now let $T \in \mathcal{E}(\mathscr{A})$. By 3.39 there exist $\lambda > 0$ and $A_1, \ldots, A_n \in \mathscr{A}$ such that $|T| \leq \lambda \sum_{i=1}^{n} |A_i|$. Let $x \in B$, we have by 4.6

$$|Tx| \leq |T||x| \leq \lambda \sum_{i=1}^{n} |A_i||x|.$$

Since x is in B, we have $|A_i||x| \in B$ by the previous. Hence we deduce $\lambda \sum_{i=1}^{n} |A_i||x| \in B$, which implies $Tx \in B$. So B is invariant under T. It follows B is $\mathcal{E}(\mathscr{A})$-invariant. Now suppose $T \in \mathcal{B}(\mathscr{A})$, then there exists a net $\{T_\alpha\}_\alpha$ in $\mathcal{E}(\mathscr{A})$ such that $0 \leq T_\alpha \uparrow |T|$ by 3.39. For $x \in B$ it yields $T_\alpha|x| \uparrow |T||x|$ by 4.10. Because B is invariant under T_α, we have $T_\alpha|x| \in B$ for each α. Since B is order closed, it follows that $|T||x| \in B$. Using the identity 4.6: $|Tx| \leq |T||x|$, we conclude $Tx \in B$. So B is T-invariant for each $T \in \mathcal{B}(\mathscr{A})$. We conclude that B is $\mathcal{B}(\mathscr{A})$-invariant. □

4.30 Corollary. *Let E be a Dedekind complete Riesz space, $\mathscr{A} \subset \mathcal{L}_b(E)$ a subset and $B \subset E$ be a band. The band B is $\mathscr{A}$-reducing if and only if B is $\mathcal{B}(\mathscr{A})$-reducing.*

4.31 Corollary. *Let E a Dedekind complete Riesz space and $B \subset E$ a band. The sets*

$$\mathscr{A}_B = \{T \in \mathcal{L}_b(E) : TB \subset B\}, \quad \widetilde{\mathscr{A}_B} = \{T \in \mathcal{L}_b(E) : B \text{ reduces } T\}$$

are band algebras.

Proof. The fact that $\mathscr{A}_B$ is an algebra is obvious. We show $\mathscr{A}_B$ is a band. The band B is $\mathscr{A}_B$-invariant and hence $\mathcal{B}(\mathscr{A}_B)$-invariant by 4.29. It follows $\mathcal{B}(\mathscr{A}_B) \subset \mathscr{A}_B$. On the other hand $\mathscr{A}_B$ is trivially contained in $\mathcal{B}(\mathscr{A}_B)$. Therefore, $\mathscr{A}_B = \mathcal{B}(\mathscr{A}_B)$ is a band. Observing $\widetilde{\mathscr{A}_B} = \mathscr{A}_B \cap \mathscr{A}_{B^{\perp}}$ the other claim directly follows. □

Clearly, $\mathrm{alg}(\mathscr{A})$ and $\mathscr{A}$ have the same invariant and reducing bands. Combined with the above we infer the following summary of results.

4.32 Proposition. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_b(E)$ a subset. Then every $\mathscr{U} \subset \mathcal{L}_b(E)$ with $\mathscr{A} \subset \mathscr{U} \subset \mathcal{B}(\mathscr{A})$ or $\mathscr{A} \subset \mathscr{U} \subset \mathrm{alg}(\mathscr{A})$ has the same invariant and reducing bands as $\mathscr{A}$. If $\mathscr{A} \subset \mathcal{L}_n(E)$ is absolutely self-majorizing, then every $\mathscr{U} \subset \mathcal{L}_n(E)$ with $\mathscr{A} \subset \mathscr{U} \subset \mathrm{bandalg}(\mathscr{A})$ has the same invariant and reducing bands as $\mathscr{A}$.*

Proof. Combine 4.26, 4.29, 4.30. □

In 2.8 we derived that for a $*$-closed subset of operators on a Hilbert space the invariant and reducing subspaces agree. For Riesz spaces we have no natural counterpart of the adjoint. However, the coincidence of the invariant and reducing subspaces plays an important role in the techniques used for answering the questions concerning the von Neumann bicommutant. So we make the following definition.

4.33 Definition. Let E be a Riesz space. A subset $\mathscr{A}$ of $\mathcal{L}(E)$ has *the $*$-property* if every $\mathscr{A}$-invariant band is $\mathscr{A}$-reducing.

An important class of instances with the $*$-property are groups of Riesz automorphisms.

4.34 Proposition. *Let E be a Riesz space and $Aut(E)$ the group of Riesz automorphisms on E. A subset $\mathscr{A} \subset Aut(E)$ closed under taking the inverse has the $*$-property.*

Proof. Suppose $\mathscr{A} \subset \mathrm{Aut}(E)$ is closed under taking the inverse. Let $B \subset E$ be a band. Take $x \in B$ and $y \in B^\perp$. The element $T^{-1}x$ is in B, since T^{-1} is in $\mathscr{A}$. Hence we have $|T^{-1}x| \wedge |y| = 0$. Now applying T on both sides yields

$$0 = T[|T^{-1}x| \wedge |y|] = |TT^{-1}x| \wedge |Ty| = |x| \wedge |Ty|$$

using 3.15. Therefore, $Ty \perp x$ holds for all $x \in B$. We conclude $Ty \in B^\perp$ for all $T \in \mathscr{A}$ and $y \in B^\perp$. So $B^\perp$ is also $\mathscr{A}$-invariant and B reduces $\mathscr{A}$. Therefore, $\mathscr{A}$ has the $*$-property. □

From 4.32 we immediately deduce the following result.

4.35 Proposition. *Let E be a Riesz space. If $\mathscr{A} \subset \mathcal{L}_b(E)$ is a subset with the $*$-property, then every $\mathscr{U} \subset \mathcal{L}_b(E)$ with $\mathscr{A} \subset \mathscr{U} \subset \mathcal{B}(\mathscr{A})$ or $\mathscr{A} \subset \mathscr{U} \subset \mathrm{alg}(\mathscr{A})$ has the $*$-property. If $\mathscr{A} \subset \mathcal{L}_n(E)$ is moreover absolutely self-majorizing, then every $\mathscr{U} \subset \mathcal{L}_n(E)$ with $\mathscr{A} \subset \mathscr{U} \subset \mathrm{bandalg}(\mathscr{A})$ has the $*$-property.*

One can also consider the dual problem. What does invariance of a subset $A \subset E$ under a set of operators $\mathscr{A}$ imply about the invariance of the ideals and bands generated by A? When A and $\mathscr{A}$ are positive, this question are easily answered.

4.36 Proposition. *Let E be a Dedekind complete Riesz space, $A \subset E^+$ and $\mathscr{A} \subset \mathcal{L}_b(E)^+$ subsets. If A is $\mathscr{A}$-invariant, then the ideal $\mathcal{E}(A)$ generated by A is $\mathscr{A}$-invariant.*

Proof. Suppose A is $\mathscr{A}$-invariant. Let $x \in \mathcal{E}(A)$. Then there exist $s_1, \ldots, s_n \in A$ and $\lambda > 0$ such that $|x| \leq \lambda \sum_{i=1}^n s_i$. Hence we have by 4.6 for $T \in \mathscr{A}$

$$|Tx| \leq T|x| \leq \lambda \sum_{i=1}^{n} Ts_i$$

with $Ts_i \in A$. The fact that $\mathcal{E}(A)$ is an ideal yields $Tx \in \mathcal{E}(A)$. It follows $\mathcal{E}(A)$ is T-invariant for all $T \in \mathscr{A}$ and thus $\mathcal{E}(A)$ is $\mathscr{A}$-invariant. □

4.37 Proposition. *Let E be a Dedekind complete Riesz space, $A \subset E$ an ideal and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. If A is $\mathscr{A}$-invariant, then the band $\mathcal{B}(A)$ generated by A is $\mathscr{A}$-invariant.*

Proof. Suppose A is $\mathscr{A}$-invariant. Let $T \in \mathscr{A}$ and $x \in \mathcal{B}(A)$. There exists a net $\{x_\alpha\}_\alpha$ in A such that $x_\alpha \xrightarrow{o} x$. By order continuity we have $Tx_\alpha \xrightarrow{o} Tx$ with $Tx_\alpha \in A$ by assumption. Hence $Tx \in \mathcal{B}(A)$ holds using $\mathcal{B}(A)$ is order closed. We conclude that $\mathcal{B}(A)$ is T-invariant for all $T \in \mathscr{A}$. Therefore, $\mathcal{B}(A)$ is $\mathscr{A}$-invariant. □

4.38 Corollary. *Let E be a Dedekind complete Riesz space, $A \subset E^+$ and $\mathscr{A} \subset \mathcal{L}_n(E)^+$ subsets. If A is $\mathscr{A}$-invariant, then the band $\mathcal{B}(A)$ generated by A is $\mathscr{A}$-invariant.*

When $\mathscr{A}$ is an ideal, we have a stronger statement. Before giving the proof, we need the following lemma.

4.39 Lemma. *Let E and F Riesz spaces with F Dedekind complete. For a positive operator $T : E \to F$ and $x \in E$ there exists $S \in \mathcal{E}(T) \subset \mathcal{L}_b(E, F)$ such that $T|x| = Sx$.*

Proof. [AL, Theorem 1.23] □

4.40 Proposition. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_b(E)$. Let A be $\mathscr{A}$-invariant. If $\mathscr{A}$ is an ideal of $\mathcal{L}_b(E)$, then $\mathcal{E}(A)$ is $\mathscr{A}$-invariant. Moreover, if $\mathscr{A}$ is an ideal of $\mathcal{L}_n(E)$, then $\mathcal{B}(A)$ is $\mathscr{A}$-invariant.*

Proof. Suppose A is $\mathscr{A}$-invariant. Let $x \in \mathcal{E}(A)$ and $T \in \mathscr{A}$, then there exists $a_1, \ldots, a_n \in A$ and $\lambda > 0$ such that $|x| \leq \lambda \sum_{i=1}^n |a_i|$. Let $i \in \{1, \ldots, n\}$, by 4.39 we have $|T||a_i| = S_i a_i$ for some $S_i \in \mathcal{E}(|T|) \subset \mathscr{A}$. So $|T||a_i| = S_i a_i$ is in A by assumption for each $i \in \{1, \ldots, n\}$. Finally,

$$|Tx| \leq |T||x| \leq \lambda \sum_{i=1}^{n} |T||a_i| \in \mathcal{E}(A)$$

holds and therefore $Tx \in \mathcal{E}(A)$, since $\mathcal{E}(A)$ is an ideal. It follows $\mathcal{E}(A)$ is invariant for all $T \in \mathscr{A}$. So $\mathcal{E}(A)$ is $\mathscr{A}$-invariant. By 4.37 the second statement follows. □

5 Orthomorphisms

In this section we zoom in on the class of orthomorphisms, which leave all bands invariant. These operators will play a key role in the definition and theorems concerning the order bicommutant.

5.1 Basis definitions and properties

First we state some well-known facts about orthomorphisms needed for later purposes. A more complete discussion can be found in [AL].

5.1 Definition. An operator T on a Riesz space E is *band preserving* whenever T leaves all bands of E invariant.

Note that sums and products of band preserving operators are again band preserving. Intersecting the algebra of band preserving operators with $\mathcal{L}_b(E)$ gives us the algebra of orthomorphisms.

5.2 Definition. A band preserving operator on a Riesz space that is also order bounded is called an *orthomorphism.* The algebra of orthomorphisms is denoted by $\mathrm{Orth}(E)$.

5.3 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. We characterize the orthomorphisms on E^X. Let $S \in \mathrm{Orth}(E^X)$. Fix $x \in X$. Consider the map $S_x : E \to E$ given by $S_x y = (Sf_y)(x)$, where $f_y \in E^X$ is a function such that $f_y(x) = y$. First we show S_x is well-defined. Take $f, g \in E^X$ such that $f(x) = g(x)$. Clearly $f - g$ is in the band $B = \{h \in E^X : h(x) = 0\} \subset E^X$. Since S is an orthomorphism, $S(f-g)$ is also in B. Therefore, we have $(Sf)(x) - (Sg)(x) = [S(f-g)](x) = 0$ implying $(Sf)(x) = (Sg)(x)$. We conclude that S_x is well-defined and independent of the choice of the function f_y.

It is a straightforward check that S_x is an order bounded operator. We show S_x is an orthomorphism. Let $B \subset E$ be a band. The subspace $A = \{f \in E^X : f(x) \in B\}$ is a band in E^X by 3.36. Let $y \in B$, then f_y is contained in A for all functions $f_y \in E^X$ with $f_y(x) = y$. We have $S_x y = (Sf_y)(x) \in B$, because Sf_y is in A. We conclude that for each $x \in X$ there exists an orthomorphism $S_x \in \mathrm{Orth}(E)$ such that $(Sf)(x) = S_x[f(x)]$. So all orthomorphisms on E^X are described in such a way. Conversely, if S_x is an orthomorphism on E for each $x \in X$, then the operator S on E^X defined by $(Sf)(x) = S_x[f(x)]$ is an orthomorphism.

Take for example $E = \mathbb{R}$. For every $m \in \mathbb{R}^X$ the multiplication map $R_m : \mathbb{R}^X \to \mathbb{R}^X$ given by $R_m f = mf$ is an orthomorphism. Moreover, all orthomorphisms are given by such a multiplication map. The maps $S_x : \mathbb{R} \to \mathbb{R}$ for $x \in X$ are in this case given by multiplication with a number $m(x)$. We obtain $\mathrm{Orth}(\mathbb{R}^X)$ is order isomorphic with $\mathbb{R}^X$ itself. ■

In fact the algebra of orthomorphisms is also a band.

5.4 Proposition. *If E is a Dedekind complete Riesz space, then* $\mathrm{Orth}(E)$ *coincides with the band generated by the identity operator I in $\mathcal{L}_b(E)$.*

Proof. [AL, Theorem 2.45] □

Now $\mathcal{L}_n(E)$ is a band in $\mathcal{L}_b(E)$ containing I by 4.14. Therefore, we have the following corollary.

5.5 Corollary. *If E is a Dedekind complete Riesz space,* $\mathrm{Orth}(E)$ *is a band algebra in $\mathcal{L}_n(E)$.*

The following example shows the inclusion of the orthomorphisms in the order continuous operators can be proper.

5.6 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7. Let $\#X \geq 3$ and $E \neq \{0\}$. Take three different elements x, y, w in X. Let $j : X \to X$ the transposition of the elements y and w. As in Example 4.28 the operators $T_j : E^X \to E^X, T_j f = f \circ j$ does not leave $B = \{f \in E^X : f(x) = 0 = f(y)\}$ invariant. By Example 4.13 T_j is order continuous, but not an orthomorphism. ■

Quite surprising is the fact that all orthomorphisms commute.

5.7 Proposition. *If E is a Dedekind complete Riesz space, then* $\mathrm{Orth}(E)$ *is a commutative full subalgebra of $\mathcal{L}_b(E)$.*

Proof. [MN, Theorems 3.1.9 and 3.1.10] □

There is an easy condition for an orthomorphism to be invertible.

5.8 Proposition. *Let E be a Dedekind complete Riesz space. Each orthomorphism $T \in$* $\mathrm{Orth}(E)$ *satisfying $T \geq I$ is invertible in $\mathcal{L}_b(E)$.*

Proof. [ZL, Theorem 146.3] □

The following proposition is an important tool for estimations involving orthomorphisms.

5.9 Proposition. *Let E be a Dedekind complete Riesz space and $T \in$* $\mathrm{Orth}(E)$*. For all $x \in E$ we have*

$$|T||x| = |T|x|| = |Tx|.$$

Proof. [AL, Theorem 2.40] □

It should be mentioned that the previous proposition is true for a more general setting. However, we refer to [AL] for this, since our use will be limited to the orthomorphisms on Dedekind complete Riesz spaces. The kernel of an orthomorphism has a nice description.

5.10 Proposition. *If E is a Dedekind complete Riesz space and $T \in$* $\mathrm{Orth}(E)$*, then* $\ker(T)$ *is a band.*

Proof. [AL, Theorem 2.48] □

Fix a positive element x. The last result of this paragraph shows that elements in the ideal generated by x are actually evaluations of orthomorphisms in x.

5.11 Proposition. *Let E be a Dedekind complete Riesz space and $x \in E^+$. For each $y \in \mathcal{E}(x)$ there exists and orthomorphism $T \in$* $\mathrm{Orth}(E)$ *such that $T(x) = y$.*

Proof. [AL, Theorem 2.49] □

This proposition will play an important role in the proof in of an order bicommutant theorem.

5.2 Order projections

Projections play an important role in the proof of the von Neumann Bicommutant Theorem. This will be of no difference for the results concerning the order bicommutant. In this paragraph we state some well-known facts. A more complete discussion about projections on Riesz spaces can be found in [AL].

If B is a band in a Dedekind complete Riesz space E, we can decompose the Riesz space as $E = B \oplus B^{\perp}$ by 3.41. Thus every $x \in E$ can be decomposed uniquely as $x = x_1 + x_2$ with $x_1 \in B$ and $x_2 \in B^{\perp}$. This gives rise to a projection P_B on B given by $P_B x = x_1$ with $\operatorname{ran}(P) = B$ and $\operatorname{ran}(I - P) = B^{\perp}$. So P and $I - P$ have orthogonal ranges. This brings us to the following equivalent statements resulting in the definition of an order projection.

5.12 Proposition. *Let E be a Dedekind complete Riesz spaces and $P : E \to E$ be an operator. The following statements are equivalent*

i. *P and $I - P$ have orthogonal ranges;*

ii. *P is a projection, $\operatorname{ran}(P)$ is a band and $\ker(P) = \operatorname{ran}(P)^{\perp}$;*

iii. *P is a projection satisfying $0 \leq P \leq I$.*

Proof. [AL, Theorem 1.44] □

5.13 Definition. An operator on a Dedekind complete Riesz space E satisfying any of the equivalent statements in 5.12 is called an *order projection.* We will denote the set of order projections by $\mathcal{P}(A)$.

5.14 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7 with $E = \mathbb{R}$. Let $Y \subset X$. Following Example 3.42 the order projection P on the band $B_Y = \{g \in \mathbb{R}^X : g(x) = 0 \text{ for all } x \in Y\}$ is given by $Pf = \chi_{X \setminus Y} f$. ■

By combining 5.4 and 5.12 we see each order projection is an orthomorphism. Moreover, by 5.7 the product of two order projections is again an order projection, which satisfies $0 \leq P \leq I$ by applying 4.17 two times. This brings us to our next statement.

5.15 Proposition. *If E is a Dedekind complete Riesz space, then $\mathcal{P}(A)$ is a multiplicatively closed subset of $\operatorname{Orth}(E) \subset \mathcal{L}_n(E)$.*

5.16 Example. Consider the Dedekind complete Riesz space E^X from Example 3.7 with $E = \mathbb{R}$. Let $m \in \mathbb{R}^X$ be a function such that $m^2 = m$. It holds $m(x)^2 = m(x)$ for each $x \in X$ and therefore $m(x)$ must be either zero or one. It follows $0 \leq m \leq \mathbf{1}$. We see the operator R_m from Example 5.3 satisfies $R_m^2 = R_m$ and $0 \leq R_m \leq I$. Hence R_m is an order projection. On the other hand, if $m \in \mathbb{R}^X$ does not satisfy $m^2 = m$, then we have $R_m^2 \mathbf{1} = m^2 \neq m = R_m \mathbf{1}$. It follows R_m does not satisfy $R_m^2 = R_m$ and is therefore not an (order) projection. Consequently, $\mathcal{P}(\mathbb{R}^X)$ can be identified with the set of characteristic functions on X. ■

Given an order projection, the range is a band. Conversely, the text above 5.12 shows that given a band B, we can find an order projection which has range B. We make the following important observation.

5.17 Proposition. *Let E be a Dedekind complete Riesz space. There exists a one-to-one correspondence between order projections and bands in E. An order projection $P : E \to E$ gives rise to a band* $\operatorname{ran}(P)$ *and conversely a band $B \subset E$ gives rise to an order projection P_B on B.*

Using Proposition 5.12 we derive that, if P is an order projection on a Dedekind complete Riesz space with range B, then $I - P$ is also an order projection with range $B^\perp$. If a band B is generated by a single element, we have a nice formula for the order projection on B.

5.18 Proposition. *Let E be a Dedekind complete Riesz space and $x \in E$. The order projection P on the band $\mathcal{B}(x)$ generated by x is given by*

$$P(y) = \sup\{y \wedge n|x| : n \in \mathbb{N}\} \text{ for } y \in E^+.$$

Proof. [AL, Theorem 1.47] □

Finally, we state a less known result. Order projections on the Dedekind complete Riesz space $\mathcal{L}_n(E)$ induce order projections on E.

5.19 Proposition. *Let E be a Dedekind complete Riesz space. If $\mathcal{P}$ is an order projection on $\mathcal{L}_b(E)$, then $P := \mathcal{P}I$ is an order projection on E.*

Proof. By 5.12 $\mathcal{P}$ and $\mathcal{I} - \mathcal{P}$ have orthogonal ranges, where $\mathcal{I}$ denotes the identity element in $\mathcal{L}_b(\mathcal{L}_b(E))$. Therefore, we have $P \wedge (I - P) = \mathcal{P}I \wedge (\mathcal{I} - \mathcal{P})I = 0$. It follows $0 \leq P \leq I$. So we only have to show $P^2 = P$ by 5.12. From $0 \leq P \leq I$ and 4.17 we deduce $0 \leq P^2 \leq P$ and $0 \leq (I - P)^2 \leq (I - P)$. So we conclude $0 \leq P^2 \wedge (I - P)^2 \leq P \wedge (I - P) = 0$. We compute

$$0 = P^2 \wedge (I - P)^2 = (P + P^2 - P) \wedge (I - P + P^2 - P) = P \wedge (I - P) + P^2 - P = P^2 - P.$$

Therefore it follows $P^2 = P$ and thus P is an order projection. □

6 Atomic Riesz spaces

We will derive an order bicommutant theorem for the class of atomic Riesz spaces, which we consider in this paragraph.

6.1 Definition. A positive element x in a Riesz space is an *atom* if the ideal $\mathcal{E}(x)$ generated by x is one-dimensional.[5]

Clearly, if $\mathcal{E}(x)$ is one dimensional, it is equal to the linear span of x. If the Riesz space is Archimedean, the linear span of x is a band.

6.2 Proposition. *Let E be an Archimedean Riesz space and x an atom in E. Then it holds $\mathcal{B}(x) = \{\alpha x : \alpha \in \mathbb{R}\}$.*

Proof. [ZA, Theorem 26.4] □

6.3 Corollary. *Let E a Dedekind complete Riesz space and x an atom in E. There exists a Riesz homomorphism $\zeta_x : E \to \mathbb{R}$ such that the projection P_x on the band $\mathcal{B}(x)$ generated by x is given by $P_x u = \zeta_x(u)x$.*

6.4 Definition. A Riesz space E is *atomic* if there exists a maximal orthogonal system of E consisting of atoms.

6.5 Example. Consider the Riesz spaces $\ell^p(X)$ and $\mathbb{R}^X$ from Example 3.35. Both spaces are atomic, since $S = \{\chi_{\{y\}} : y \in X\} \subset \ell^p(X) \subset \mathbb{R}^X$ is a maximal orthogonal system consisting of atoms. Here $\chi_{\{y\}}$ denotes the characteristic function of the subset $\{y\}$ of X. ■

In atomic Dedekind complete Riesz spaces the identity operator is approached by a net of sums of rank one order projections. This is an important facet in the proof of the order bicommutant theorem for atomic Riesz spaces.

6.6 Proposition. *Let E be an atomic Dedekind complete Riesz space. Denote by S a maximal orthogonal system consisting of atoms. For $z \in E$ denote by P_z the projection on the band $\mathcal{B}(z)$ generated by z. Denote by S_H the operator $S_H = \sum_{v\in H} P_v$ for $H \subset S$ finite. Then $S_H \uparrow I$ holds, where I denotes the identity operator.*

Proof. Take $x \in E^+$ and define $x_{n,H} = \sum_{u\in H} x \wedge nu$ for $n \in \mathbb{N}$ and $H \subset S$ finite. By 3.32 it holds $x_{n,H} \uparrow_{n,H} x$. Since two different atoms $u, v \in S$ are orthogonal, we have that elements $x \wedge nu \in \mathcal{E}(u)$ and $x \wedge nv \in \mathcal{E}(v)$ are orthogonal. By 3.30 it yields for $n \in \mathbb{N}$ and $H \subset S$ finite

$$x_{n,H} = \sum_{u\in H} x \wedge nu = \bigvee_{u\in H} x \wedge nu.$$

With the aid of 3.30 and 5.18, we calculate for $H \subset S$ finite

$$\sup_n x_{n,H} = \sup_n \bigvee_{u\in H} [x \wedge nv] = \bigvee_{u\in H} \sup_n [x \wedge nv] = \bigvee_{u\in H} P_u x = \sum_{u\in H} P_u x$$

[5] Some authors call this a 'discrete element' and have a more general definition for an atom, see [ZA, Definition 26.1]. However, for Archimedean Riesz spaces both terms coincides by [ZA, Theorem 26.4].

Using the above facts we derive

$$x = \sup_{n,H} x_{n,H} = \sup_H \sup_n x_{n,H} = \sup_H \sum_{u \in H} P_u x = \sup_H S_H x.$$

Furthermore, $S_{H_1} x \leq S_{H_2} x$ holds for $H_1 \subset H_2$, since order projections are positive by 5.12. Combining these facts we obtain $S_H x \uparrow x$ for all $x \in E^+$. By applying 4.10 the claim follows. $\square$

Every atomic Dedekind complete Riesz space E, with maximal orthogonal system S consisting of atoms, is order isomorphic to a Riesz subspace of $\mathbb{R}^S$. The order isomorphism $\psi : E \to \mathbb{R}^S$ is given by $\psi(x) = (\zeta_u(x))_{u \in S}$, where ζ_u is as in 6.3. If $\psi(x) = 0$ for some $x \in E$, then $x \perp u$ holds for all $u \in S$. Since S is a maximal orthogonal system, it follows $x = 0$. Therefore, ψ is indeed an injective Riesz homomorphism and E is order isomorphic with the Riesz subspace $\psi(E) \subset \mathbb{R}^S$.

7 Freudenthal's Spectral Theorem

The Freudenthal Spectral Theorem is an important approximation result for elements of a singly generated band. This result will be of significant importance in the proof of the reflexivity result concerning the order bicommutant.

7.1 Theorem. *Let E be a Dedekind complete Riesz space and $x \in E^+$. Let y be an element in the band $\mathcal{B}(x)$ generated by x. For $\alpha \in \mathbb{R}$ define P_α to be the order projection on the band $\mathcal{B}((\alpha x - y)^+)$ generated by $(\alpha x - y)^+$. Moreover, take intervals $[a_n, b_n] \subset \mathbb{R}$ for every $n \in \mathbb{N}$ such that $0 \geq a_n \downarrow -\infty$ and $0 \leq b_n \uparrow \infty$. Let $\pi_n(\alpha_{n,0}, \ldots, \alpha_{n,m_n})$ be a partition of the interval $[a_n, b_n]$ such that the restriction of π_{n+1} to $[a_n, b_n]$ is a refinement of π_n and for the mesh we have $|\pi_n| \downarrow 0$ if $n \to \infty$. Define*

$$y_n = \sum_{k=1}^{m_n} \alpha_{n,k-1}(P_{\alpha_{n,k}} - P_{\alpha_{n,k-1}})x \text{ for } n \in \mathbb{N}.$$

The sequence $\{y_n\}_{n\in\mathbb{N}}$ converges in order to y.

Proof. [ZA, Theorem 40.3] □

We apply the Freudenthal Spectral Theorem to the approximation of orthomorphisms.

7.2 Corollary. *Let E be a Dedekind complete Riesz space. Then $\mathrm{Orth}(E)$ coincides with the set of order limits of sequences of linear combinations of projections. More precisely, every orthomorphism $S \in \mathrm{Orth}(E)$ is the order limit of a sequence from the linear span of $\{P_\alpha : \alpha \in \mathbb{R}\}$, where $P_\alpha = \sup\{I \wedge n(\alpha I - S)^+ : n \in \mathbb{N}\}$ is an order projection on E.*

Proof. Let $S \in \mathrm{Orth}(E)$ be an orthomorphism. The orthomorphism algebra $\mathrm{Orth}(E)$ is the band generated by the identity operator I in $\mathcal{L}_b(E)$ by 5.4. Hence we can apply Freudenthal's Spectral Theorem to $S \in \mathcal{B}(I) \subset \mathcal{L}_b(E)$. For $\alpha \in \mathbb{R}$ denote by $\mathcal{P}_\alpha$ the order projection on the band $\mathcal{B}((\alpha I - S)^+) \subset \mathcal{L}_b(E)$ generated by $(\alpha I - S)^+ \in \mathcal{L}_b(E)$ and write $P_\alpha = \mathcal{P}_\alpha I$. By 5.19 the operator P_α is an order projection for each $\alpha \in \mathbb{R}$, moreover, by 5.18 we have

$$P_\alpha = \mathcal{P}_\alpha I = \sup\{I \wedge n(\alpha I - S)^+ : n \in \mathbb{N}\}.$$

By applying Freudenthal's spectral Theorem 7.1, there exists a sequence of operators $\{S_n\}_n$ converging in order to S, where, for each $n \in \mathbb{N}$, the operator S_n is a linear combination of elements $\mathcal{P}_\alpha I, \alpha \in \mathbb{R}$. It follows S is the order limit of the sequence $\{S_n\}_n$ contained in the linear span of $\{P_\alpha : \alpha \in \mathbb{R}\}$. □

8 The commutant

Before going into detail about the order bicommutant, we stress some important properties about commutants in $\mathcal{L}_b(E)$.

8.1 Commuting operators

Before we state some results about the commutant of a set of operators in $\mathcal{L}_n(E)$, we will need some preliminary results about commuting operators. First of all order convergence behaves well with respect to commuting operators.

8.1 Proposition. *Let E be a Dedekind complete Riesz space and $T \in \mathcal{L}_n(E)$. Let $S_\alpha \xrightarrow{o} S$ in $\mathcal{L}_b(E)$. If T commutes with S_α for each α, then T commutes with S.*

Proof. By definition there exists some net $\{R_\alpha\}_\alpha$ such that $|S_\alpha - S| \leq R_\alpha \downarrow 0$. Let $y \in E$, then we have $|S_\alpha y - Sy| \leq |S_\alpha - S||y| \leq R_\alpha|y| \downarrow 0$ by 4.10. It follows $S_\alpha y \xrightarrow{o} Sy$ for all $y \in E$. We obtain by order continuity of T and 3.21

$$0 = TS_\alpha y - S_\alpha Ty \xrightarrow{o} TSy - STy.$$

Therefore, $TSy = STy$ holds for all $x \in E$. We conclude $TS = ST$. □

In the results about the von Neumann bicommutant the projection Lemma 2.9 was a main tool. This will also be the case when concerning the order bicommutant. However, the statement will be somewhat different, since we have no natural notion of an adjoint in Riesz spaces. Commuting with an order projection is equivalent to being reduced by the corresponding band (see 5.17).

8.2 Lemma (Projection Lemma)**.** *Let E be a Dedekind complete Riesz space and $T \in \mathcal{L}_b(E)$. A band $B \subset E$ is T-reducing if and only if $PT = TP$ for the order projection P on B.*

Proof. Suppose B is T-reducing. Take $x \in E$ and let $I - P$ the order projection on $B^\perp$. Since $TPx \in B$ and $T(I-P)x \in B^\perp$, we derive

$$TPx = PTPx = PTPx + PT(I-P)x = PTx.$$

Therefore, P commutes with T. Conversely, suppose P commutes with T. We have $TB = TPB = PTB \subset B$. The fact that the projection $I - P$ also commutes with T yields $TB^\perp \subset B^\perp$ by a completely analogous argument. It follows that B is T-reducing. □

8.2 The commutant

In section 2 we took the commutant inside the bounded operators. For Riesz spaces the order continuous operators seem to be the most fruitful context to work in, because for positive group representations, the subject that motivated our study of the order bicommutant, the natural setting is $\mathcal{L}_n(E)$.

8.3 Definition. Let E be a Dedekind complete Riesz space. For a subset $\mathscr{A} \subset \mathcal{L}_n(E)$. We write $\mathscr{A}^c = \{S \in \mathcal{L}_n(E) : ST = TS \text{ for all } T \in A\}$ for *the commutant of $\mathscr{A}$ in the order continuous operators.*

When $\mathscr{A}$ is contained in the orthomorphisms, the commutant has a nice structure.

8.4 Theorem. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathrm{Orth}(E)$ be a subset. The commutant $\mathscr{A}^c$ in $\mathcal{L}_n(E)$ is a band algebra containing I.*

Proof. The fact that $\mathscr{A}^c$ is an algebra is trivial. We show $\mathscr{A}^c$ is a band. By combining 5.4 and 5.5 $\mathrm{Orth}(E)$ is the band generated by the identity operator I in $\mathcal{L}_n(E)$. For $M \in \mathrm{Orth}(E)$ consider the right and left multiplication operators $\mathcal{R}_M, \mathcal{L}_M$ on $\mathcal{L}_n(E)$ given by $\mathcal{R}_M T = TM, \mathcal{L}_M T = MT$. We are going to show $\mathcal{R}_M, \mathcal{L}_M \in \mathrm{Orth}(\mathcal{L}_n(E))$. Let $S \in \mathrm{Orth}(E)$ be positive, then by 3.39 there exists a net $\{S_\alpha\}_\alpha$ in the ideal generated by I such that $0 \leq S_\alpha \uparrow S$. Fix α, there exists $\lambda_\alpha \in \mathbb{R}_{>0}$ such that $S_\alpha \leq \lambda_\alpha I$. Notice that for $B \in \mathcal{L}_n(E)^+$ by 4.17 and 4.18 we have $|BS_\alpha| \leq |B|S_\alpha \leq \lambda_\alpha |B|$. For $T \in \mathcal{L}_n(E)^+$ we deduce using 4.5

$$|\mathcal{R}_{S_\alpha}|T = \sup\{|BS_\alpha| : |B| \leq T\} \leq \lambda_\alpha \sup\{|B| : |B| \leq T\} = \lambda_\alpha T.$$

It follows $|\mathcal{R}_{S_\alpha}| \leq \lambda_\alpha \mathcal{I}$ with $\mathcal{I}$ the identity operator on $\mathcal{L}_n(E)$. Therefore, we conclude $\mathcal{R}_{S_\alpha} \in \mathrm{Orth}(\mathcal{L}_n(E))$ by applying 5.4. Furthermore, for $T \in \mathcal{L}_n(E)$ positive

$$(\mathcal{R}_S - \mathcal{R}_{S_\alpha})T = T(S - S_\alpha) \downarrow 0$$

holds by 4.19. We conclude $\mathcal{R}_{S_\alpha} \uparrow \mathcal{R}_S$ by 4.10. It follows $\mathcal{R}_S \in \mathrm{Orth}(\mathcal{L}_n(E))$. Now take $S \in \mathrm{Orth}(E)$ arbitrarily and write $S = S^+ - S^-$. We have $S^+, S^- \in \mathrm{Orth}(E)^+$ and thus the previous yields $\mathcal{R}_{S^+}, \mathcal{R}_{S^-} \in \mathrm{Orth}(\mathcal{L}_n(E))$. It is clear $\mathcal{R}_S = \mathcal{R}_{S^+} - \mathcal{R}_{S^-}$ is also an element of $\mathrm{Orth}(\mathcal{L}_n(E))$. By completely similar arguments one shows $\mathcal{L}_S \in \mathrm{Orth}(\mathcal{L}_n(E))$ for $S \in \mathrm{Orth}(E)$. Note that we have

$$\mathscr{A}^c = \{T \in \mathcal{L}_n(E) : \mathcal{R}_S T = \mathcal{L}_S T \text{ for all } S \in \mathrm{Orth}(E)\} = \bigcap_{S \in \mathrm{Orth}(E)} \ker(\mathcal{R}_S - \mathcal{L}_S).$$

By 5.10 $\ker(\mathcal{R}_S - \mathcal{L}_S)$ is a band in $\mathcal{L}_n(E)$, since it holds $\mathcal{R}_S - \mathcal{L}_S \in \mathrm{Orth}(\mathcal{L}_n(E))$ for $S \in \mathrm{Orth}(E)$. Being an intersection of bands, it follows $\mathscr{A}^c$ is a band. □

8.5 Example. Let $m \in \mathbb{N}$ and consider the Dedekind complete Riesz space $\mathbb{R}^m$ (see Example 3.7). Let $D \in \mathcal{L}(\mathbb{R}^m)$ be an $m \times m$-diagonal matrix. Following 5.3 we know D is an orthomorphism. The fact that $\{D\}^c$ is an ideal is a consequence of the above theorem. However, this can also be shown by elementary means.

Suppose $A \in \mathcal{L}_n(\mathbb{R}^m) = \mathcal{L}(\mathbb{R}^m)$ commutes with D. Let E be an eigenspace of D for some eigenvalue λ. Take $v \in E$. We have $DAv = ADv = \lambda Av$. Therefore, $Av \in E$ holds and A leaves all eigenspaces of D invariant. Decomposing $\mathbb{R}^m = \bigoplus_{i=1}^n E_i$ in a direct sum of eigenspaces E_i of D yields a decomposition $A = \bigoplus_{i=1}^n A_i$ with $A_i : E_i \to E_i$.

Now suppose $B \in \mathcal{L}(\mathbb{R}^m)$ satisfies $|B| \leq |A|$. Write $B = [B_{ij}]_{i,j=1,\ldots,n}$ with $B_{ij} : E_i \to E_j$ as a blockmatrix acting on the eigenspaces E_i. The inequality $|B| \leq |A|$ implies each matrix entry of B is smaller or equal to the corresponding matrix entry of A. So we have $B_{ij} = 0$ for $i \neq j$. Therefore, B is also of the form $B = \bigoplus_{i=1}^n B_i$ with $B_i : E_i \to E_i$. So B leaves every eigenspace of D invariant. Let E be an eigenspace of D for some eigenvalue λ. Take $v \in E$. We have $BDv = \lambda Bv = DBv$, because Bv is in E. Since $\mathbb{R}^m$ is a direct sum of eigenspaces of D, it follows $BD = DB$. We conclude $B \in \{D\}^c$ and therefore $\{D\}^c$ is an ideal of $\mathcal{L}(\mathbb{R}^m)$. ■

8.3 The commutant taken in the orthomorphisms

For defining the order bicommutant we also need the notion of the commutant taken inside the orthomorphisms.

8.6 Definition. Let E be a Dedekind complete Riesz space. For a subset $\mathscr{A} \subset \mathcal{L}_n(E)$ we define $\mathscr{A}^o = \{S \in \mathrm{Orth}(E) : ST = TS \text{ for all } T \in \mathscr{A}\} = \mathrm{Orth}(E) \cap \mathscr{A}^c$ to be *the commutant of $\mathscr{A}$ taken in the orthomorphisms.*

For this commutant as well we obtain a nice structure.

8.7 Theorem. *Let E be a Dedekind complete Riesz space and let $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. The commutant $\mathscr{A}^o$ is an order closed full Riesz subalgebra*[6] *of $\mathcal{L}_b(E)$ and the linear span of its order projections is sequentially order dense.*

8.8 Example. Let $m \in \mathbb{N}$ and consider the Dedekind complete Riesz space $\mathbb{R}^m$ (see Example 3.7). By Example 5.3 $\mathrm{Orth}(\mathbb{R}^m)$ equals the algebra of diagonal matrices. Let $\mathscr{A} \subset \mathcal{L}_n(\mathbb{R}^m) = \mathcal{L}(\mathbb{R}^m)$. By the above theorem $\mathscr{A}^o$ is a Riesz subspace. However, this can also be shown by elementary means.

Take diagonal matrices $D_1, D_2 \in \mathscr{A}^o$. Fix $A \in \mathscr{A}$. Following Example 8.5 all eigenspaces of D_1 and D_2 are left invariant by A. Therefore, there is a decomposition $\mathbb{R}^m = \bigoplus_{i=1}^n E_i$ such that D_1 and D_2 act as a scalar on each E_i and E_i is left invariant by A. Write $D_1 = \mathrm{diag}(a_1, \dots, a_n)$ and $D_2 = \mathrm{diag}(b_1, \dots, b_n)$. The supremum $D_1 \vee D_2$ is given by $\mathrm{diag}(a_1 \vee b_1, \dots, a_n \vee b_n)$. Hence the diagonal matrix $D_1 \vee D_2$ works also as a scalar on each E_i. Since the E_i are left invariant by A, we know $D_1 \vee D_2$ commutes with A following 8.5. We conclude $D_1 \vee D_2 \in \mathscr{A}^o$ and therefore $\mathscr{A}^o$ is a Riesz subspace.

Here we prove the result for the case where $\mathscr{A}$ is a Riesz subspace of $\mathcal{L}_n(E)$. Since we need properties of the order bicommutant for showing the general statement, we defer the full proof to paragraph 9.1.

8.9 Lemma. *Let E be a Dedekind complete Riesz space and let $\mathscr{A} \subset \mathcal{L}_n(E)$ a Riesz subspace. The commutant $\mathscr{A}^o$ is an order closed full Riesz subalgebra of $\mathcal{L}_b(E)$.*

Proof. It is obvious $\mathscr{A}^o$ is an algebra. The fact that $\mathscr{A}^o$ is order closed follows immediately from 8.1. We show $\mathscr{A}^o$ is full in $\mathcal{L}_b(E)$. Take $T \in \mathscr{A}^o$ invertible in $\mathcal{L}_b(E)$. For each $S \in \mathscr{A}$ we have $TS = ST$. Applying T^{-1} on both sides of the previous identity yields $ST^{-1} = T^{-1}S$. Therefore, T^{-1} is contained in $\mathscr{A}^o$ by 5.7. Hence $\mathscr{A}^o$ is a full subalgebra of $\mathcal{L}_b(E)$.

We still have to show $\mathscr{A}^o$ is closed under the lattice operations. By the last two identities of 3.12 it is enough to show $\mathscr{A}^o$ is closed under taking the absolute value. Suppose $S \in \mathscr{A}^o$, we are going to show $|S| \in \mathscr{A}^o$. Since $\mathscr{A}$ is a Riesz subspace, every $T \in \mathscr{A}$ is a sum $T = T^+ - T^-$ of two positive elements of $\mathscr{A}$. Therefore, it is enough to show that $|S|$ commutes with all positive operators in $\mathscr{A}$. Take $A \in \mathscr{A}^+$. We prove $|S|$ commutes with A. Let $x \in E^+$. By the third identity of 3.12 we have $(Sx)^+ \in \mathcal{B}((Sx)^-)^\perp$ and $(Sx)^- \in \mathcal{B}((Sx)^+)^\perp$. Let P_+ be the order projection on the band $\mathcal{B}((Sx)^+)$ and, similarly, let P_- be the order projection on $\mathcal{B}((Sx)^-)$. Define $B = AP_+ - AP_- \in \mathcal{L}_n(E)$. We derive

$$A|Sx| = A[(Sx)^+] + A[(Sx)^-] = AP_+[(Sx)^+ - (Sx)^-] - AP_-[(Sx)^+ - (Sx)^-] = BSx.$$

[6] That is, a Riesz subspace and an algebra.

By 5.7 S commutes with P_+ and P_- and thus S commutes with $B = AP_+ - AP_-$. Moreover, B satisfies $-A \leq -AP_- \leq B \leq AP_+ \leq A$ by 4.17, 5.12 and the positivity of A. Hence $|B| \leq A$ holds. We obtain with the aid of 4.6 and 5.9 the following chain of inequalities

$$\begin{aligned} A|Sx| &= BSx = SBx \leq |SBx| \leq |S||Bx| \leq |S||B|x \\ &\leq |S|Ax = |S||Ax| = |SAx| = |ASx| \leq A|Sx|. \end{aligned}$$

It follows we actually have equality and, in particular, $A|Sx| = |S|Ax$. Using 5.9 once again, we conclude $A|S|x = A|Sx| = |S|Ax$ for all $x \in E^+$. Since an operator is completely determined by its action on the positive cone, we have $A|S| = |S|A$. Therefore, $|S|$ commutes with all positive $A \in \mathscr{A}$. So $|S|$ is in $\mathscr{A}^o$ and the claim follows. $\square$

Although the statement in 8.9 is not of full generality, we will need it as auxiliary result for proving 8.7.

9 Order bicommutant theorems

In the previous section we derived in Theorem 8.4 that the commutant in $\mathcal{L}_n(E)$ of a subset of the orthomorphisms is a band algebra. Moreover, when $\mathcal{H}$ is a Hilbert space and $\mathscr{D} \subset \mathcal{L}_b(\mathcal{H})$ a $*$-closed subset, the von Neumann bicommutant $\mathscr{D}''$ equals $\mathcal{P}(\mathscr{D}')'$ by 2.11. Since the sets $\mathscr{D}''$ and $\mathcal{P}(\mathscr{D}')'$ agree, there are multiple possibilities to define an analogue of the von Neumann bicommutant for Riesz spaces. In a Riesz space the band generated by the order projections is equal to the orthomorphism algebra by 5.4 and 5.15. Since, furthermore, order projections in the commutant correspond bijectively to reducing bands by the Projection Lemma 8.2, the following definition seems to be the most natural and promising analogue of the von Neumann bicommutant.

9.1 Definition. Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. The *order bicommutant* is the set $(\mathscr{A}^o)^c$.

Similarly as for the von Neumann bicommutant, we obtain that $(\mathscr{A}^o)^c$ equals its own order bicommutant.

9.2 Proposition. *Let E be a Dedekind complete Riesz space, $\mathscr{A} \subset \mathcal{L}_n(E)$ be a subset and $\mathscr{U} = (\mathscr{A}^o)^c$ be its order bicommutant. Then we have $\mathscr{U}^o = \mathscr{A}^o$ and $(\mathscr{U}^o)^c = \mathscr{U}$.*

Proof. Since $\mathscr{A}$ is contained in the order bicommutant $\mathscr{U}$, we have $\mathscr{U}^o \subset \mathscr{A}^o$. This shows one inclusion. Conversely, suppose $T \in \mathscr{A}^o$, then $T \in \mathrm{Orth}(E)$ commutes with all operators in $(\mathscr{A}^o)^c = \mathscr{U}$. Therefore, T is in $\mathscr{U}^o$. This shows the other inclusion. It follows $\mathscr{U}^o = \mathscr{A}^o$ and, taking the commutant once again, yields the second claim immediately. □

9.1 Q1: a description of the order bicommutant

The results of the previous section bring us directly to our first goal: describing the structure of the order bicommutant.

9.3 Theorem. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. The order bicommutant $(\mathscr{A}^o)^c$ is a band algebra containing the identity operator I.*

Proof. This follows immediately from 8.4. □

With this fact we are able to prove Theorem 8.7.

Proof of Theorem 8.7. Define $\mathscr{U} = (\mathscr{A}^o)^c$. By 9.3 this is a band, so in particular a Riesz subspace. So 8.9 yields $\mathscr{U}^o$ is an order closed full Riesz subalgebra of $\mathcal{L}_b(E)$. We have $\mathscr{A}^o = \mathscr{U}^o$ by 9.2. In combination with 9.5 below this gives the desired result. □

9.2 Q2: reflexivity

Similarly to the definition in paragraph 2.2, reflexive operator algebras are characterized by their invariant bands.

9.4 Definition. Let E be a Dedekind complete Riesz space. A subset $\mathscr{A} \subset \mathcal{L}_b(E)$ is *reflexive*, if it is equal to the algebra of order bounded operators, which leave invariant each band left invariant by every operator in $\mathscr{A}$.

To obtain our reflexivity result, we first need a consequence of the Freudenthal Spectral Theorem.

9.5 Lemma. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. Every orthomorphism $S \in \mathscr{A}^o$ is the order limit of a sequence in the linear span of $\mathcal{P}(\mathscr{A}^o)$.*

Proof. Let $S \in \mathscr{A}^o$ be an arbitrary orthomorphism. By 7.2 we know S is the order limit of a sequence in the linear span of $\{P_\alpha : \alpha \in \mathbb{R}\}$ with

$$P_\alpha = \sup\{I \wedge n(\alpha I - S)^+ : n \in \mathbb{N}\}$$

an order projection on E. Fix $\alpha \in \mathbb{R}$. Since I and S are in $\mathscr{A}^o$ and $\mathscr{A}^o$ is a Riesz subspace by 8.7, the elements $I \wedge n(\alpha I - S)^+$ are in $\mathscr{A}^o$. So P_α is in the commutant $\mathscr{A}^o$ by 8.1, since $I \wedge n(\alpha I - S)^+$ increases to P_α as $n \to \infty$. The result now follows. □

The above result may be compared with 2.10. Our reflexivity result will be a consequence of the following theorem.

9.6 Theorem. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(\mathcal{H})$ a subset. The order bicommutant $(\mathscr{A}^o)^c$ equals $\mathcal{P}(\mathscr{A}^o)^c$.*

Proof. The inclusion $(\mathscr{A}^o)^c \subset \mathcal{P}(\mathscr{A}^o)^c$ is trivial, because $\mathcal{P}(\mathscr{A}^o)$ is contained in $\mathscr{A}^o$. For the other inclusion take $T \in \mathcal{P}(\mathscr{A}^o)^c$. Then T commutes with all projections in $\mathscr{A}^o$. Let $S \in \mathscr{A}^o$. By 9.5 we know S is the order limit of a sequence $\{S_n\}_n$ in the linear span of $\mathcal{P}(\mathscr{A}^o)$. Clearly T commutes with S_n for each $n \in \mathbb{N}$. Therefore, T commutes with S by applying 8.1. Hence, we have $T \in (\mathscr{A}^o)^c$. Now the desired equality $(\mathscr{A}^o)^c = \mathcal{P}(\mathscr{A}^o)^c$ is obtained. □

Applying the Projection Lemma yields a description of the order bicommutant in terms of reducing bands, similar to 2.12 with 'invariant closed subspace' replaced by 'reducing band'.

9.7 Theorem. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. Denote by*

$$\mathscr{A}^{red} := \{T \in \mathcal{L}_n(E) : T \textit{ is reduced by every } \mathscr{A}\textit{-reducing band}\}.$$

We have $(\mathscr{A}^o)^c = \mathscr{A}^{red}$.

Proof. Suppose $T \in (\mathscr{A}^o)^c$. Let B be an $\mathscr{A}$-reducing band and denote by P the projection on B. By 8.2 P commutes with every $S \in \mathscr{A}$ and it follows $P \in \mathscr{A}^o$. So P commutes with T and therefore B is T-reducing by 8.2 again. It follows $T \in \mathscr{A}^{red}$, which shows one inclusion. Conversely, suppose we have $T \in \mathscr{A}^{red}$. Let $P \in \mathscr{A}^o$ be a projection on a band B. By Lemma 8.2 B is an $\mathscr{A}$-reducing band and therefore B reduces T. Again by 8.2, T commutes with P. Hence T commutes with all projections $P \in \mathscr{A}^o$. When applying 9.6, it follows T is in $\mathcal{P}(\mathscr{A}^o)^c = (\mathscr{A}^o)^c$, which implies the other inclusion. □

9.8 Example. Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathrm{Orth}(E)$. Since $\mathrm{Orth}(E)$ is commutative by 5.7, we have $\mathrm{Orth}(E) = \mathscr{A}^o$ and $\mathrm{Orth}(E) \subset \mathrm{Orth}(E)^c$. On the other hand, if we take $T \in \mathrm{Orth}(E)^c$, then T commutes with every order projection. Therefore, T leaves every band invariant by 8.2. So T is an orthomorphism. Consequently, we have $\mathrm{Orth}(E) = \mathrm{Orth}(E)^c$ and, moreover, $(\mathscr{A}^o)^c = \mathrm{Orth}(E)$. This shows $\mathrm{Orth}(E)$ is a maximal abelian subalgebra of $\mathcal{L}_b(E)$. Note the equality $(\mathscr{A}^o)^c = \mathrm{Orth}(E)$ also becomes clear by applying Theorem 9.7 above and observing $\mathscr{A}^{red} = \mathrm{Orth}(E)$. ■

However, in 2.12 we obtained a description in terms of invariant closed subsets. For this we used $*$-invariant subsets are reduced by invariant subspaces. To obtain a similar statement we require $\mathscr{A}$ to have the $*$-property.

9.9 Corollary. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ have the $*$-property. Denote by*

$$\mathscr{A}^{inv} := \{T \in \mathcal{L}_n(E) : T \text{ leaves every } \mathscr{A}\text{-invariant band invariant}\}.$$

We have $(\mathscr{A}^o)^c = \mathscr{A}^{inv} = \mathscr{A}^{red}$.

Proof. Let $T \in \mathscr{A}^{red}$. Suppose $B \subset E$ is an $\mathscr{A}$-invariant band. Since $\mathscr{A}$ has the $*$-property, it follows B reduces $\mathscr{A}$. So B reduces T and it follows T leaves B in particular invariant. We conclude $T \in \mathscr{A}^{inv}$. Conversely, let $T \in \mathscr{A}^{inv}$. Suppose a band $B \subset E$ reduces $\mathscr{A}$, then B and $B^\perp$ are $\mathscr{A}$-invariant. So B and $B^\perp$ are invariant under T. We conclude that B reduces T and therefore $T \in \mathscr{A}^{red}$ holds. Hence $\mathscr{A}^{red} = \mathscr{A}^{inv}$. By 9.7 we have $(\mathscr{A}^o)^c = \mathscr{A}^{red} = \mathscr{A}^{inv}$. □

Finally, we gain reflexivity for the order bicommutant of a subset of $\mathcal{L}_n(E)$ with the $*$-property.

9.10 Corollary. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ have the $*$-property. The order bicommutant $\mathscr{U} = (\mathscr{A}^o)^c$ is reflexive.*

Proof. First we show $\mathscr{U}$ has the $*$-property. Let therefore B be a band invariant under $\mathscr{U}$. Since $\mathscr{A}$ is clearly contained in $\mathscr{U}$, the band B is also $\mathscr{A}$-invariant. Since $\mathscr{A}$ has the $*$-property, B reduces $\mathscr{A}$. By 9.7 every $T \in \mathscr{U} = \mathscr{A}^{red}$ is reduced by B. We infer B reduces $\mathscr{U}$. Combining 9.2 and 9.9 we obtain $\mathscr{U} = \mathscr{U}^{inv}$ and hence $\mathscr{U}$ is reflexive. □

By combining the last two results, we obtain all reflexive subsets of $\mathcal{L}_n(E)$ with the $*$-property are order bicommutants.

9.11 Corollary. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ have the $*$-property. The equality $\mathscr{A} = (\mathscr{A}^o)^c$ holds if and only if $\mathscr{A}$ is reflexive.*

Theorem 9.7 yields another elegant proof of 9.3, without using 8.4.

Alternative proof for 9.3. By applying 9.7 we observe

$$(\mathscr{A}^o)^c = \mathscr{A}^{red} = \bigcap_{\substack{B \subset E \\ \text{reduces } \mathscr{A}}} \{T \in \mathcal{L}_n(E) : B \text{ reduces } T\}$$

is a band algebra, since it is an intersection of band algebras $\{T \in \mathcal{L}_n(E) : B \text{ reduces } T\}$ by 4.31. The fact that $(\mathscr{A}^o)^c$ contains the identity operator I is obvious. □

9.3 Q3: Schur's Lemma

For the commutant, taken inside the orthomorphisms, we now derive Schur's Lemma. It can immediately be deduced from the results obtained in the previous paragraph. As in the previous paragraph we again need the $*$-property, which is the analogue of $*$-invariance for Riesz spaces, to replace reducing bands by invariant bands.

9.12 Theorem (Schur's Lemma)**.** *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. The following statements are equivalent.*

i. The only reducing bands for $\mathscr{A}$ are the trivial ones: $\{0\}$ and E.

ii. The commutant taken in the orthomorphisms $\mathscr{A}^o$ consists of multiples of the identity operator $I \in \mathcal{L}_n(E)$.

Proof. Suppose (i) holds. Let $P \in \mathcal{P}(\mathscr{A}^o)$ be a projection in $\mathscr{A}^o$ and the band B its range. By the projection Lemma 8.2 B is $\mathscr{A}$-reducing. By assumption B is trivial and hence P is either 0 or I. Now applying 9.5 every $S \in \mathscr{A}^o$ is the order limit of some sequence in the linear span of the identity operator I in $\mathcal{L}_n(E)$. By 3.27 it follows S must be a multiple of the identity. We conclude that $\mathscr{A}^o$ consists of multiples of the identity operator. Conversely, assume (ii). Let $B \subset E$ be an $\mathscr{A}$-reducing band. By 8.2 the projection P on B is in $\mathscr{A}^o$. By assumption we have $\mathcal{P}(\mathscr{A}^o) = \{0, I\}$ and thus P is either 0 or I. We conclude that B is a trivial band. So the only bands that reduce $\mathscr{A}$ are the trivial ones. $\square$

Since the reducing and invariant bands for $\mathscr{A}$ coincide when $\mathscr{A}$ has the $*$-property, we immediately derive the following analogue of 2.15.

9.13 Corollary (Schur's Lemma $*$)**.** *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset with the $*$-property. The following statements are equivalent.*

i. The only invariant bands for $\mathscr{A}$ are the trivial ones: $\{0\}$ and E.

ii. The commutant taken in the orthomorphisms $\mathscr{A}^o$ consists of multiples of the identity operator $I \in \mathcal{L}_n(E)$.

9.14 Example. Consider Example 3.35 with F either $\ell^p(X)$ or $\mathbb{R}^X$ and $X = \mathbb{N}$. Consider the permutation $h : \mathbb{N} \to \mathbb{N}$, given by $h(1) = 2$, $h(n) = n - 2$ if $n \neq 1$ is odd and $h(n) = n + 2$ if n is even. Observe the cycle notation for h is given by

$$\ldots \to 9 \to 7 \to 5 \to 3 \to 1 \to 2 \to 4 \to 6 \to 8 \to \ldots.$$

Let $\mathscr{A} = \{T_h^k : k \in \mathbb{Z}\}$ with $T_h \in \mathcal{L}_n(F)$ given by $T_h f = f \circ h$.

Let B be an $\mathscr{A}$-reducing band. By 3.36 B is of the form $B_I = \{f \in F : f(n) = 0 \text{ for all } n \in I\}$ for some $I \subset \mathbb{N}$. Clearly, $\{0\} = B_{\mathbb{N}}$ and $F = B_{\emptyset}$ are $\mathscr{A}$-reducing. Now suppose $I \neq \emptyset$ is a proper subset of $\mathbb{N}$. There exists some $n, m \in I$ with $n \in I$ and $m \notin I$. From the cycle notation of h we see each number can be reached by applying h and h^{-1} a finite amount of times. Hence we have $m = h^k(n)$ for some $k \in \mathbb{Z}$. Observe the characteristic function $\chi_{\{m\}}$ on $\{m\}$ is contained in B_I. We have $T_h^k \chi_{\{m\}} = \chi_{\{n\}} \notin B_I$, hence T_h^k does not leave B_I invariant. It follows B_I cannot be $\mathscr{A}$-reducing. We conclude that the only $\mathscr{A}$-reducing bands are the trivial ones. By 9.12 it follows $\mathscr{A}^o$ consists of multiples of the identity operator. Consequently, we have $(\mathscr{A}^o)^c = \mathcal{L}_n(E)$. $\blacksquare$

9.4 Q4: approximation results

We follow a similar approach to obtain pointwise approximation as in 2.16. So for this we need the order equivalent 8.2 of the projection Lemma 2.9. In 2.9 we used that for a $*$-closed subset $\mathscr{A}$ of the bounded operators on a Hilbert space every $\mathscr{A}$-invariant closed subspace is $\mathscr{A}$-reducing. For Riesz spaces will therefore need the analogue of $*$-invariance: the $*$-property.

9.15 Proposition. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a unital band algebra with the $*$-property. For all $T \in (\mathscr{A}^o)^c$ and $y \in E$ there exists a net $\{T_\alpha\}_\alpha$ in $\mathscr{A}$ such that $T_\alpha y \xrightarrow{o} Ty$. Moreover, if T and y are positive there exists a net $\{T_\alpha\}_\alpha$ in $\mathscr{A}$ such that $0 \leq T_\alpha y \uparrow Ty$.*

Proof. Since $\mathscr{A}$ is a unital band, we have $\mathrm{Orth}(E) \subset \mathscr{A}$ by 5.4. Take $T \in (\mathscr{A}^o)^c$ and $y \in E$ arbitrary. Now consider the set $G = \{Ry : R \in \mathscr{A}\}$. We show G is an ideal of E. Firstly, observe $|Ry| = (Ry)^+ + (Ry)^-$ and $(Ry)^+ \perp (Ry)^-$ by 3.12. For $z \in E$ denote by $P_z \in \mathrm{Orth}(E)$ the projection on the band $\mathcal{B}(z)$ generated by z. Hence for $R \in \mathscr{A}$ we have

$$|Ry| = (P_{(Ry)^+} + P_{(Ry)^-})Ry \in G,$$

using $\mathscr{A}$ is an algebra and $\mathrm{Orth}(E) \subset \mathscr{A}$. Now, consider $z \in E$ such that $0 \leq |z| \leq |Ry|$ for some $R \in \mathscr{A}$. By 5.11 there exists $U \in \mathrm{Orth}(E) \subset \mathscr{A}$ such that $U|Ry| = z$. Since $|Ry|$ is in G and $\mathscr{A}$ is an algebra, we have $z = U|Ry| \in G$. It follows G is an ideal. Since G is $\mathscr{A}$-invariant, the band $\mathcal{B}(G)$ generated by G is $\mathscr{A}$-invariant by 4.37. Hence $\mathcal{B}(G)$ is $\mathscr{A}$-reducing, because $\mathscr{A}$ has the $*$-property. The band $\mathcal{B}(G)$ is also T-reducing by 9.7. Since $\mathscr{A}$ is unital, we have $y \in G \subset \mathcal{B}(G)$. We derive $Ty \in \mathcal{B}(G)$. Hence there exists a net $\{T_\alpha\}_\alpha$ in $\mathscr{A}$ such that $T_\alpha y \xrightarrow{o} Ty$, since $\mathcal{B}(G)$ is the order closure of the ideal G. The first claim follows. Now if T and y are both positive it follows by 3.39 there exists a net $\{T_\alpha\}_\alpha$ in $\mathscr{A}$ such that $0 \leq T_\alpha y \uparrow Ty$. This implies the second claim. $\square$

In contrast to the global approximation result 2.18 for the von Neumann bicommutant, we obtain an approximation result perturbed by orthomorphisms. This has to do with the fact that the diagonal algebra of a band algebra is not an ideal, let alone a band. So Proposition 9.15 cannot be applied on the diagonal algebra.

9.16 Lemma. *Let E be a Dedekind complete Riesz space and X a set. Consider the Riesz space E^X of functions $f : X \to E$. Let $\mathscr{A} \subset \mathcal{L}_b(E^X)$. For $R \in \mathcal{L}_b(E)$ define $R^\infty \in \mathcal{L}_b(E^X)$ given by $[R^\infty f](x) = R[f(x)], x \in X$. For $\mathscr{U} \subset \mathcal{L}_b(E^X)$ define $\mathscr{U}^\infty = \{R^\infty \in \mathcal{L}_b(E^X) : R \in \mathscr{A}\}$. The inclusion $((\mathscr{A}^o)^c)^\infty \subset ((\mathscr{A}^\infty)^o)^c$ holds.*

Proof. Let $A^\infty \in ((\mathscr{A}^o)^c)^\infty$ for some $A \in (\mathscr{A}^o)^c$ and $B \in (\mathscr{A}^\infty)^o \subset \mathrm{Orth}(E^X)$. By 5.3 there exists for each $x \in X$ an orthomorphism $B_x \in \mathrm{Orth}(E)$ such that B is given by $(Bf)(x) = B_x[f(x)]$. Let $C^\infty \in \mathscr{A}^\infty$ for some $C \in \mathscr{A}$, then B commutes with C^∞. Fix $x \in X$. For $y \in E$ let f_y be a function such that $f_y(x) = y$. We have for each $y \in E$

$$CB_x y = C[(Bf_y)(x)] = [C^\infty Bf_y](x) = [BC^\infty f_y](x) = B_x[(C^\infty f_y)(x)] = B_x C y.$$

So C commutes with B_x for all $C \in \mathscr{A}$. Therefore, B_x is an element of $\mathscr{A}^o$ for each $x \in X$. It follows A commutes with B_x for each $x \in X$. Now observe for all $f \in E^X$ and $x \in X$

$$[A^\infty Bf](x) = A[(Bf)(x)] = AB_x[f(x)] = B_x A[f(x)] = B_x[(A^\infty f)(x)] = [BA^\infty f](x).$$

Therefore, A^∞ commutes with B for each $B \in (\mathscr{A}^\infty)^o$. We conclude that A^∞ is contained in $((\mathscr{A}^\infty)^o)^c$ and the claim follows. $\square$

9.17 Lemma. *Let E be a Dedekind complete Riesz space and X a set, such that $\mathscr{A} \subset \mathcal{L}(E)$ has the $*$-property. Then $\mathscr{A}^\infty \subset \mathcal{L}(E^X)$ (as in 9.16) has the $*$-property too.*

Proof. Let $B \subset E^X$ be an $\mathscr{A}^\infty$-invariant band. By 3.36 there exists bands $B_x \subset E$ for each $x \in X$ such that B has the form

$$B = \{f \in E^X : f(x) \in B_x \text{ for all } x \in X\}.$$

Now let $T \in A$ and $y \in B_x$ for some $x \in X$. Let $f_y \in E^X$ be the function such that $f_y(x) = y$ and $f_y(z) = 0$ for $z \neq x$. Then we have $f_y \in B$ and therefore we have $T^\infty f_y \in B$. We conclude $B_x \ni [T^\infty f_y](x) = T[f_y(x)] = Ty$. So T leaves B_x invariant for each $x \in X$. Therefore, B_x is $\mathscr{A}$-invariant for all $x \in X$. Hence B_x is $\mathscr{A}$-reducing for each $x \in X$ as $\mathscr{A}$ has the $*$-property. Now let $T^\infty \in \mathscr{A}^\infty$ for some $T \in \mathscr{A}$. By 3.36 we know $B^\perp = \{f \in E^X : f(x) \in B_x^\perp \text{ for all } x \in X\}$. For all $f \in B^\perp$ we have $(T^\infty f)(x) = T[f(x)] \subset B_x^\perp$ using T leaves $B_x^\perp$ invariant. We conclude that $B^\perp$ is $\mathscr{A}^\infty$-invariant and therefore that $\mathscr{A}^\infty$ has the $*$-property. $\square$

9.18 Theorem. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a unital Riesz subalgebra with the $*$-property. For all $T \in ((\mathscr{A})^o)^c$ there exists a net $\{S_\alpha\}_\alpha$ in $\mathscr{A}$ such that for all $x \in E$ there exists a net $\{R_\alpha^x\}_\alpha$ of orthomorphisms such that $R_\alpha^x S_\alpha x \xrightarrow{o} Tx$.*

Proof. Define $\mathscr{U} = \mathscr{A}^+$ to be the positive part of $\mathscr{A}$. By 4.17 $\mathscr{U}$ is closed under multiplication. Consider the identity function $f \in E^E$ given by $f(x) = x$. Let $\mathscr{U}^\infty \subset \mathcal{L}_n(E^E)$ be as in 9.16. It is clear $\mathscr{U}^\infty$ consists of positive operators and is multiplicatively closed, since $\mathscr{U}$ satisfies those two properties. Consider the set $\mathscr{U}^\infty f \subset (E^E)^+$. Since $\mathscr{U}^\infty$ is closed under multiplication, $\mathscr{U}^\infty f$ is $\mathscr{U}^\infty$-invariant. By 4.38 the band $\mathscr{D} = \mathcal{B}(\mathscr{U}^\infty f)$ is $\mathscr{U}^\infty$-invariant. Let $T^\infty \in \mathscr{A}^\infty$ for some $T \in \mathscr{A}$. Then decompose $T = T^+ - T^-$ with $T^+, T^- \in \mathscr{A}^+ = \mathscr{U}$, using $\mathscr{A}$ is a Riesz subspace. We derive $T^\infty = (T^+)^\infty - (T^-)^\infty$ with $(T^+)^\infty, (T^-)^\infty \subset \mathscr{U}^\infty$. It follows $\mathscr{D}$ is invariant under T^∞ for all $T \in \mathscr{A}$. So $\mathscr{D}$ is $\mathscr{A}^\infty$-invariant.

Since $\mathscr{A}^\infty$ has the $*$-property by 9.17, $\mathscr{D}$ is $\mathscr{A}^\infty$-reducing. Let $T \in (\mathscr{A}^o)^c$. By Lemma 9.16 T^∞ is an element of $((\mathscr{A}^\infty)^o)^c$. By 9.7 it follows the band $\mathscr{D}$ reduces T^∞. The identity operator I is in $\mathscr{A}$ and thus also in $\mathscr{U}$. Hence I^∞ is an element of $\mathscr{U}^\infty$. Therefore, it follows f is in $\mathscr{D}$ and hence also $T^\infty f$ is an element of $\mathscr{D}$. So there exists a net $\{g_\alpha\}_\alpha$ in $\mathcal{E}(\mathscr{U}^\infty f)$ such that $g_\alpha \xrightarrow{o} T^\infty f$ implying $g_\alpha(x) \xrightarrow{o} [T^\infty f](x) = T[f(x)] = Tx$ for all $x \in E$ by 3.20. Fix α. By 3.39 there exists $S_1^\alpha, \dots, S_{n_\alpha}^\alpha \in \mathscr{U}$ and $\lambda_\alpha > 0$ such that

$$|g_\alpha| \le \lambda_\alpha \sum_{i=1}^{n_\alpha} (S_i^\alpha)^\infty f = S_\alpha^\infty f \text{ for } S_\alpha := \lambda_\alpha \sum_{i=1}^{n_\alpha} S_i^\alpha \in \mathscr{U},$$

where we used $\mathscr{A}$ is in particular a subspace and the sum S_α is in $\mathscr{A}^+ = \mathscr{U}$. Now fix $x \in E$. We have

$$|g_\alpha(x)| = |g_\alpha|(x) \le [S_\alpha^\infty f](x) = S_\alpha[f(x)] = S_\alpha x$$

By 5.11 there exists an orthomorphism $R_\alpha^x \in \mathrm{Orth}(E)$ such that $g_\alpha(x) = R_\alpha^x S_\alpha x$. We conclude that $R_\alpha^x S_\alpha x = g_\alpha(x)$ converges to Tx for all $x \in E$. $\square$

Currently, it is not clear whether a converse of the above theorem holds true. A promising approach is the following. Let T in $\mathcal{L}_n(E)$ and suppose there exists a net $\{S_\alpha\}_\alpha$ in $\mathscr{A}$ such that for all $x \in E$ there exists a net $\{R_\alpha^x\}_\alpha$ of orthomorphisms such that $R_\alpha^x S_\alpha x \xrightarrow{o} Tx$. Then for each $U \in \mathscr{A}^o$ we have $U(R_\alpha^x - R_\alpha^{Ux}) S_\alpha x \xrightarrow{o} TUx - UTx$ for all $x \in E$. So in order for T and U to commute, we need R_α^x and R_α^{Ux} to agree on $S_\alpha x$. Hence, we need a better understanding of the orthomorphisms R_α^x to answer this question. This is left for further research.

9.5 Q5: an order bicommutant theorem for atomic Riesz spaces

Since we do not obtain a global approximation result for the order bicommutant, we can not proceed in a similar way as in 2.19. In [PI] a bicommutant theorem is shown for atomic σ-complete Boolean algebras of projections on a Banach space[7]. This motivates us to focus on atomic Riesz spaces from now on. Here it is possible to avoid the diagonalization process.

9.19 Theorem. *Let E be a Dedekind complete atomic Riesz space. Let $\mathscr{A} \subset \mathcal{L}_n(E)$ be a subset with the $*$-property. We have $(\mathscr{A}^o)^c = \mathscr{A}$ if and only if $\mathscr{A}$ is a unital band algebra.*

Proof. If $(\mathscr{A}^o)^c = \mathscr{A}$ holds, then by Corollary 9.3 $\mathscr{A}$ is a band algebra with $I \in \mathscr{A}$. Conversely, suppose $\mathscr{A}$ is a unital band algebra. First, note $\mathrm{Orth}(E) \subset \mathscr{A}$ by 5.4, since $\mathscr{A}$ is a unital band. The inclusion $\mathscr{A} \subset (\mathscr{A}^o)^c$ is trivial. For the other inclusion take $T \in (\mathscr{A}^o)^c$ positive. Since E is atomic, there exists a maximal orthogonal system $S \subset E$ consisting of atoms. Let $x \in S$ be an atom. By 9.15 there exists a net $\{S^x_\alpha\}_\alpha$ in $\mathscr{A}$ such that $0 \le S^x_\alpha x \uparrow Tx$. For $x \in S$ denote by P_x the projection on the band $\mathcal{B}(x)$ generated by x. By 6.3 there exists a positive functional ζ_x such that $P_x y = \zeta_x(y)x$. Consider the net $\{S^x_\alpha P_x\}_\alpha$ in $\mathscr{A}$, where we used $\mathscr{A}$ contains $\mathrm{Orth}(E)$ and $\mathcal{P}(E) \subset \mathrm{Orth}(E)$ by 5.15. For all positive $y \in E$ we have

$$S^x_\alpha P_x y = \zeta_x(y) S^x_\alpha x \uparrow \zeta_x(y) Tx = TP_x y.$$

By 4.10 we conclude $S^x_\alpha P_x \uparrow TP_x$. We have $TP_x \in \mathscr{A}$ for each $x \in S$, since $\mathscr{A}$ is a band. Now by 6.6 the operators

$$S_H = \sum_{x \in H} P_x, H \subset S \text{ finite}$$

increase to the identity operator I. Since TP_x is in $\mathscr{A}$ for each $x \in S$, it follows TS_H is an element of $\mathscr{A}$. By 4.19 it follows $TS_H \uparrow T$. Since $\mathscr{A}$ is a band, T is in $\mathscr{A}$. We derive $((\mathscr{A}^o)^c)^+ \subset \mathscr{A}$. Now take $T \in (\mathscr{A}^o)^c$ arbitrary. Write $T = T^+ - T^-$ by 3.12. Because $(\mathscr{A}^o)^c$ is a band by 9.3, we observe T^+ and T^- are in $((\mathscr{A}^o)^c)^+$ and therefore in $\mathscr{A}$. It follows T is an element of $\mathscr{A}$. The other inclusion $(\mathscr{A}^o)^c \subset \mathscr{A}$ follows. We conclude $\mathscr{A} = (\mathscr{A}^o)^c$. □

Combining the above result with 9.11 we obtain the following corollary.

9.20 Corollary. *Let E be a Dedekind complete atomic Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ have the $*$-property. The subset $\mathscr{A}$ is reflexive if and only if $\mathscr{A}$ is a unital band algebra.*

We can sharpen 9.19, as is done in 2.21.

9.21 Corollary. *Let E be a Dedekind complete atomic Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ absolutely self-majorizing with the $*$-property. Then $(\mathscr{A}^o)^c$ equals* $\mathrm{bandalg}(\mathscr{A} \cup \{I\}) = \mathcal{B}(\mathrm{alg}(\mathscr{A} \cup \{I\}))$.

Proof. We denote $\mathscr{U} = \mathrm{bandalg}(\mathscr{A} \cup \{I\})$. The set $\mathscr{A} \cup \{I\}$ has the $*$-property, because $\mathscr{A}$ has the $*$-property and I leaves every band invariant. So $\mathscr{U}$ has the $*$-property by 4.35. It follows $(\mathscr{U}^o)^c = \mathscr{U}$ by 9.19. Moreover, $\mathscr{A}$ is contained in $\mathscr{U}$ and therefore we have $(\mathscr{A}^o)^c \subset (\mathscr{U}^o)^c = \mathscr{U}$. This shows one inclusion. For the other inclusion observe that $\mathscr{A}$ and I are contained in $(\mathscr{A}^o)^c$ and by 9.3 $(\mathscr{A}^o)^c$ is a band algebra. Hence $\mathscr{U} = \mathrm{bandalg}(A \cup \{I\})$ is contained in $(\mathscr{A}^o)^c$. This shows the other inclusion. Finally, the equality $\mathrm{bandalg}(\mathscr{A} \cup \{I\}) = \mathcal{B}(\mathrm{alg}(\mathscr{A} \cup \{I\}))$ is the content of Proposition 4.26. □

[7]See section 11 and [BA].

Combining Proposition 4.34 with the above we derive the following theorem.

9.22 Theorem. *Let E be an atomic Dedekind complete Riesz space and denote by $\mathrm{Aut}_c(E)$ the group of order continuous Riesz automorphisms on E. Let $\mathscr{A} \subset \mathrm{Aut}_c(E)$ a subgroup. The order bicommutant $(\mathscr{A}^o)^c$ equals $\mathcal{B}(\mathscr{A}) = \mathrm{bandalg}(\mathscr{A})$. Moreover, the band $\mathcal{B}(\mathscr{A})$ generated by $\mathscr{A}$ equals its order bicommutant.*

Proof. Following 4.34 $\mathscr{A}$ has the $*$-property. Furthermore, $\mathscr{A}$ consists of positive operators and is therefore absolutely self-majorizing. Consequently, we have $(\mathscr{A}^o)^c = \mathrm{bandalg}(\mathscr{A})$ by applying 9.21 and noting $\mathscr{A}$ contains the identity element I of $\mathrm{Aut}_c(E)$. Finally, realizing $\mathscr{A}$ is multiplicatively closed yields that $\mathcal{B}(\mathscr{A})$ is a band algebra containing $\mathscr{A}$ by 4.24. Hence we have $\mathrm{bandalg}(\mathscr{A}) = \mathcal{B}(\mathscr{A})$ and the first claim follows. By 4.35 the unital band algebra $\mathcal{B}(\mathscr{A}) = \mathrm{bandalg}(\mathscr{A})$ has the $*$-property. Applying 9.19 it follows $\mathcal{B}(\mathscr{A})$ equals its order bicommutant, which is the second claim. □

9.23 Example. Consider Example 3.35 with F either $\ell^p(X)$ or $\mathbb{R}^X$. With the aid of Example 6.5 we know F is an atomic Dedekind complete Riesz space. Let $\mathcal{S}(X)$ be the group of bijections on X. Let $H \subset \mathcal{S}(X)$ be a subgroup and consider the subset $\mathscr{A} = \{T_h : h \in H\}$, where $T_h \in \mathcal{L}_n(F)$ is given by $T_h f = f \circ h$. By Example 3.17 it follows that $\mathscr{A}$ is a subgroup of the order continuous Riesz automorphisms $\mathrm{Aut}_c(F)$. Following 9.22 the order bicommutant $(\mathscr{A}^o)^c$ equals the band $\mathcal{B}(\mathscr{A})$ generated by $\mathscr{A}$. Moreover, $\mathcal{B}(\mathscr{A})$ equals its order bicommutant. ■

It can be that $(\mathscr{A}^o)^c$ becomes the whole space $\mathcal{L}_n(E)$ as is shown in Example 9.14. However, this is not always the case.

9.24 Example. Consider the previous Example 9.23 and let $\#X \geq 3$. Let $h \in \mathcal{S}(X)$ a transposition of two elements $x, y \in X$ and consider the subgroup generated by H. Then $H = \{\mathrm{id}, h\}$ has two elements. Let $\mathscr{A} = \{T_h : h \in H\}$. The band $B = \{f \in F : f(x) = 0 = f(y)\}$ is $\mathscr{A}$-reducing. Let $z \in X$ with $z \neq x, y$ and $g \in \mathcal{S}(X)$ the transposition of the elements y and z. Clearly, T_g leaves $\mathscr{A}$ not invariant and by 9.7 T_g can not be contained in $(\mathscr{A}^o)^c$. This shows $(\mathscr{A}^o)^c$ is not the whole space $\mathcal{L}_n(E)$. ■

In 2.22 it is shown that each von Neumann algebra arises as the commutant of a group of unitaries. Also the order bicommutant is always the commutant of a group of orthomorphisms.

9.25 Proposition. *Let E be a Dedekind complete Riesz space and $\mathscr{A} \subset \mathcal{L}_n(E)$ a subset. The order bicommutant $(\mathscr{A}^o)^c$ equals $U(\mathscr{A}^o)^c$, where $U(\mathscr{A}^o)$ denotes the group of invertible orthomorphisms in $\mathscr{A}^o$.*

Proof. The statement that $U(\mathscr{A}^o)$ is a group, follows immediately from the fact that $\mathscr{A}^o$ is a unital full algebra in $\mathcal{L}_b(E)$ by 8.7. Since $U(\mathscr{A}^o)$ is contained in $\mathscr{A}^o$, it follows $(\mathscr{A}^o)^c \subset U(\mathscr{A}^o)^c$. Conversely, let $T \in U(\mathscr{A}^o)^c$ and $S \in \mathscr{A}^o$ and write $S = S^+ - S^-$. By 8.7 $\mathscr{A}^o \subset \mathcal{L}_n(E)$ is a Riesz subspace. So the operators $R_1 := S^+ + I$ and $R_2 := S^- + I$ are in $\mathscr{A}^o$. We have $R_1, R_2 \geq I$, so R_1 and R_2 are invertible in $\mathcal{L}_b(E)$ by 5.8. Since $\mathscr{A}^o$ is a full algebra in $\mathcal{L}_b(E)$, R_1 and R_2 are invertible in $\mathscr{A}^o$. Therefore, T commutes with R_1, R_2 and in particular with $S = R_1 - R_2$. This implies $T \in (\mathscr{A}^o)^c$. Finally, the other inclusion $U(\mathscr{A}^o)^c \subset (\mathscr{A}^o)^c$ follows. □

A consequence of the order bicommutant theorem for atomic Riesz spaces 9.19 is that every unital band algebra with the $*$-property arises as the commutant of a group of orthomorphisms.

9.26 Theorem. *Let E be a Dedekind complete atomic Riesz space. A subset $\mathscr{A} \subset \mathcal{L}_n(\mathcal{H})$ with the $*$-property is a unital band algebra if and only if $\mathscr{A}$ is the commutant of a group of orthomorphisms.*

Proof. Suppose $\mathscr{A}$ is a unital band algebra. It follows $\mathscr{A} = (\mathscr{A}^o)^c = U(\mathscr{A}^o)^c$ by 9.19 and 9.25. This shows $\mathscr{A}$ is the commutant of the group $U(\mathscr{A}^o)$ of orthomorphisms. Conversely, suppose $\mathscr{A} = G^c$ is the commutant of a group G of orthomorphisms on E. Proposition 8.4 yields $\mathscr{A} = G^c$ is a unital band algebra. $\square$

10 Conclusion

10.1 Summary of results

Let E be a Dedekind complete Riesz space. We answer the questions formulated in the introduction.

Q1: Description of the bicommutant

For each subset $\mathscr{A} \subset \mathcal{L}_n(E)$ the order bicommutant $(\mathscr{A}^o)^c$ is a unital band algebra. It is interesting that, if E is a Banach lattice with order continuous norm[8], then every $\mathscr{A} \subset \mathcal{L}_n(E)$ closed in the strong operator topology is order closed.[9] Hence it could be expected that the order bicommutant $(\mathscr{A}^o)^c$ is an order closed algebra.

Q2: Reflexivity

With the aid of the Freudenthal Spectral Theorem we retrieved that the order bicommutant of a subset $\mathscr{A} \subset \mathcal{L}_n(E)$ is completely determined by its reducing bands. To obtain reflexivity, we need more. For the von Neumann bicommutant this is proved by means of the adjoint in a Hilbert space. In Riesz spaces we have no natural counterpart of the adjoint. If $\mathcal{H}$ is a Hilbert space, then each closed subspace, that is invariant under a $*$-closed subset $\mathscr{U}$ of $\mathcal{L}_b(\mathcal{H})$, reduces $\mathscr{U}$. Therefore, we zoomed in on subsets $\mathscr{A}$ of $\mathcal{L}_n(E)$ such that each $\mathscr{A}$-invariant band is in fact $\mathscr{A}$-reducing. In that case $\mathscr{A}$ is said to have the $*$-property. The order bicommutant of subsets $\mathscr{A} \subset \mathcal{L}_n(E)$ with the $*$-property is indeed reflexive, similar to the situation for the von Neumann bicommutant. Moreover, a subset $\mathscr{A} \subset \mathcal{L}_n(E)$ with the $*$-property is reflexive if and only if $\mathscr{A}$ equals its order bicommutant.

Q3: Schur's Lemma

For subsets $\mathscr{A} \subset \mathcal{L}_n(E)$ we obtained a slightly different version of Schur's Lemma than known for Hilbert spaces. To wit, the commutant $\mathscr{A}^o$ consists of multiples of the identity if and only if the only $\mathscr{A}$-reducing bands are the trivial ones. Again we need the $*$-property to regain the original version of Schur's Lemma. That is, for subsets $\mathscr{A} \subset \mathcal{L}_n(E)$ with the $*$-property the commutant $\mathscr{A}^o$ consists of multiples of the identity if and only if the only $\mathscr{A}$-invariant bands are the trivial ones.

Q4: Approximation results

We derived the bicommutant of $\mathscr{A} \subset \mathcal{L}_n(E)$ is a unital band algebra. Approximation in order of an operator $S \in (\mathscr{A}^o)^c$ by a net of operators in $\mathscr{A}$ may therefore be expected, if $\mathscr{A}$ is a unital band algebra. Similarly, as for the von Neumann bicommutant, we obtain for each $x \in E$ there exists a net in $\mathscr{A}x$ converging in order to Sx, if $\mathscr{A}$ is a unital band algebra with the $*$-property. However, the diagonalization process carried out to obtain global approximation of the von Neumann bicommutant, does not work well for the order bicommutant. This has to do with the fact that the diagonal algebra $\mathscr{A}_\infty$ is not a band. We still gain an approximation result, which is however pointwise perturbed by orthomorphisms.

[8] That is, E is a Riesz space and a Banach space, such that $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ and $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$.

[9] Order converge of a net $\{S_\alpha\}_\alpha$ to S in $\mathcal{L}_b(E)$ implies $S_\alpha x \xrightarrow{o} Sx$ for all $x \in E$. By order continuity of the norm we have norm convergence $S_\alpha x \to Sx$ for all $x \in E$.

Q5: Bicommutant theorem

> Since the diagonalization process does not help us to get global approximation in order out of pointwise approximation, we searched for other methods. If E is atomic, the identity on E is approximated by finite sums of rank one order projections. This gets us around the obstruction mentioned above. We do retrieve global approximation here. So, if E is atomic and $\mathscr{A} \subset \mathcal{L}_n(E)$ has the $*$-property, then $\mathscr{A}$ equals its own order bicommutant $(\mathscr{A}^o)^c$ if and only if $\mathscr{A}$ is a unital band algebra. Moreover, if $\mathscr{A} \subset \mathcal{L}_n(E)$ is absolutely self-majorizing with the $*$-property, then the order bicommutant is the unital band algebra generated by $\mathscr{A}$. So, in the case E is atomic, we obtain an analogue of the von Neumann Bicommutant Theorem. Moreover, we derived the order bicommutant $(\mathscr{A}^o)^c$ is the commutant of the group $U(\mathscr{A}^o)$ of invertible orthomorphisms in $\mathscr{A}^o$. Combining facts gives that, if E is atomic, then each $\mathscr{A} \subset \mathcal{L}_n(E)$ with the $*$-property is a unital band algebra if and only if $\mathscr{A}$ is the commutant of some group of invertible orthomorphisms.

To obtain the above results we studied the commutant taken in the continuous operators $\mathscr{A}^c$ and taken in the orthomorphisms $\mathscr{A}^o$. It is interesting in its own right that $\mathscr{A}^o$ is an order closed Riesz space and full subalgebra of $\mathcal{L}_b(E)$ for subsets $\mathscr{A} \subset \mathcal{L}_n(E)$. Moreover, $\mathscr{A}^c$ is a band algebra, if $\mathscr{A}$ is a subset of $\mathrm{Orth}(E)$. Further, we mention there is progress made in operator algebras on Riesz spaces. Thus, we have a better understanding of the multiplicative structure on $\mathcal{L}_b(\mathcal{H})$ and introduced the concept of a band algebra. Furthermore, we answered some questions about invariance under a set of operators on a Riesz space. For the precise results we refer to section 4.

10.2 Further research

Perhaps the most pressing question is whether the bicommutant Theorem 9.19 can be extended beyond atomic spaces. It is still an open problem whether 9.19 holds for all Dedekind complete Riesz spaces. Riesz spaces with a weak order unit could be a good starting point for the extension of the theorem, because the order structure in these spaces is somewhat more convenient. Another option is to study alternative proofs of the von Neumann Bicommutant Theorem, which avoid the diagonalization process. This process is the main obstruction in generalizing the proof of the von Neumann bicommutant Theorem 2.19 to an order bicommutant theorem.

The second question, which immediately comes to mind, is whether a converse of Theorem 9.18 holds true (for certain classes of Riesz spaces). A first promising attempt below 9.18 shows we need a better understanding of the orthomorphisms R^x_α to answer this question. Classes of Riesz spaces for which the orthomorphisms are known seem to be a good starting point.

In the study of unitary representations on a Hilbert space, the von Neumann Bicommutant Theorem is a basic result. For atomic Riesz spaces we have shown there is also such a notion. Is this a good starting point for a study of groups representations? What can we derive from combining this with the study of group representations on Banach lattices, done in [WO]?

Furthermore, it is interesting to change the setting. For example, one could take the order bicommutant in the bounded operators $\mathcal{L}_b(E)$, instead of the order continuous operators $\mathcal{L}_n(E)$. Do our results remain valid? If E is a Banach lattice, then E has also a topological structure induced by a norm. Does this additional structure help us in proving more precise results?

Finally, there is a large variety of results on von Neumann algebras inferred from the von Neumann Bicommutant Theorem. In this thesis we only considered the five results, which in our opinion are the most fundamental ones. Of course, one could go further and study the possible existence of other analogues for Riesz spaces. There is also a lot of theory known for reflexive subsets of $\mathcal{L}_b(\mathcal{H})$. Do these results also have an analogue for Riesz spaces?

11 Discussion of related literature

In paragraph 1.2 of the introduction we have given a short overview of related work on the subject. In this section we will discuss it more thoroughly, moreover we will compare it to our own results.

It is a well-known problem, whether an analogue of the von Neumann Bicommutant Theorem holds for a set of operators $\mathscr{A}$ on a vector space X, which is not a priori a Hilbert space. We have been studying this question for X a Dedekind complete Riesz space. However, all literature on the subject treats the case for X a Banach space. Moreover, there are multiple ways to define the bicommutant in Hilbert spaces, since $\mathscr{D}''$ and $\mathcal{P}(\mathscr{D}')'$ agree for a $*$-closed subset $\mathscr{D}$ of the bounded operators on a Hilbert space. We decided to define the bicommutant by $(\mathscr{A}' \cap \mathrm{Orth}(E))'$, whereas all literature considers the classical bicommutant $\mathscr{A}''$. Thus, in the overview of related work given below we take X a Banach space and consider the classical bicommutant $\mathscr{A}''$ of a subset $\mathscr{A} \subset \mathcal{L}_b(X)$.

The first direction we will touch upon, is closely linked to our work. Here X is a Banach lattice, which is in particular a Riesz space. We consider the bicommutant $\mathscr{A}''$ for subsets $\mathscr{A}$ consisting of the multiplication operators in $\mathcal{L}_b(X)$. Here the commutant is defined by $\mathscr{A}' = \{S \in \mathcal{L}_b(X) : ST = TS \text{ for all } T \in \mathscr{A}\}$. In [PR] de Pagter and Ricker take the space X to be the Banach lattice $L^p(\mu)$ for $1 \leq p \leq \infty$. Here (Ω, Σ, μ) is a Maharam measure space, meaning that the associated measure algebra is a complete Boolean algebra and μ has the property that, whenever $\mu(F) > 0$, there is $E \in \Sigma$ with $E \subset F$ such that $0 < \mu(E) < \infty$. The following bicommutant theorems are derived [PR].

11.1 Theorem (Case $1 \leq p < \infty$)**.** *Let (Ω, Σ, μ) be a Maharam measure space and $1 \leq p < \infty$. Denote by M_ϕ for $\phi \in L^\infty(\mu)$ the multiplication operator on $L^p(\mu)$. Let $\mathscr{U}$ be a subalgebra of $L^\infty(\mu)$. The bicommutant of the subset*

$$R_p(\mathscr{U}) := \{M_\phi : \phi \in \mathscr{U}\} \subset \mathcal{L}_b(L^p(\mu))$$

is given by $R_p(\mathscr{D})$, where $\mathscr{D}$ is the closure of $U \cup \{\mathbf{1}\}$ in the weak-star topology $\sigma(L^\infty(\mu), L^1(\mu))$.

11.2 Theorem (Case $p = \infty$)**.** *Let (Ω, Σ, μ) be a Maharam measure space and $p = \infty$. Denote by M_ϕ for $\phi \in L^\infty(\mu)$ the multiplication operator on $L^p(\mu)$. Let $\mathscr{U}$ be a subalgebra of $L^\infty(\mu)$. The bicommutant of the subset*

$$R_p(\mathscr{U}) := \{M_\phi : \phi \in \mathscr{U}\} \subset \mathcal{L}_b(L^p(\mu))$$

is given by $R_p(\mathscr{D})$, where $\mathscr{D}$ is the Dedekind closure of $\mathscr{U} \cup \{\mathbf{1}\}$.

Note the Dedekind closure of a set $\mathscr{U} \subset L^\infty(\mu)$ is given by

$$\{f \in L^\infty(\mu) : \sup\{g \in \mathscr{U} : g \leq f\} = \inf\{g \in \mathscr{U} : f \leq g\}\}.$$

In [KI] Kitover takes the Banach lattice X equal to the continuous functions $C(K)$ on K. The following bicommutant theorem is derived

11.3 Theorem. *Let K be a metrizable, connected and locally connected compact space. Take $f \in C(K)$ and let M_f be the corresponding multiplication operator on $C(K)$. The bicommutant $\{M_f\}''$ equals the closure of the algebra generated by M_f and the identity operator I in the strong operator topology.*

In each of the three theorems above the bicommutant of a set $\mathscr{A} \subset \mathcal{L}_b(X)$ equals the closure of the algebra generated by $\mathscr{A} \cup \{I\}$ in a certain topology. Therefore these results are comparable with 9.21, in case of the order bicommutant, and 2.21, in case of the von Neumann bicommutant. Furthermore, in [KI] a necessary condition on K, for the above theorem to hold, is derived.

11.4 Theorem. *Let K be a metrizable compact space and S be the set of all isolated points of K. Suppose for all $f \in C(K)$ we have that $\{M_f\}''$ equals the strong closure of the algebra generated by M_f and the identity operator I. Then the set $K \setminus S$ is connected.*

Furthermore, Kitover considers in [KI] the case when the multiplier is either a polynomial or a non-decreasing (or non-increasing) function. Since the statements are rather technical and do not contribute to the scope of our discussion here, the interested reader is referred to [KI]. Finally, in [DI] Dieudonné showed there are Banach lattices X and algebras of multiplication operators $\mathscr{A}$ for which a bicommutant theorem does not hold.

Since the Banach lattices $L_p(\mu)$ for $1 \leq p \leq \infty$ as considered are in particular Dedekind complete Riesz spaces, we will investigate the applicability of the above statements to the order bicommutant. However, there is a fundamental obstruction. The multiplication operators are contained in the orthomorphism algebra. We have shown in 9.8 that in a Dedekind complete Riesz space E the order bicommutant $(\mathscr{A}^o)^c$ of a subset $\mathscr{A} \subset \mathrm{Orth}(E)$ always equals $\mathrm{Orth}(E)$. Therefore, the above results are not applicable to the order bicommutant. However, it leads to an interesting observation: the bicommutants $\mathscr{A}''$ and $(\mathscr{A}^o)^c = \mathcal{P}(\mathscr{A}^o)^c$ (see 9.6) do not agree in general, whereas they do coincide for a Hilbert space by 2.11. Finally, it is worthwhile noticing that in the proofs of the above facts, besides the algebraic and topological structure, the lattice structure on the Banach space X is used. A structure we also considered in our study of the order bicommutant.

Another direction in which research on an analogue of the von Neumann Bicommutant Theorem has evolved, is the case where X is a reflexive Banach space. Here there is no use of an order structure. Instead, some theory about modules is used. Let $\mathscr{U}$ be a unital Banach algebra. Recall a left $\mathscr{U}$-module X is cyclic, if there exists $x \in X$ with $\mathscr{U}x$ dense in X. Further, a left $\mathscr{U}$-module X is self-generating, if for each closed cyclic submodule $K \subset X$ the linear span of $\{T(X) : T : X \to K \text{ is an } \mathscr{U}\text{-module homomorphism}\}$ is dense in K. In [DA] Daws takes $\mathscr{A}$ to be the range of a bounded homomorphism, from a unital Banach algebra into $\mathcal{L}_b(X)$. We state the results from this article.

11.5 Theorem. *Let $\mathscr{U}$ be a unital Banach algebra and X a reflexive Banach space. Let $\pi : \mathscr{U} \to \mathcal{L}_b(X)$ a bounded homomorphism. Use π to turn E into a left $\mathscr{U}$-module. Suppose the left $\mathscr{U}$-module $\ell^2(E)$ is self-generating. Then $\pi(\mathscr{U})''$ agrees with the weak-star closure of $\pi(\mathscr{U})$ in $\mathcal{L}_b(E)$.*

We see the bicommutant of a unital algebra $\mathscr{A}$ equals the closure of $\mathscr{A}$ in a certain topology. Again, this can be compared with results 9.21 and 2.21. Furthermore, given a unital dual Banach algebra $\mathscr{U}$, there exists a reflexive Banach space X and an isometric homomorphism $\mathscr{U} \to \mathcal{L}_b(X)$ such that the range $\mathscr{A}$ equals its own bicommutant.

11.6 Theorem. *Let $\mathscr{U}$ be a unital dual Banach algebra. There exists a reflexive Banach space X and an isometric weak-star-continuous homomorphism $\pi : \mathscr{U} \to \mathcal{L}_b(X)$ such that $\pi(\mathscr{U})'' = \pi(\mathscr{U})$.*

In the last considered direction of research X is an arbitrary Banach space. The projections $\mathcal{P}(X)$ on X can be ordered by range inclusion. A Boolean algebra of projections $\mathscr{A} \subset \mathcal{P}(X)$ is σ-complete, if for each sequence $\{E_\alpha\}_\alpha$ in $\mathscr{A}$ the projections on the closed linear span of $\{E_n X : n \in \mathbb{N}\}$ and onto $\bigcap_n E_n X$ are in $\mathscr{A}$. For a complete discussion on σ-completeness see [BA]. For the bicommutant of a σ-complete Boolean algebra of projections, there are several bicommutant theorems. We first consider the theorems stated by de Pagter and Ricker in [PI].

11.7 Theorem. *Let X a Banach space and $\mathscr{A} \subset \mathcal{L}_b(X)$ be an atomic, σ-complete Boolean algebra of projections. Then the bicommutant $\mathscr{A}$ equals the strong operator closed algebra generated by $\mathscr{A}$.*

The above result did suggest us to look at an order bicommutant theorem 9.19 for atomic Riesz spaces. Another bicommutant theorem is obtained, when looking at the so called Lat($\mathscr{A}'$)-condition. A subset $\mathscr{A} \subset \mathcal{L}_b(X)$ satisfies the Lat($\mathscr{A}'$)-condition, if every closed $\mathscr{A}'$-invariant subspace, is the range of some projection from the commutant $\mathscr{A}'$.

11.8 Theorem. *Let X a Banach space and let $\mathscr{A} \subset \mathcal{L}_b(X)$ a σ-complete Boolean algebra of projections, satisfying the Lat($\mathscr{A}'$)-condition. Then the bicommutant $\mathscr{A}''$ equals the strong operator closed algebra generated by $\mathscr{A}$.*

Again the above two results may be compared with 9.21 and 2.21. In [PI] de Pagter and Ricker also give an example of a σ-complete Boolean algebra of projections, which does not satisfy the Lat($\mathscr{A}'$)-condition, but the bicommutant $\mathscr{A}''$ equals the strong operator closed algebra generated by $\mathscr{A}$. So the Lat($\mathscr{A}'$)-condition is sufficient, but not necessary. In the article [RO] Rosenthal and Sourour consider the above setting with the additional assumption that the Boolean algebra is cyclic. Observe a Boolean algebra $\mathscr{A}$ is cyclic, if there exists a vector $x \in X$ such that $\mathscr{A}x$ is all of X. The following result is stated in [RO].

11.9 Theorem. *Let X be a Banach space and $\mathscr{A} \subset \mathcal{L}_b(X)$ a strongly closed algebra of operators, which contains a σ-complete Boolean algebra of projections. If every invariant subspace of $\mathscr{A}$ has an invariant complement, then $\mathscr{A}''$ is equal to $\mathscr{A}$. Moreover, $\mathscr{A}$ is reflexive.*

Indeed a bicommutant theorem is retrieved as in 9.19 and 2.19. Moreover, the reflexivity statement is inferred as in 9.11 and 2.14. The fact that every invariant subspace of $\mathscr{A}$ has an invariant complement, may be compared with -invariance and the -property for Hilbert spaces, respectively Riesz spaces. This result is generalized to quasi-cyclic Boolean algebras of projections in [DB]. Observe that in our case it is also possible to take the order bicommutant of a set of projections. However, 9.8 shows $(\mathscr{A}^o)^c$ equals Orth(E) in that case. So these theorems on the bicommutant of a Boolean algebra of projections do not have an analogue for the order bicommutant.

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