G. L. van der Sluijs

Change ringing

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Supervisor: dr. M. J. Bright

Mathematisch Instituut, Universiteit Leiden

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Introduction

This bachelor thesis will be concerned with the old English art of ringing church bells called *change ringing*. The development of change ringing in the early 17th century was mainly due to the invention of the full-circle wheel on which the bells were mounted. By pulling a rope, a bell would make a rotation of almost 360 degrees with a period of approximately two seconds. The time between two strikes of the same bell could be controlled rather accurately, which made it possible to ring a certain number of bells all after each other and keep repeating this in the same order.

A practised bell ringer could also adjust the rotation period of his bell slightly. Two neigboring bells might be swapped by increasing the rotation period of the first bell and decreasing the rotation period of the second. In this fashion one could change the order in which the bells were rung. In such a transition from one ordering of the bells to the order, each bell could not move more than one place, since the rotation period of a bell could only be adjusted a little bit. Bell ringers became interested in ringing the bells in every possible order after each other, whereby the bells were not allowed to be rung more than once in the same order; that is, they wanted to ring an *extent*. Thus the art of change ringing was born.

Along with the development of change ringing came the (mathematical) study of ringing patterns, which is sometimes called *campanology* (from the Latin word *campana*, which means 'bell'). Change ringers tried to create *methods* to ring all possible orderings of the bells in a smart way. An important 17th century study on change ringing was done by Fabian Stedman in his book *Campanalogia* (1677, see [3]). Initially, most of the campanologists were change ringers, but in the 20th century also mathematicians became interested in this field of study. For an extensive mathematical description of change ringing, the reader may consult [7] and [1]; [2] elaborates on Fabian Stedman as someone using group theoretical tools before the actual development of group theory.

This thesis gives a mathematical analysis of change ringing. In Section 2 we will introduce the important notion of a Cayley graph and prove the existence of an extent for an arbitrary number of bells. A historically interesting question about a method called Grandsire Triples will be answered in Section 3, and Section 4 covers Rankin's campanological theorem, which is closely related to Section 3. Section 5 will be concerned with the existence and construction of extents using specific methods.

1 Preliminaries

1.1 Change ringing terminology

Suppose we have a set of $n \in \mathbb{Z}_{>0}$ bells. In the art of change ringing, an ordering of the bells is called a *row*. There are $n!$ different rows, each of them corresponding to a specic order in which the bells are rung. More interesting than the rows are the transitions from one row to the other. For technical reasons, a transition is only allowed if each bell moves no more than one place; such a transition is called a *change*. More formally, rows and changes are defined as follows.

Definition 1.1. A *row* is a bijection $r : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and a *change* is a permutation $\sigma \in S_n \setminus \{(1)\}$ which is the product of disjoint transpositions of the form $(k\; k+1)$.

We view a row as a map that assigns a unique bell to each bell position and a change as a permutation of the set of bell positions. Precomposition defines a right action of S_n on the the set of rows; this action defines the application of a change to a row. Despite of the different interpretation of rows and changes, we sometimes want to treat them both as elements of the same group S_n. From this perspective, applying a change to a row comes down to right multiplication: applying the change σ to the row r yields the new row $r\sigma$.

The following notation visualizes the positions of the bells in a row.

Notation 1.2. A row r is denoted as $r(1)r(2)\ldots r(n)$.

Examples of special rows are $12\ldots n$ (*rounds*), which is the identity map on $\{1, \ldots, n\}$, and $n \ldots 21$ (*back rounds*), which is the map $x \mapsto n + 1 - x$.

Bells that do not move in a change are said to 'make places'. Given the number of bells n that is used, a change is determined by its fixed points (the positions in which the bells make places). This observation gives rise to the so-called *place notation* for changes, which will often be used instead of the usual cycle notation.

Notation 1.3. The change with fixed points $i_1, \ldots, i_k$ is denoted by $a_{i_1 \ldots i_k}$ and the change without any fixed points is denoted by $a_\times$.

Obviously, not every choice of positions in which the bells make places defines a change. If n is odd, for example, the change $a_\times$ is ill-defined. In general, the parity of the number of fixed points of a change equals the parity of n.

Example 1.4. Consider the following successive rows (for $n = 6$ bells), starting with rounds:

$$\begin{gathered} 123456 \\ 214365 \\ 241356 \\ 142356 \end{gathered}$$

Note that the first two transitions $\sigma_1 = (12)(34)(56)$ and $\sigma_2 = (23)(56)$ are changes; in place notation we can write $\sigma_1 = a_\times$ and $\sigma_2 = a_{14}$. The transition from the first row r_1 to the third row r_3 is described by the identity $r_3 = r_1\sigma_1\sigma_2$. The transition (13) from the third to the fourth row is not a change since the bells in position 1 and 3 move more than one place.

The following relation between the number c_n of possible changes using n bells and the $(n+1)$th Fibonacci number F_{n+1} can be found in [1, Theorem 2.1].

Theorem 1.5. *For $n \geq 1$ we have $c_n = F_{n+1} - 1$.*

Proof. We proceed by induction on n. Note that $c_1 = 0 = F_2 - 1$ and $c_2 = 1 = F_3 - 1$. Assume that the result is true for all $k \leq n$ for some $n \geq 2$ and consider the number of possible changes c_{n+1} for $n+1$ bells. There are c_n changes fixing position $n+1$ and c_{n-1} changes exchanging the positions n and $n+1$ and at least two other positions. Finally, there is the transposition $(n\ n+1)$. This yields

$$c_{n+1} = c_n + c_{n-1} + 1 = (F_{n+1} - 1) + (F_n - 1) + 1 = F_{n+2} - 1.$$

□

The aim of change ringing is to ring all possible rows in a sequence without any repetition. Traditionally one starts and ends with rounds, thus getting a sequence of $n! + 1$ rows of which only the first and the last are identical. Such a sequence is called an *extent*. In more mathematical terms it can be defined as follows.

Definition 1.6. An *extent* on n bells is a $n! + 1$-tuple $(r_1, \ldots, r_{n!+1}) \in (S_n)^{n!+1}$ satisfying

1. $r_1 = r_{n!+1} = (1)$;
2. $r_i \neq r_j$ for all $1 \leq i < j \leq n!$;
3. There are changes $\sigma_1, \ldots, \sigma_{n!} \in S_n$ such that $r_{i+1} = r_i\sigma_i$ for $i = 1, \ldots, n!$.

A shorter sequence of rows satisfying the above conditions is called a *touch* on n bells.

Ringing an extent on n bells requires remembering a sequence of $n!$ changes, which can be a difficult job, even for relatively small n like 6 or 7 (we have $6! = 720$ and $7! = 5040$). In order to make this easier, change ringers often use a *method*, a sequence of changes, described by a specific algorithm that is relatively simple. A method usually consists of repeating one or more short blocks of changes; a sequence of rows produced by such a block is called a *lead*. The first row of a lead is called the *lead head* and the last row the *lead end*. Typically, the first bell, called the *treble*, is in first position in these rows.

The following lemma will be used several times.

Lemma 1.7. *Let $n \geq 4$ and define the changes*

$$\alpha = \begin{cases} a_\times, & \text{if } n \text{ is even} \\ a_n, & \text{if } n \text{ is odd} \end{cases}, \quad \beta = \begin{cases} a_{1n}, & \text{if } n \text{ is even} \\ a_1, & \text{if } n \text{ is odd} \end{cases}$$

for n bells. Then we have $\langle \alpha, \beta \rangle = D_n$.

Proof. We consider the regular n-sided polygon with its vertices labelled as $2, 4, \ldots, n, n-1, n-3, \ldots, 1$ for even n and $2, 4, \ldots, n-1, n, n-2, \ldots, 1$ for odd n. Then α and β are reflections of this polygon. The reflection α and the rotation

$$
\begin{aligned}
\alpha\beta &= \begin{cases} (12)\cdots(n-1\ n)(23)\cdots(n-2\ n-1), & \text{if } n \text{ is even} \\ (12)\cdots(n-2\ n-1)(23)\cdots(n-1\ n), & \text{if } n \text{ is odd} \end{cases} \\
&= \begin{cases} (2\,4\ldots n\ n-1\ n-3\ldots 1), & \text{if } n \text{ is even} \\ (2\,4\ldots n-1\ n\ n-2\ldots 1), & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

of order n together generate D_n. It follows that $\langle \alpha, \beta \rangle = D_n$. $\square$

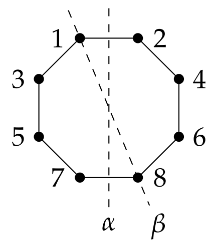

Figure 1: For $n = 8$ the reflections $\alpha = a_\times$ and $\beta = a_{18}$ generate D_8.

Example 1.8. Consider the method *Plain Bob Minimus* on 4 bells (see also [2, p. 774]), which is shown in Figure 2. A line denotes the end of a lead so that each column corresponds to one lead. The rows of the first column, which are produced by an alternation of the changes $\alpha = a_\times = (12)(34)$ and $\beta = a_{14} = (23)$, make up the dihedral group $D_4 = \langle \alpha, \beta \rangle$ (Lemma 1.7).

The first lead is followed by the change $\gamma = a_{12} = (34)$ and after that the same sequence of alternating α's and β's. This means that the second lead is the left coset bD_4 where $b = (\alpha\beta)^3\alpha\gamma = (234)$. Likewise, the third lead is the left coset b^2D_4. So the Plain Bob Minimus method can be seen as the decomposition of S_4 into the three left cosets D_4, bD_4 and b^2D_4 of D_4. Since these cosets form a partition of S_4, we know immediately that this is indeed an extent.

1234	1342	1423
2143	3124	4132
2413	3214	4312
4231	2341	3421
4321	2431	3241
3412	4213	2314
3142	4123	2134
1324	1432	1243
		1234

Figure 2: Plain Bob Minimus.

1.2 Words

The notion of *words* will be helpful to describe the succession of changes.

Definition 1.9. Let G be a group and $X \subset G$ a subset. A *word* in G is a formal expression $w = g_1.g_2 \dots g_k$ with *letters* $g_1, g_2, \dots, g_k \in G$. The *length* of w is $|w| = k$ and a word $u = g_i.g_{i+1} \dots g_j$ with $1 \leq i \leq j \leq k$ is called a *subword* of w. By $W(G, X)$ we denote the set of words in G, made up of letters from X.

If a word contains repetitions of subwords, we use exponential notation. For example, the word $g_1.g_1.g_1.g_2.g_3.g_2.g_3.g_1$ can be written as $g_1^3.(g_2.g_3)^2.g_1$.

Let G be a group and $X \subset G$ a subset. A product operation on $W(G, X)$ can be defined by concatenation:

$$g_1.g_2 \dots g_k \cdot h_1.h_2 \dots h_l = g_1.g_2 \dots g_k.h_1.h_2 \dots h_l.$$

Also we define the evaluation map

$$\begin{aligned} \overline{\cdot} : \quad & W(G, X) \to G \\ & g_1.g_2 \cdots g_k \mapsto \overline{g_1.g_2 \cdots g_k} = g_1 g_2 \cdots g_k. \end{aligned}$$

It is surjective if and only if X is a generating set for G.

A word $w \in W(G, X)$ can be considered as the product of its letters. However, we will always separate its letters by dots to avoid confusion with its evaluation $\overline{w} \in G$.

Definition 1.10. Let G be a group with neutral element e. A word w in G is called an *irreducible identity word* if $\overline{w} = e$ and $\overline{u} \neq e$ for every proper subword u of w.

2 The existence of an extent

2.1 Plain changes

An important question one might ask, is whether ringing an extent on n bells is possible for all $n \geq 2$. Notice that there is no extent on 1 bell since the neutral element $(1) \in S_1$ is not a change. Obviously, an extent on 2 bells does exist, and as early as 1677, Fabian Stedman demonstrated in [3, pp. 16–20] how to ring an extent on 3, 4 and 5 bells by using a method called *plain changes*, which is nowadays known as the (Steinhaus–)Johnson–Trotter algorithm (see [4] for S. M. Johnson's original description of it). The name *plain changes* indicates that in every change only two bells are exchanged. The treble *plain hunts*, which means that it occupies the successive positions $1, 2, \ldots, n, n, n-1, \ldots, 1, 1, 2, \ldots$ While the treble is moving, the other bells remain in the same relative order. When the treble makes places, the other bells behave as if they were doing plain changes with one fewer bell (i.e. the second bell plain hunts).

If r is a row of $n-1$ bells, we define $b_i = r(i) + 1$ for $i = 1, \ldots, n-1$. Now for $j = 1, \ldots, n$ we construct rows

$$r_j = b_1 b_2 \ldots b_{j-1} 1 b_j b_{j+1} \ldots b_{n-1}$$

of n bells by 'inserting' bell 1 in every possible position. Suppose that an extent of plain changes with $n-1$ bells is given. By replacing each even row r by the successive rows $r_1, \ldots, r_n$ and each odd row r by the successive rows $r_n, \ldots, r_1$, we obtain an extent of plain changes for n bells (for the case $n = 4$ this construction is shown in Figure 3). This argument could be turned into a proof that an extent on n bells exists for all $n \geq 2$ by using induction on n. However, in Section 2.3 we choose an alternative route, proving in addition the existence of an extent that uses only three different changes.

234	342	423		**1**234	**1**342	**1**423
				2**1**34	3**1**42	4**1**23
				23**1**4	34**1**2	42**1**3
				234**1**	342**1**	423**1**
324	432	243	$\rightarrow$	324**1**	432**1**	243**1**
				32**1**4	43**1**2	24**1**3
				3**1**24	4**1**32	2**1**43
				1324	**1**432	**1**243
		234				**1**234

Figure 3: The construction of plain changes with 4 bells out of plain changes with 3 bells. The bell in bold is the inserted treble.

2.2 The Cayley graph

We now introduce the notion of a Cayley graph. It will appear to be closely related to words.

Definition 2.1. Let G be a group and $X \subset G$ a subset. The *Cayley graph* $\Gamma(G, X)$ is the directed graph with vertex set G and arrow set $\{(g, gx) : g \in G, x \in X\}$.

If X only contains elements of order 2 (for example if its elements are changes), there is an arrow from vertex v to vertex w if and only if there is one from w to v. In this case we can consider the Cayley graph as an undirected graph.

Remark 2.2. If G is a group and $X \subset G$ a subset, the Cayley graph $\Gamma(G, X)$ is connected if and only if X is a generating set of G.

Definition 2.3. Let $\Gamma = (V, E)$ be an undirected graph. A *k-cycle* in Γ is a $k+1$-tuple $(v_1, v_2, \dots, v_k, v_1)$ with $v_1, \dots, v_k \in V$ all distinct such that $\{v_i, v_{i+1}\} \in E$ for $i = 1, \dots, k \mod k$. A k-cycle in a directed graph is defined analogously.

Definition 2.4. A $\#V$-cycle in a graph $\Gamma = (V, E)$ is called a *Hamiltonian cycle*. A graph is called *Hamiltonian* if it contains a Hamiltonian cycle.

The next theorem, which clarifies the relation between extents, words and Cayley graphs, is crucial (see for instance [1, Theorem 2.2] for the equivalence (1) $\Leftrightarrow$ (3)).

Theorem 2.5. *Let n be a positive integer and $X \subset S_n$ a set of changes. The following are equivalent:*

1. *There is an extent on n bells using only changes from X.*
2. *There is an irreducible identity word $w \in W(S_n, X)$ of length $|w| = n!$.*
3. *The Cayley graph $\Gamma(S_n, X)$ is Hamiltonian.*

Proof. Let $(r_1, \dots, r_{n!+1})$ be an extent on n bells using changes $g_i = r_i^{-1} r_{i+1} \in X$ for $i = 1, \dots, n!$. The word $w = g_1.g_2 \dots g_{n!} \in W(S_n, X)$ satisfies $\overline{w} = (1)$ since we have $r_1 = r_{n!+1} = r_1 \overline{w}$. Furthermore, if a proper subword $u = g_l.g_{l+1} \cdots g_m$ of w satisfied $\overline{u} = (1)$, then we would have $r_l = r_l \overline{u} = r_{m+1}$, which is a contradiction. Hence w is an irreducible identity word, which proves (1) $\Rightarrow$ (2). If $w = g_1.g_2 \dots g_{n!} \in W(S_n, X)$ is an irreducible identity word, then the subwords $g_1.g_2 \dots g_k$ of w for $k = 1, \dots, n!$ evaluate to $n!$ different elements of S_n. Hence $((1), g_1, g_1 g_2, \dots, g_1 g_2 \cdots g_{n!})$ is a Hamiltonian cycle in $\Gamma(S_n, X)$, which proves (2) $\Rightarrow$ (3). Finally, the implication (3) $\Rightarrow$ (1) is clear. $\square$

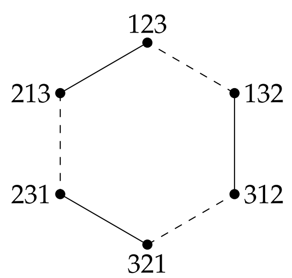

Figure 4: The Cayley graph $\Gamma(S_3, \{(12), (23)\})$. The (12) changes are denoted by solid lines and the (23) changes by dashed lines.

Example 2.6. There are two possible extents on 3 bells. They correspond to the two Hamiltonian cycles in the Cayley graph $\Gamma(S_3, \{(12), (23)\})$ (see Figure 4) and the two irreducible identity words $((12).(23))^3$ and $((23).(12))^3$ of length 3! in $W(S_3, \{(12), (23)\})$.

2.3 Existence of an extent using only three changes

We will now prove the existence of an extent on $n \geq 2$ bells using only three different changes (compare [1, pp. 206–209]). Following the proof of [5], we need the lemma below.

Lemma 2.7. *Let $\Gamma = (V, E)$ be a connected 3-regular undirected graph. Suppose that there exist a set C of cycles and a set Q of 4-cycles, each partitioning V, such that there is no $c \in C$ containing all vertices of some $q \in Q$. Then Γ is Hamiltonian.*

Proof. We define the following 'gluing operation'. Suppose we have a 4-cycle $q = (v_1, v_2, v_3, v_4, v_1) \in Q$ and two different cycles $c_1, c_2 \in C$ such that the edge $\{v_1, v_2\}$ is contained in c_1 and the edge $\{v_3, v_4\}$ is contained in c_2. Then we glue c_1 and c_2 together by removing the edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ and adding the edges $\{v_2, v_3\}$ and $\{v_1, v_4\}$, thus making a new cycle c. By repeating this gluing operation until it cannot be done any more, we get a set C' of cycles, still partitioning V.

We claim that C' contains only one cycle, which then has to be Hamiltonian. If not, C' must contain two different cycles c'_1 and c'_2, joined by an edge e that connects the vertices x_1 in c'_1 and x_2 in c'_2. By 3-regularity and the assumption that c'_1 and c'_2 cannot be glued together, it follows that e is not contained in any 4-cycle of Q. But then the 4-cycle $q \in Q$ that contains x_1 must also contain the two edges $\{a, x_1\}$ and $\{b, x_1\}$ in c'_1 incident to x_1. Note that the vertices a, b, x_1 belong to the same cycle $c_1 \in C$ and to the same 4-cycle $q \in Q$. Therefore, the fourth vertex of q cannot belong to c_1. But by 3-regularity it cannot be contained in another cycle $c \in C$ either, which is a contradiction. $\square$

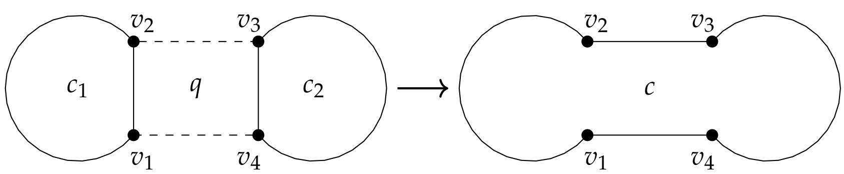

Figure 5: The gluing operation in the proof of Lemma 2.7.

Lemma 2.8. *For $n \geq 2$ the permutations*

$$\alpha = (12), \quad \beta = (12)(34)(56)\cdots, \quad \gamma = (23)(45)(67)\cdots$$

generate S_n.

Proof. Since the case $n = 2$ is clear, we assume that $n \geq 3$. Fix $a \in \{2, 3, \ldots, n\}$. It follows from

$$\alpha\beta\gamma = \begin{cases} (2\,4\,6\ \ldots\ n\ n-1\ n-3\ \ldots\ 3), & \text{if } n \text{ is even} \\ (2\,4\,6\ \ldots\ n-1\ n\ n-2\ \ldots\ 3), & \text{if } n \text{ is odd} \end{cases}$$

that we can choose a positive integer k such that $(\alpha\beta\gamma)^k$ maps 2 to a, yielding $(\alpha\beta\gamma)^k\alpha(\alpha\beta\gamma)^{-k} = (1a)$. Since any transposition (ab) with $a, b \neq 1$ can be written as $(ab) = (1a)(1b)(1a)$ and S_n is generated by its transpositions, this concludes the proof. □

Theorem 2.9. *For $n \geq 4$ the Cayley graph $\Gamma(S_n, \{\alpha, \beta, \gamma\})$ with*

$$\alpha = (12), \quad \beta = (12)(34)(56)\cdots, \quad \gamma = (23)(45)(67)\cdots$$

is Hamiltonian.

Proof. The Cayley graph $\Gamma = \Gamma(S_n, \{\alpha, \beta, \gamma\})$ is 3-regular, and by Lemma 2.8 and Remark 2.2 it is connected. We will show that the other conditions of Lemma 2.7 are satisfied. For any vertex the irreducible identity word $(\alpha.\beta)^2$ corresponds to a 4-cycle in Γ, and if two of such 4-cycles share a common vertex, they are identical. This yields a partition Q of S_n consisting of 4-cycles. Likewise, the irreducible identity word $(\alpha.\gamma)^3$ (for $n = 4$) or $(\alpha.\gamma)^6$ (for $n > 4$) corresponds to a partition C of S_n, consisting of 6-cycles or 12-cycles, respectively. If all vertices of a 4-cycle in Q were contained in some cycle in C, we could write β as a product of alternating α's and γ's. However, this is not possible. Hence we can apply Lemma 2.7 and find that Γ is Hamiltonian. □

Corollary 2.10. *An extent on n bells exists for all $n \geq 2$. Moreover, this can be accomplished by using only the changes*

$$\alpha = (12), \quad \beta = (12)(34)(56)\cdots, \quad \gamma = (23)(45)(67)\cdots.$$

Proof. The case $n = 2$ is trivial and the case $n = 3$ follows from Example 2.6. For $n \geq 4$, Theorems 2.5 and 2.9 together give the result. □

3 Grandsire Triples

3.1 Description and basic properties

We now consider a method on 7 bells called *Grandsire Triples* (for a more extensive description of Grandsire Triples see [6, pp. 13–22]). Three different lead types occur in this method: *plain leads*, *bob leads* and *single leads*, each consisting of 14 rows. These three lead types are described by the words

$$L_P = a_3.(a_1.a_7)^6.a_1, \quad L_B = a_3.(a_1.a_7)^5.a_1.a_3.a_1, \quad L_S = a_3.(a_1.a_7)^5.a_1.a_3.a_{123},$$

respectively. The successive letters of the word that defines a certain lead type are the changes producing a lead of that type, where we read from left to right. Notice that a lead is determined by its lead head and lead type. We will write $P = \overline{L_P}$, $B = \overline{L_B}$ and $S = \overline{L_S}$ for the evaluated values in S_7 of each lead type.

Plain lead	**Bob lead**	**Single lead**
1234567	1234567	1234567
2135476	2135476	2135476
2314567	2314567	2314567
3241657	3241657	3241657
3426175	3426175	3426175
4362715	4362715	4362715
4637251	4637251	4637251
6473521	6473521	6473521
6745312	6745312	6745312
7654132	7654132	7654132
7561423	7561423	7561423
5716243	5716243	5716243
5172634	5172634	5172634
<u>1527364</u>	<u>1576243</u>	<u>1576243</u>
1253746	1752634	1572634

Figure 6: The plain, bob and single lead of Grandsire Triples starting with rounds.

Change ringers prefer to use plain leads and bob leads and will only use single leads if strictly necessary. Thus the question arises whether ringing an extent of Grandsire Triples is possible by using only plain and bob leads. As we will see in Section 3.2, the answer is negative. Before we come to this, there is some preliminary work to be done.

Lemma 3.1. *The following identities hold:*

$$P = (35764), \quad B = (274)(356), \quad PB^{-1} = (23467).$$

Proof. We compute

$$\begin{aligned}
P &= a_3(a_1a_7)^6a_1 = (12)(45)(67)(1357642)^6(23)(45)(67) = (35764);\\
B &= a_3(a_1a_7)^5a_1a_3a_1\\
&= (12)(45)(67)(1357642)^5(23)(45)(67)(12)(45)(67)(23)(45)(67)\\
&= (274)(356);\\
PB^{-1} &= (35764)((274)(356))^{-1} = (23467).
\end{aligned}$$

□

Lemma 3.2. *Let r be a row of 7 bells and define*

$$n_r = \begin{cases} r^{-1}(1), & \textit{if } r \textit{ and } r^{-1}(1) \textit{ have reverse parity} \\ 15 - r^{-1}(1), & \textit{if } r \textit{ and } r^{-1}(1) \textit{ have equal parity} \end{cases}.$$

Then in any touch of Grandsire Triples using only plain and bob leads in which r occurs, r is the n_rth row of its lead.

Proof. Suppose that we have a touch of Grandsire Triples using only plain and bob leads in which r occurs as the kth row of its lead. Since the treble plain hunts, we must have $k = r^{-1}(1)$ or $k = 15 - r^{-1}(1)$. Now notice that r and $r^{-1}(1)$ have reverse parity if and only if $k \in \{1, \dots, 7\}$: every lead head is even with the treble in first position, and the parity of the rows reverses every change, whereas the parity of the treble positions reverses every change except the 7th and 14th one. Therefore we have $k = n_r$. □

Notation 3.3. For $n \geq 3$ we let G_n denote the subgroup of S_n consisting of all permutations that keep 1 fixed and $H_n = G_n \cap A_n$ the subgroup of S_n consisting of all even permutations that keep 1 fixed.

Notice that we have $G_n \cong S_{n-1}$ and $H_n \cong A_{n-1}$, where A_{n-1} is the alternating group on $n-1$ elements.

Remark 3.4. In the notation of Lemma 3.2 we have $H_7 = \{r \in S_7 : n_r = 1\}$. So the elements of H_7 occurring in a touch of Grandsire Triples using only plain and bob leads, are precisely the lead heads occurring in the touch.

Every lead head in a touch of Grandsire Triples using only plain and bob leads is a product of P's and B's. So if $\{P, B\}$ did not even generate H_7, this would immediately disprove the existence of an extent of Grandsire Triples using only plain and bob leads. However, as the following lemma shows, $\{P, B\}$ does generate H_7 (see [7, Lemma 4.14]).

Lemma 3.5. *We have $\langle P, B\rangle = H_7$.*

Proof. The inclusion $\langle P, B\rangle \subset H_7$ is clear from Lemma 3.1. It is a well-known fact that A_n is generated by $\{(12k) : k = 3, \dots, n\}$ (see for example [7, Proposition 4.2]). So for the inclusion $H_7 \subset \langle P, B\rangle$ it suffices to show that

$(23k) \in \langle P, B\rangle$ for $k = 4, \ldots, 7$. We indeed have

$$\begin{aligned}
(234) &= (35764)^{-2}(274)(356) = P^{-2}B;\\
(235) &= (35764)(234)^{-1}(35764)^{-1} = P(234)^{-1}P^{-1};\\
(23764) &= (235)^{-1}(35764)(235) = (235)^{-1}P(235);\\
(237) &= (23764)(234)(23764)^{-1};\\
(236) &= (35764)^{-2}(237)^{-1}(35764)^{2} = P^{-2}(237)^{-1}P^{2}.
\end{aligned}$$

□

3.2 Thompson's proof

In 1886 W. H. Thompson succeeded in disproving the existence of an extent of Grandsire Triples using only plain and bob leads (see [8]). As T. J. Fletcher remarks in [9, p. 624], Thompson seems not to be a mathematician, nor does his proof suggest that he was aware of using group theoretical tools. Therefore we will use [9, pp. 624–625] to present Thompson's interesting but rather long-winded proof in a streamlined version. The terminology is Thompson's except Definition 3.8.

Definition 3.6. A *round block* of length $k > 0$ is a k-tuple $(x_1, \ldots, x_k) \in (H_7)^k$ with $x_i \neq x_j$ for $1 \leq i < j \leq k$ such that for all $i = 1, \ldots, k \mod k$ we have either $x_{i+1} = x_i P$ or $x_{i+1} = x_i B$. A *P-block* is a round block $(x, xP, xP^2, xP^3, xP^4)$ and a *B-block* is a round block (x, xB, xB^2).

Remark 3.7. Since no product of one or two elements of $\{P, B\}$ equals the neutral element, there are no round blocks of length 1 and 2. The smallest round block is a B-block, which has length 3.

Definition 3.8. An *H_7-decomposition* is a set of round blocks such that every element of H_7 appears exactly once as coefficient in one of the round blocks.

One can think about an H_7-decomposition as a set of cycles in the Cayley graph $\Gamma(H_7, \{P, B\})$ that forms a partition of H_7.

Remark 3.9. It follows from Remark 3.4 that the existence of an extent of Grandsire Triples by using only plain and bob leads would imply the existence of a round block of length $\#H_7 = 360$. This is equivalent to the existence of an H_7-decomposition containing only 1 element, which is again equivalent to the existence of a Hamiltonian cycle in the Cayley graph $\Gamma(H_7, \{P, B\})$.

Definition 3.10. The *Q-set* of a row $x \in H_7$ is the left coset $x\langle PB^{-1}\rangle$. Given an H_7-decomposition, a *plained Q-set* is a Q-set of which each element is followed by a plain lead and a *bobbed Q-set* is a Q-set of which each element is followed by a bob lead. We say that a bobbed Q-set is *plained* if the bob leads following its elements are replaced by plain leads; *bobbing* a plained Q-set is defined analogously.

Notice that a Q-set contains exactly 5 elements since $PB^{-1} = (23467)$ has order 5. In Figure 7 a Q-set is shown as part of the Cayley graph $\Gamma(H_7, \{P, B\})$.

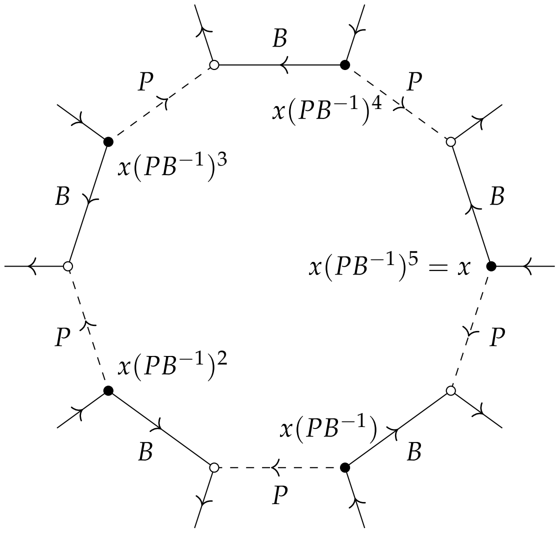

Figure 7: In this part of the Cayley graph $\Gamma(H_7, \{P, B\})$ the black vertices form a bobbed Q-set. The solid arrows are part of the cycles that form the H_7-decomposition.

Remark 3.11. By plaining a bobbed Q-set in an H_7-decomposition R_b, one obtains a different H_7-decomposition R_p.

The following observation is essential.

Lemma 3.12. *Given an H_7-decomposition, a Q-set $x\langle PB^{-1}\rangle$ is either plained or bobbed.*

Proof. Suppose that $x(PB^{-1})^i$ is followed by a plain lead while $x(PB^{-1})^{i+1}$ is followed by a bob lead for some $i = 1, \ldots, 5 \mod 5$. Then the row $x(PB^{-1})^i P = x(PB^{-1})^{i+1} B$ appears more than once in the H_7-decomposition, which is a contradiction. $\square$

The key ingredient for the proof of the main result of this section is the following lemma.

Lemma 3.13. *Plaining a bobbed Q-set in an H_7-decomposition does not change the parity of the number of round blocks.*

Thompson proves Lemma 3.13 by listing the 28 different ways in which the elements of a Q-set can be distributed over round blocks (see [8, pp. 15–17]) and even Fletcher says in [9, p. 625] that "it is very difficult to see any means by which this could have been avoided." However, D. J. Dickinson provided a rather short proof in [10]. We will give a slightly different proof, suggested by H. W. Lenstra, which is perhaps even more elegant. For this we use the following property of the sign map (see [11, p. 25]).

Lemma 3.14. *Let σ be a permutation of a finite set X, having cycle type $(k_1, \ldots, k_r)$. Then $sgn(\sigma) = (-1)^{\#X-r}$.*

Proof. It follows from $\sum_{i=1}^{r} k_i = \#X$ that

$$\operatorname{sgn}(\sigma) = \prod_{i=1}^{r}(-1)^{k_i-1} = (-1)^{\sum_{i=1}^{r}(k_i-1)} = (-1)^{\#X-r}.$$

□

Proof of Lemma 3.13. Let $R_b = \{(x_{1,1}, \dots, x_{1,k_1}), \dots, (x_{r,1}, \dots, x_{r,k_r})\}$ be an H_7-decomposition and $\sigma_b = (x_{1,1} \dots x_{1,k_1}) \cdots (x_{r,1} \dots x_{r,k_r})$ the corresponding permutation of H_7. Suppose that $X = x\langle PB^{-1}\rangle$ is a bobbed Q-set in R_b. Let R_p be the H_7-decomposition obtained by plaining X and define σ_p analogously to σ_b. Writing $x_i = x(PB^{-1})^i$ for $i = 1, \dots, 5$, we have

$$\sigma_p(x_i) = x(PB^{-1})^i P = x(PB^{-1})^{i+1} B = \sigma_b(x_{i+1})$$

for $i = 1, \dots, 5 \mod 5$ and $\sigma_p(x) = \sigma_b(x)$ for $x \in H_7 \setminus X$; hence we find $\sigma_p = \sigma_b \circ (x_1 \dots x_5)$. It follows by Lemma 3.14 that

$$(-1)^{5-\#R_p} = \operatorname{sgn}(\sigma_p) = \operatorname{sgn}(\sigma_b) \cdot \operatorname{sgn}((x_1 \dots x_5)) = (-1)^{5-\#R_b} \cdot 1 = (-1)^{5-\#R_b}$$

and thus $\#R_b \equiv \#R_p \mod 2$. □

Theorem 3.15. *There is no extent of Grandsire Triples using only plain leads and bob leads.*

Proof. Suppose there were an extent of Grandsire Triples using only plain leads and bob leads and thus a round block of length 360 (Remark 3.9). Let R denote the corresponding H_7-decomposition and T the H_7-decomposition consisting of only B-blocks, so that $\#R = 1$ and $\#T = \frac{360}{3} = 120$. It follows from Lemma 3.12 that R can be obtained from T by repeatedly plaining bobbed Q-sets. However, since the parity of the number of round blocks is invariant under plaining bobbed Q-sets by Lemma 3.13, this is a contradiction. □

3.3 The largest possible touch

Since any H_7-decomposition contains at least two round blocks of which the smallest has length at least 3 (Remark 3.7), the length of a round block in an H_7-decomposition cannot exceed 357. As early as 1751, more than a century before Thompson's proof was published, John Holt had attained a touch of this length (see [10]). In his extent of Grandsire Triples, which can be found in [12, part III, p. 73], only the 357th and 360th lead are single leads. Notice that the number of single leads cannot be reduced to one: the lead heads following the first single lead are contained in $G_7 \setminus H_7$ instead of H_7, so that at least one other single lead is needed to get back to rounds.

Is there a round block of length 358 or 359 (and thus a larger touch than Holt's one)? Such a round block would not fit in an H_7-decomposition, but it is not a priori clear that it does not exist. However, as will be shown below, there is no round block of length greater than 357.

Definition 3.16. The *extended Q-set* of a row $x \in H_7$ is $x\langle PB^{-1}\rangle \cup x\langle PB^{-1}\rangle B$.

Figure 7 might clarify this definition: the extended Q-set of a row $x \in H_7$ consists of 5 elements from $x\langle PB^{-1}\rangle$, forming the (ordinary) Q-set of x (the black vertices), and 5 elements from $x\langle PB^{-1}\rangle B = x\langle PB^{-1}\rangle P$ (the white vertices). Suppose we have a round block with corresponding cycle c in $\Gamma(H_7, \{P, B\})$. Then c contains as many arrows going into the extended Q-set as arrows going out of it. As a consequence we have the following remark.

Remark 3.17. Let $x \in H_7$ be a row. A round block contains an equal number of elements from $x\langle PB^{-1}\rangle$ and $x\langle PB^{-1}\rangle B$.

Remark 3.18. A row $x \in H_7$ is contained in exactly two extended Q-sets, namely the extended Q-sets of x and xB^{-1}.

Theorem 3.19. *There is no round block of length greater than* 357.

Proof. Let R be a round block. By Theorem 3.15 there is a row $x \in H_7$ that is not contained in R. Let

$$\begin{aligned} X &= x\langle PB^{-1}\rangle \cup x\langle PB^{-1}\rangle B; \\ X' &= xB^{-1}\langle PB^{-1}\rangle \cup xB^{-1}\langle PB^{-1}\rangle B = x\left(\langle PB^{-1}\rangle B\right)^{-1} \cup x\langle B^{-1}P\rangle; \end{aligned}$$

be the two extended Q-sets in which x is contained. It follows from

$$\begin{aligned} \langle PB^{-1}\rangle \cap B^{-1}\langle PB^{-1}\rangle &= \varnothing; \\ \langle PB^{-1}\rangle \cap \langle B^{-1}P\rangle &= \langle (23467)\rangle \cap \langle (24675)\rangle = \{(1)\}; \\ \langle PB^{-1}\rangle B \cap \left(\langle PB^{-1}\rangle B\right)^{-1} &= \langle (23467)\rangle (274)(356) \cap (\langle (23467)\rangle (274)(356))^{-1} \\ &= \{B, (35764), (235), (35)(2467), (26)(3547)\} \\ &\quad \cap \{B, (35764), (235), (35)(2467), (26)(3547)\}^{-1} \\ &= \varnothing; \\ \langle PB^{-1}\rangle B \cap B^{-1}\langle PB^{-1}\rangle B = \varnothing; \end{aligned}$$

that $X \cap X' = \{x\}$. By Remark 3.17 there are rows $y \in X \setminus \{x\}$ and $z \in X' \setminus \{x\}$ that are not contained in R. It follows from $X \cap X' = \{x\}$ that $y \neq z$. So we have found three distinct rows $x, y, z \in H_7$ that are not contained in R. Hence R has length at most 357. $\square$

We conclude that Holt's touch is indeed the largest possible touch of Grandsire Triples using only plain and bob leads.

4 Rankin's campanological theorem

4.1 Rankin's theorem

Thompson's result concerning Grandsire Triples, which we discussed in Section 3.2, was in 1948 generalized by R. A. Rankin (see [13]). He proved a theorem that provides a tool to show in a more general setting that certain leads can never form an extent. We will follow the proof of [14].

Definition 4.1. Let X be a finite set and S a set of permutations of X. An *S-cycle* in X is a nonempty tuple $(x_1, \ldots, x_k)$ with $x_i \in X$ and $x_i \neq x_j$ for $1 \leq i < j \leq k$ such that there exist $\sigma_i \in S$ with $x_{i+1} = \sigma_i(x_i)$ for all $i = 1, \ldots, k \mod k$.

Lemma 4.2. *Let X be a finite set and $S = \{\sigma_1, \sigma_2\}$ a set of permutations of X, where σ_1 and σ_2 have a_1 and a_2 orbits, respectively. Suppose that $\rho = \sigma_2^{-1}\sigma_1$ has odd order and that X can be partitioned into r distinct S-cycles. Then $r \equiv a_1 \equiv a_2 \mod 2$.*

Proof. If $(x_{1,1}, \ldots, x_{1,k_1}), \ldots, (x_{r,1}, \ldots, x_{r,k_r})$ are the r disjoint S-cycles partitioning X, we define the permutation $\pi = (x_{1,1} \ldots x_{1,k_1}) \cdots (x_{r,1} \ldots x_{r,k_r})$ of X. Also define the sets $A_i = \{x \in X : \pi(x) = \sigma_i(x)\}$ for $i = 1, 2$ and $\tau = \sigma_2^{-1}\pi$. Then we have $\tau|_{A_1} = \rho|_{A_1}$ and $\tau|_{A_2} = \mathrm{id}_{A_2}$. It follows from $A_1 \cup A_2 = X$ that A_1 is stable under τ and therefore also under ρ. So $\tau|_{A_1} = \rho|_{A_1}$ has odd order since ρ has. Hence τ, being the identity on $X \setminus A_1 \subset A_2$, has odd order as well and we find $\mathrm{sgn}(\sigma_2^{-1})\mathrm{sgn}(\pi) = \mathrm{sgn}(\tau) = 1$. So we get by Lemma 3.14

$$(-1)^{\#X - r} = \mathrm{sgn}(\pi) = \mathrm{sgn}(\sigma_2^{-1}) = \mathrm{sgn}(\sigma_2) = (-1)^{\#X - a_2}$$

and therefore $r \equiv a_2 \mod 2$. If $\sigma_2^{-1}\sigma_1$ has odd order, then $\sigma_1^{-1}\sigma_2 = (\sigma_2^{-1}\sigma_1)^{-1}$ has odd order as well. So by interchanging the roles of σ_1 and σ_2 we find $r \equiv a_1 \mod 2$. □

In Definition 4.1 we can choose X to be the finite group G. If $S = \{s_1, s_2\} \subset G$ is a subset, then for $i = 1, 2$ the element s_i induces the permutation $\sigma_i : g \mapsto gs_i$ of G. The $\{\sigma_1, \sigma_2\}$-cycles we just call *S-cycles*. For $i = 1, 2$ the orbits of σ_i are the left cosets of $\langle s_i \rangle$, so that σ_i has $[G : \langle s_i \rangle]$ orbits. Furthermore, $s_1 s_2^{-1}$ and $\sigma_2^{-1}\sigma_1 : g \mapsto gs_1 s_2^{-1}$ have the same order. Thus we find the following result as a consequence of Lemma 4.2.

Theorem 4.3 (Rankin, 1948)**.** *Let G be a finite group and $S = \{s_1, s_2\} \subset G$ a subset such that $s_1 s_2^{-1}$ has odd order. Suppose that G can be partitioned into r distinct S-cycles. Then $r \equiv [G : \langle s_1 \rangle] \equiv [G : \langle s_2 \rangle] \mod 2$.* □

4.2 Application to Grandsire Triples

Rankin's theorem can now be used to easily prove that there is no extent of Grandsire Triples using only plain and bob leads (compare [13, p. 24]).

Alternative proof of Theorem 3.15. Suppose there were an extent of Grandsire Triples using only plain and bob leads. Then we know by Remark 3.4 that the group H_7 is a $\{P, B\}$-cycle; or in terms of Theorem 4.3: $r = 1$. Furthermore, by Lemma 3.1 $PB^{-1} = (23467)$ has odd order. Therefore

$$[H_7 : \langle B \rangle] = \frac{\#H_7}{\#\langle B \rangle} = \frac{\#A_6}{\#\langle (274)(356) \rangle} = \frac{360}{3} = 120 \not\equiv 1 = r \mod 2$$

is a contradiction by Theorem 4.3. □

Let us compare Thompson's original proof of Theorem 3.15 with the proof using Rankin's theorem. The notion of a round block used in Thompson's proof appears in generalized form as a $\{P, B\}$-cycle in Rankin's proof, and an H_7-decomposition is just a partition of H_7 into $\{P, B\}$-cycles. Thompson's key observation, Lemma 3.13, yields that the parity of the number of round blocks in the H_7-decomposition of B-blocks equals that of an arbitrary H_7-decomposition. This also follows from Theorem 4.3, which says that the parity of the number $[H_7 : \langle B \rangle]$ of $\{B\}$-cycles equals the parity of the number of $\{P, B\}$-cycles in an arbitrary partition of H_7 into $\{P, B\}$-cycles.

4.3 Application to Double Norwich Court Bob Major

In [13, pp. 24–25] Rankin applies his theorem to several other methods, showing that there is no extent using plain and bob leads only. As an example, we consider the method on 8 bells called *Double Norwich Court Bob Major* (for a more extensive description see [6, pp. 41–48]). Its plain and bob leads consist of 16 rows and are produced by the words

$$L_P = a_\times.a_{14}.a_\times.a_{36}.a_\times.a_{58}.a_\times.a_{18}.a_\times.a_{58}.a_\times.a_{36}.a_\times.a_{14}.a_\times.a_{18};$$
$$L_B = a_\times.a_{14}.a_\times.a_{36}.a_\times.a_{58}.a_\times.a_{18}.a_\times.a_{58}.a_\times.a_{36}.a_\times.a_{14}.a_\times.a_{16}.$$

It can be checked that

$$P = \overline{L_P} = (2836547),\ B = \overline{L_B} = (2854736),\ PB^{-1} = (253),$$

and we have the following analogue of Lemma 3.2.

Lemma 4.4. *Let r be a row of 8 bells and define*

$$n_r = \begin{cases} r^{-1}(1), & \text{if } r^{-1}(1) \equiv 1,2 \mod 4 \text{ and } r \text{ is even} \\ r^{-1}(1), & \text{if } r^{-1}(1) \equiv 0,3 \mod 4 \text{ and } r \text{ is odd} \\ 17 - r^{-1}(1), & \text{else} \end{cases}.$$

Then in any touch of Double Norwich Court Bob Major using only plain and bob leads in which r occurs, r is the n_rth row of its lead.

Proof. Suppose that we have a touch of Double Norwich Court Bob Major using only plain and bob leads in which r occurs as the kth row of its lead. Since the treble plain hunts, we must have $k = r^{-1}(1)$ or $k = 17 - r^{-1}(1)$. If $r^{-1}(1) \equiv 1, 2 \mod 4$ then r is even if and only if $k \in \{1, \ldots, 8\}$, and if $r^{-1}(1) \equiv 0, 3 \mod 4$ then r is odd if and only if $k \in \{1, \ldots, 8\}$. Therefore we have $k = n_r$. □

Like in Remark 3.4 we have $H_8 = \{r \in S_8 : n_r = 1\}$, and analogously to the case of Grandsire Triples it follows that the existence of an extent using only plain and bob leads would imply that H_8 is an $\{P, B\}$-cycle. Therefore, noticing that

$$[H_8 : \langle B \rangle] = \frac{\#H_8}{\#\langle B \rangle} = \frac{\#A_7}{\#\langle (2854736) \rangle} = \frac{2520}{7} = 360 \not\equiv 1 \mod 2$$

and that $PB^{-1} = (253)$ has odd order gives a contradiction. Thus we find the following result.

Theorem 4.5. *There is no extent of Double Norwich Court Bob Major using only plain leads and bob leads.* □

5 The existence and construction of extents

5.1 Extent existence theorems

In the previous sections we have mainly focused on how to show that certain lead types cannot produce an extent. Now we will study some cases in which it is possible to prove the existence of an extent or even to construct one. By Corollary 2.10 we know that an extent on n bells exists for all $n \geq 2$, but the extent that can be derived from the proof is far too unstructured to be used for real change ringing. It would be better to construct extents divided into leads. Sometimes this can be done by brute force, but hopefully the use of mathematical tools can make things easier. This section, which is mainly based on [7, pp. 735–743], describes some of these tools.

Definition 5.1. Let G be a group and H a subgroup. A *right transversal* for H in G is a subset $S \subset G$ containing exactly one element of every right coset of H. A *left transversal* for H in G is defined analogously.

The following theorem gives for two special lead types a necessary and sufficient condition for the existence of an extent using only leads of these two types.

Theorem 5.2. *Let $L_y = x_1.x_2 \ldots x_k.y$ and $L_z = x_1.x_2 \ldots x_k.z$ be two lead types for n bells and suppose that the leads of type L_y and L_z are right transversals for $H = \langle \overline{L_y}, \overline{L_z} \rangle$ in S_n. Then there is an extent on n bells using only lead types L_y and L_z if and only if $\Gamma(H, \{\overline{L_y}, \overline{L_z}\})$ is Hamiltonian.*

Proof. Define $w_i = x_1 x_2 \cdots x_{i-1}$ for $i = 2, \ldots, k+1$ and $w_1 = (1)$. Then the right cosets $Hw_1, \ldots, Hw_{k+1}$ of H form a partition of S_n and a row $r \in Hw_i$ occurring in a touch using only lead types L_y and L_z must be the ith row of its lead. Let rw_i and sw_j be rows occurring in leads of type L_y or L_z with different lead heads $r, s \in H$, respectively. In the case $i = j$ we obviously have $rw_i \neq sw_j$. Otherwise, Hw_i and Hw_j are disjoint and it follows from $rw_i \in Hw_i$ and $sw_j \in Hw_j$ that $rw_i \neq sw_j$. Hence leads with different lead heads are disjoint.

Suppose there is an extent on n bells using only lead types L_y and L_z. Since the lead heads are precisely the elements of $H = Hw_1$, this implies the existence of a Hamiltonian cycle in $\Gamma(H, \{\overline{L_y}, \overline{L_z}\})$. Conversely, assume that there is a Hamiltonian cycle in $\Gamma(H, \{\overline{L_y}, \overline{L_z}\})$. Since leads with different lead heads are disjoint, this yields a touch on n bells using only lead types L_y and L_z. The touch visits $\#H$ different elements of every right coset of H and is therefore an extent. □

The proof of the following lemma is similar to the proof of Lemma 2.7.

Lemma 5.3. *Let G be a finite group containing elements $g, h \in G$ such that $gh^{-1} = hg^{-1}$ and $\langle g, h \rangle = G$. Then $\Gamma(G, \{g, h\})$ is Hamiltonian.*

Proof. Suppose that g is of order k. The irreducible identity word g^k yields a partition C of G consisting of k-cycles in $\Gamma = \Gamma(G, \{g, h\})$. It follows from

$\langle g, h\rangle = G$ that Γ is connected. Therefore, every two distinct cycles $c_1, c_2 \in C$ must be connected by an arrow (v, vh) from c_1 to c_2. It follows from $vgh^{-1} = vhg^{-1}$ that the cycles c_1 and c_2 can be glued together by removing the arrows (v, vg) and (vhg^{-1}, vh) and adding the arrows (v, vh) and (vhg^{-1}, vg), thus making a new cycle c. By repeating this gluing operation until it cannot be done any more, we get a set C' of cycles, still partitioning G.

We claim that C' contains only one cycle, which then has to be Hamiltonian. If not, there must be two distinct cycles $c'_1, c'_2 \in C'$ and an arrow $(v', v'h)$ from c'_1 to c'_2. Notice that the arrows $(v', v'h)$ and $(v'hg^{-1}, v'g)$ are not contained in c'_1 and c'_2, whereas $(v', v'g)$ is contained in c'_1 and $(v'hg^{-1}, v'h)$ in c'_2. Therefore, c'_1 and c'_2 can be glued together as before, which is a contradiction. $\square$

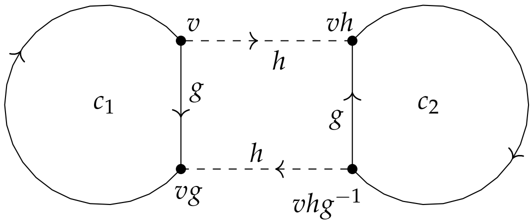

Figure 8: The situation in the proof of Lemma 5.3.

Theorem 5.4. *Let $L_y = x_1.x_2 \ldots x_k.y$ and $L_z = x_1.x_2 \ldots x_k.z$ be two lead types for n bells and suppose that the leads of type L_y and L_z are right transversals for $H = \langle \overline{L_y}, \overline{L_z}\rangle$ in S_n. If y and z satisfy $yz = zy$, then there is an extent on n bells using only lead types L_y and L_z.*

Proof. It follows from $yz = zy$ and the fact that every change is of order 2 that

$$\overline{L_y}\,\overline{L_z}^{-1} = (x_1x_2\cdots x_ky)(zx_k\cdots x_2x_1) = (x_1x_2\cdots x_kz)(yx_k\cdots x_2x_1) = \overline{L_z}\,\overline{L_y}^{-1}.$$

By Lemma 5.3 we find that $\Gamma(H, \{\overline{L_y}, \overline{L_z}\})$ is Hamiltonian, so that Theorem 5.2 gives the result. $\square$

We find the following corollary (compare [7, Theorem 4.11]).

Corollary 5.5. *Let $n > 0$ be an integer such that $n \equiv 0 \mod 4$ or $n \equiv 3 \mod 4$ and let $\alpha, \beta, \gamma, \delta$ be changes of n bells such that $\langle \alpha, \beta\rangle = D_n$ and $\gamma\delta = \delta\gamma$. Suppose that the two lead types $L_\gamma = (\alpha.\beta)^{n-1}.\alpha.\gamma$ and $L_\delta = (\alpha.\beta)^{n-1}.\alpha.\delta$ satisfy $\langle \overline{L_\gamma}, \overline{L_\delta}\rangle = H_n$. Then there is an extent on n bells using only lead types L_γ and L_δ.*

Proof. By Theorem 5.4 it suffices to prove that leads of type L_γ and L_δ are right transversals for H_n in S_n. Notice that a lead of type L_γ or L_δ with lead head $x \in H_n$ equals the left coset $xD_n = x\{(1), \alpha, \alpha\beta, \ldots, (\alpha\beta)^{n-1}\alpha\}$ of D_n. Suppose that two rows xr and xs of this lead with $r, s \in D_n$ are both contained in the same right coset of H_n. It follows from $H_nr = H_ns$ that $rs^{-1} \in H_n \cap D_n$. Since we have $n \equiv 0 \mod 4$ or $n \equiv 3 \mod 4$, the neutral element (1) is the only even element of D_n that fixes 1. Therefore, we find $rs^{-1} = (1)$ and $xr = xs$.

Hence different rows of the same lead are contained in different right cosets of H_n. Since a lead has $2n = \#S_n/\#A_{n-1} = [S_n : H_n]$ rows, this concludes the proof. $\square$

5.2 Existence of Plain Bob Major extent with special bob leads

Let us try to find changes $\alpha, \beta, \gamma, \delta$ for 8 bells such that the conditions of Corollary 5.5 are fulfilled. We follow the reasoning of [7, pp. 736–737]. If we take $\alpha = a_\times$ and $\beta = a_{18}$, then we know by Lemma 1.7 that $\langle \alpha, \beta \rangle = D_8$. Inspired by Plain Bob Minimus (see Example 1.8) we take $\gamma = a_{12}$. Finally, $\delta = (78)$ might be a good guess since we have $\gamma\delta = (34)(56) = \delta\gamma$. Thus we get lead types $L_\gamma = (\alpha.\beta)^7.\alpha.\gamma$ and $L_\delta = (\alpha.\beta)^7.\alpha.\delta$. The word L_γ produces the plain leads of Plain Bob Major. However, for obtaining the usual bob leads of Plain Bob Major, $\delta = (78)$ in L_δ should be replaced by a_{14}. So the method under examination is a variation of Plain Bob Major. Define the evaluations $X = \overline{L_\gamma}$ and $Y = \overline{L_\delta}$ of L_γ and L_δ.

Lemma 5.6. *We have* $X = (3578642)$, $Y = (23)(45)(678)$ *and* $\langle X, Y \rangle = H_8$.

Proof. We have

$$X = (\alpha\beta)^7\alpha\gamma = (\alpha\beta)^8\beta\gamma = \beta\gamma = (23)(45)(67)(34)(56)(78) = (3578642);$$
$$Y = (\alpha\beta)^7\alpha\delta = (\alpha\beta)^8\beta\delta = \beta\delta = (23)(45)(67)(78) = (23)(45)(678);$$

and therefore $\langle X, Y \rangle \subset H_8$. Since A_n is generated by $\{(12k) : k = 3, \ldots, n\}$ (see for example [7, Proposition 4.2]), H_8 is generated by $\{(68k) : k = 2, 3, 4, 5, 7\}$. These elements can be obtained by conjugating $Y^2 = (687)$ by powers of $XY^2 = (23574)$, which shows that $H_8 \subset \langle X, Y \rangle$. $\square$

Theorem 5.7. *Define the changes* $\alpha = a_\times$, $\beta = a_{18}$, $\gamma = a_{12}$ *and* $\delta = (78)$ *for 8 bells. There is an extent on 8 bells using only lead types* $L_\gamma = (\alpha.\beta)^{n-1}.\alpha.\gamma$ *and* $L_\delta = (\alpha.\beta)^{n-1}.\alpha.\delta$.

Proof. By Lemmas 5.6 and 1.7 and the observations $8 \equiv 0 \mod 4$ and $\gamma\delta = \delta\gamma$ the conditions of Corollary 5.5 are satisfied. $\square$

5.3 Extent construction of Plain Bob Doubles

Example 5.8. Reconsider the method Plain Bob Minimus, which we have discussed in Example 1.8. By taking $\delta = \gamma$ we have only one lead type $L_\gamma = L_\delta = (\alpha.\beta)^3.\alpha.\gamma$, which satisfies $\langle \overline{L_\gamma} \rangle = \langle (234) \rangle = H_4$. Also we have $\langle \alpha, \beta \rangle = D_4$ and $\gamma\delta = \gamma^2 = \delta\gamma$. Therefore, Corollary 5.5 guarantees that the word $L_\gamma^3 = ((\alpha.\beta)^3.\alpha.\gamma)^3$ corresponds to an extent. Put the rows of Figure 2 (except for the last one) in a matrix $R = (r_{ij})_{1 \le i \le 8,\, 1 \le j \le 3}$. The columns of R are the left cosets of D_4 and the rows of R are the right cosets of H_4. By considering the 6 half leads in Figure 2 we can find a third coset decomposition of S_4. Each half lead is a right transversal for $G_4 \cong S_3$ in S_4. Notice that the lead heads and lead ends belong to G_4 and that the subword

$w = (\alpha.\beta)^3.\alpha$ of L_γ is palindromic. Therefore, the 4 right cosets of G_4 are the sets $\{r_{kj} : j = 1,2,3 \text{ and } k = i, 9-i\}$ for $i = 1,2,3,4$.

Unfortunately, there are many cases in which Theorem 5.4 cannot be applied. However, sometimes the third coset decomposition described in Example 5.8 helps us to find an extent. We illustrate this by explicitly constructing an extent of *Plain Bob Doubles* (see [7, pp. 730–732]). The plain and bob leads of this method on 5 bells are produced by the words $L_P = (a_5.a_1)^4.a_5.a_{125}$ and $L_B = (a_5.a_1)^4.a_5.a_{145}$, respectively. We have $P = \overline{L_P} = (2354)$ and $B = \overline{L_B} = (45)$. Each lead consists of 10 rows and is a left coset of D_5 (Lemma 1.7; see also Examples 1.8 and 5.8). Since P, B, a_{125}, a_{145} are all odd permutations fixing 1, both the lead heads and lead ends are contained in G_5, but not necessarily in H_5.

Remark 5.9. Let $r \in S_5$ be a row occurring in a touch of Plain Bob Doubles using only plain and bob leads. Since the treble plain hunts, r is either the $r^{-1}(1)$th or the $(11 - r^{-1}(1))$th row of its lead.

Theorem 5.10. *Let $(h_1, h_2, \ldots, h_{12}, h_1)$ be a cycle in $\Gamma(G_5, \{P, B\})$ such that $h_1 = (1)$ and $h_i \neq h_j(P^2B)^2$ for all $i, j \in \{1, \ldots, 12\}$. Then the succession of plain and bob leads with lead heads $h_1, \ldots, h_{12}$ forms an extent of Plain Bob Doubles.*

Proof. Denote the ith row of the jth lead by $r_{i,j}$ for $i = 1, \ldots, 10$ and $j = 1, \ldots, 12$. It suffices to show that $r_{i,j} \neq r_{k,l}$ if $(i,j) \neq (k,l)$. Assume to the contrary that $r_{i,j} = r_{k,l}$ for some $(i,j) \neq (k,l)$. Since the elements of a lead are all distinct, we have $j \neq l$. If $i = k$ then the lead heads $h_j = r_{1,j}$ and $h_l = r_{1,l}$ would be identical, which is a contradiction. So we have $i \neq k$ and by Remark 5.9 it follows that $k = 11 - i$. Since the subword $w = (a_5.a_1)^4.a_5$ of L_P and L_B is palindromic, it follows that the lead head $h_j = r_{1,j}$ of the lead containing $r_{i,j}$ equals the lead end $r_{10,l}$ of the lead containing $r_{k,l}$. However, this is a contradiction:

$$h_j = r_{10,l} = r_{1,l}\overline{w} = h_l(23)(45) = h_l(P^2B)^2.$$

We find that $r_{i,j} \neq r_{k,l}$ if $(i,j) \neq (k,l)$, which concludes the proof. □

An extent of Plain Bob Doubles can now be constructed as follows. Write $G_5 = \{x_1, x_2, \ldots, x_{12}, y_1, y_2, \ldots, y_{12}\}$ such that $x_1 = (1)$ and $x_i = y_i(P^2B)^2$ for $i = 1, \ldots, 12$. Since $(P^2B)^2 = (23)(45)$ is of order 2, we also have $y_i = x_i(P^2B)^2$ for $i = 1, \ldots, 12$. By Theorem 5.10 it is now sufficient to find a cycle $(h_1, h_2, \ldots, h_{12}, h_1)$ in $\Gamma(G_5, \{P, B\})$ with the property that for all $i \in \{1, \ldots, 12\}$ there is exactly one $j \in \{1, \ldots, 12\}$ such that $h_j \in \{x_i, y_i\}$. If we look at Figure 9 (see also [7, Figure 4.1]), this is an easy job. Examples are the cycles given by the words $w_1 = (P^3.B)^3$ and $w_2 = B.(P^3.B)^2.P^3$, beginning at $x_1 = (1)$. The words $w_1, w_2 \in W(G_5, \{P, B\})$ are irreducible identity words of length 12, not containing a subword u with $\overline{u} = (P^2B)^2$, which is another way of formulating the condition in Theorem 5.10 on the 12-cycle in $\Gamma(G_5, \{P, B\})$.

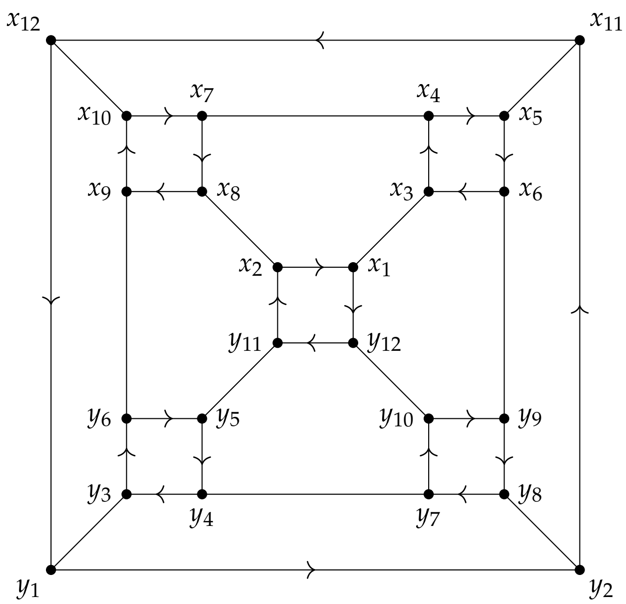

Figure 9: The Cayley graph $\Gamma(G_5, \{P, B\})$. The arrows correspond to the plain leads and the undirected edges to the bob leads (since $B = (45)$ has order 2).

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