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# Ultrafilters and Topology

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# 1 Introduction

In this thesis, we will explore a simple, but rich concept, called *ultrafilters*. This concept has utilities in many areas, such as model theory, logic, and topology (this can be read in [2]). As the title suggests, this thesis has its focus on the latter area, topology.

On topological spaces, ultrafilters can converge. Many topological properties can be formulated in terms of this convergence. They work in is similar fashion to the ordinary sequence convergence, with one notable difference: it works for *every* topological space. This comes with a nice bonus: *Tychonov's theorem*, a theorem that is otherwise hard to prove!

The main result of this thesis will a reconstruction of the category of compact Hausdorff spaces. Originally proven by Ernest Manes, this construction enables a very different way of creating topological spaces, what will come in handy last chapter.

When finding a major result, one could do two things with it: generalizing or finding an application. I went for both. The fourth chapter yields a reconstruction of the category of *all* topological spaces. The result is not much different. However, this will involve taking a look at the category of sets and *relations*, which has the structure of a *2-category*. As such, everything will be slightly harder.

The last chapter will cover a utility of this reconstruction: the *Stone-Čech compactification* of a topological space. This is a natural way to make a space compact Hausdorff. The resulting space is rich of mathematical structure, and some unusual applications. This can be read here [5] and here [6].

This thesis is very categorical, and so, the reader will be assumed to know about functors and natural transformations. Other concepts, like *monads*, will be introduced throughout the thesis.

Before starting, I would like to thank my supervisor Owen Biesel for his knowledge and support throughout the project.

# 2 Ultrafilters and topology

## 2.1 Fundamental matters

In this section, we will give a definition for ultrafilters, and a way to construct them.

**Definition 2.1.** Let $S$ be a set. A collection of subsets $F \subseteq \mathcal{P}(S)$ is an *ultrafilter* on $S$ if it satisfies the following properties:

(1) For every subset $A \subseteq S$, $F$ contains either $A$ or its complement $A^{\mathsf{c}}$, but not both.

(2) If $A \in F$, and $A \subseteq B \subseteq S$, then $B \in F$.

(3) For all $A, B \in F$, $A \cap B \in F$ as well.

We denote the set of ultrafilters on $S$ by $\mathcal{U}(S)$.

There is one easy, but important class of ultrafilters.

**Example 2.2.** Let $x \in S$. Then the set $\{A \subseteq S\colon x \in A\}$ is an ultrafilter. We denote this set by $P_x$.

An ultrafilter that can be written as $P_x$, for some $x \in S$, is called a *principal ultrafilter*. These are the only ultrafilters we can construct explicitly. Others will require the *Axiom of choice*. For the construction, we will need a new concept:

**Definition 2.3.** A collection of subsets $C \subseteq \mathcal{P}(S)$ *has FIP* (Finite intersection Property) if each finite subcollection of $C$ has a nonempty intersection, i.e. for every finite subcollection $\{A_1, ..., A_n\} \subseteq C$, we have: $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

The following theorem will give us the ability to create free ultrafilters. It will be very fundamental throughout the thesis. And so, many proofs will involve verifying whether a given collection of sets has FIP.

**Theorem 2.4.** *Let $C \subseteq \mathcal{P}(S)$ be any collection with FIP. Then there exists an ultrafilter $F \in \mathcal{U}(S)$ such that $C \subseteq F$.*

*Proof.* Let $\mathfrak{C} \subseteq \mathcal{P}(\mathcal{P}(S))$ be the collection of supersets of $C$ with FIP. Applying Zorn's Lemma (the hypothesis is easy to verify) shows this set has a maximal element; call it $M$.
We now have to verify $M$ is an ultrafilter. Conditions (2) and (3) are clear from maximality. Suppose $M$ does not meet condition (1). Then there exists an $A \subseteq S$ such that $A, A^{\mathsf{c}} \notin M$ (they cannot be both contained in $M$, as $M$ has FIP). Then $M \cup \{A\}$ does not have FIP. As $M$ meets condition (2), this would mean that there exists a $B \in M$ such that $B \cap A = \emptyset$. But that means that $B \subseteq A^{\mathsf{c}}$. But since $M$ meets condition (3) as well, this means that

$A^{\mathsf{c}} \in M$, which is a contradiction to our assumption. So $M$ meets condition (1) as well. So $M$ is an ultrafilter.
So $C$ is contained within an ultrafilter, $M$. □

**Remark 2.5.** This proof shows that ultrafilters can also be interpreted as maximal sets with FIP.

### 2.2 Convergence and topological properties

Ultrafilters play a very interesting role in *topology*: ultrafilters can *converge*! In this section, we will introduce this type of convergence, and derive some powerful properties. All of these properties will be proven with Theorem 2.4, so they rely on the Axiom of Choice!

**Definition 2.6.** Let $S$ be a topological space. Consider an ultrafilter $F \in \mathcal{U}(S)$ and a point $x \in S$. We say $F$ *converges to* $x$ if $F$ contains all open neighborhoods of $x$. We denote $F \searrow x$.

Several examples of convergent ultrafilters are fairly easy to derive.

**Example 2.7.** Let $S$ be an arbitrary topological space, and let $x \in S$. The ultrafilter $P_x$ contains all sets containing $x$, let alone the open neighborhoods thereof. And so, we have the convergence relation $P_x \searrow x$.

**Example 2.8.** When $S$ is discrete, there are no convergent ultrafilters outside of the ones described in Example 2.7. On the other hand, when $S$ is indiscrete, each ultrafilter converges to every point.

Ultrafilters shine more in an abstract sense. The property we are going to prove now allows us to tell exactly whether a given set is open or not.

**Theorem 2.9.** *Let $S$ be a topological space, and $U \subseteq S$. The following are equivalent:*

(1) *$U$ is open in $S$.*

(2) *$U$ appears in each ultrafilter that converges to some point in $U$.*

*Proof.* (1) $\Rightarrow$ (2) Let $F \in \mathcal{U}(S)$ be an ultrafilter that converges to, say, $x \in U$. Then $U$ is an open neighborhood of $x$, so $U \in F$ by definition.
(2) $\Rightarrow$ (1) Let $x \in U$. We are going to prove first that $U$ is a superset of an open neighborhood of $x$. Suppose it is not. Then the following subset of $\mathcal{P}(S)$ has FIP:

$$C = \{\text{open neighborhoods of } x\} \cup \{U^{\mathsf{c}}\}$$

So we can extend $C$ to an ultrafilter $F \in \mathcal{U}(S)$. This converges to $x$, but lacks $U$, which is a contradiction to our assumption. So $U$ is a superset of an open neighborhood of $x$.
Now we can construct $U$ by taking an open neighborhood $U_x \subseteq U$, for all $x \in U$. Then $U$ is the union of them, so $U$ is indeed open in $S$. □

This means that the underlying topology, and therefore its topological properties, can be fully derived out of a given convergence relation! Some properties have a very nice and simple ultrafilter interpretation. We will start with *compactness*.

**Theorem 2.10.** *Let $S$ be a topological space. The following are equivalent:*

(1) *$S$ is compact.*

(2) *Each ultrafilter $F \in \mathcal{U}(S)$ converges to at least one point.*

*Proof.* (1) $\Rightarrow$ (2) Suppose there exists on ultrafilter $F \in \mathcal{U}(S)$ that does not converge anywhere. Then we can choose for each $x \in S$ an open neighborhood $U_x \subseteq S$ such that $U_x \notin F$. This leaves us with an open cover $(U_x)_{x \in S}$ of $S$. Since $S$ is compact, it has a finite subcover $(U_{x_i})_{i=1}^n$. As none of the sets of the cover are contained in $F$, $F$ fully contains $(U_{x_i}^{\mathsf{c}})_{i=1}^n$. But $\bigcap_{i=1}^n U_{x_i}^{\mathsf{c}} = \emptyset$. Since ultrafilters are closed under finite intersections, this means that $\emptyset \in F$, which is a contradiction. So each ultrafilter in $\mathcal{U}(S)$ converges to some point in $A$.

(2) $\Rightarrow$ (1) Suppose $S$ is not compact. Let $C$ be an open cover of $S$ with no finite subcover. Then $C'$, the collection of complements of sets in $C$, has FIP, and can be extended to an ultrafilter $F \in \mathcal{U}(S)$. This converges somewhere, by assumption, to say, $x \in S$. As $C$ covers $S$, there is an open set $U \in C$, such that $x \in U$. By convergence, this means that $U \in F$. But $U^{\mathsf{c}} \in C'$, which is fully contained in $F$, so $F$ contains both $U$ and $U^{\mathsf{c}}$, which is a contradiction. So $S$ is compact. □

The *Hausdorff* property also has a very interesting interpretation.

**Theorem 2.11.** *Let $S$ be a topological space. The following are equivalent:*

(1) *$S$ is Hausdorff.*

(2) *Each ultrafilter $F \in \mathcal{U}(S)$ converges to at most one point.*

*Proof.* (1) $\Rightarrow$ (2) Suppose there exists an ultrafilter $F \in \mathcal{U}(S)$ that converges to multiple points, say $x$ and $y$. Let $U_x$ and $U_y$ be disjoint open neighborhoods of $x$ and $y$ respectively. Then $F$ contains both $U_x$ and $U_y$, and therefore contains $\emptyset$. This is a contradiction, so each ultrafilter in $\mathcal{U}(S)$ converges to at most one point.

(2) $\Rightarrow$ (1) Suppose $S$ is not Hausdorff. Then there exists an $x, y \in S$ without pairwise disjoint neighborhoods. Let $C$ be the set of all open neighborhoods of $x$ and $y$. This set has FIP, as each finite intersection of open neighborhoods can be written as an intersection of an open neighborhood of $x$ and an open neighborhood of $y$, which by assumption are never disjoint. So $C$ can be extended to an ultrafilter $F \in \mathcal{U}(S)$. However, this ultrafilter converges to both $x$ and $y$; a contradiction to our assumption. So $S$ is Hausdorff. □

Combining the previous theorems yields the following characterization.

**Corollary 2.12.** *Let $S$ be a topological space. The following are equivalent:*

(1) *$S$ is compact Hausdorff.*

(2) *Each ultrafilter $F \in \mathcal{U}(S)$ converges to precisely one point.*

This shows that the compact Hausdorff spaces are precisely those whose convergence relation is a *function* $\mathcal{U}(S) \to S$. We will give a closer look to it next chapter.

The last property we will give an ultrafilter interpretation is *continuity.* But we will need one new concept, of *pushforward ultrafilters.*

**Definition 2.13.** Let $f: S \to S'$ be any function, and $F \in \mathcal{U}(S)$ be an ultrafilter. The *pushforward ultrafilter* $f_*(F) \in \mathcal{U}(S')$ is defined by:

$$f_*(F) := \{A \subseteq S' : f^{-1}(A) \in F\}$$

As we see from this definition, every function $f: S \to S'$ induces a function $f_*: \mathcal{U}(S) \to \mathcal{U}(S')$. This will play a major role next chapter, when talking about categories. But it also allows us to have an ultrafilter interpretation of continuity.

**Theorem 2.14.** *Let $S$ and $S'$ be topological spaces, and let $f: S \to S'$ be a map. Let $x \in S$. The following are equivalent:*

(1) *$f$ is continuous at $x$.*

(2) *For each ultrafilter $F \in \mathcal{U}(S)$ where $F \searrow x$, we have $f_*(F) \searrow f(x)$.*

*Proof.* (1) $\Rightarrow$ (2) Let $F \in \mathcal{U}(S)$ such that $F \searrow x$. Let $U \subseteq S'$ be an open neighborhood of $f(x)$. Since $f$ is continuous at $x$, $f^{-1}(U)$ is an open neighborhood of $x$. So $f^{-1}(U) \in F$ and hence $U \in f_*(F)$. So $f_*(F)$ contains all open neighborhoods of $f(x)$. Hence $f_*(F) \searrow f(x)$.

(2) $\Rightarrow$ (1) Let $U \subseteq S'$ be an open neighborhood of $f(x)$. Let $F \in \mathcal{U}(S)$ be an ultrafilter converging to $x$. By assumption, $f_*(F) \searrow f(x)$ and hence $U \in f_*(F)$. By definition, $f^{-1}(U) \in F$. This holds for all ultrafilters converging to a point of $f^{-1}(U)$. In follows that $f^{-1}(U)$ is open in $S$ by Theorem 2.9. So $f$ is continuous. □

**Remark 2.15.** There are similarities between ultrafilter convergence and the ordinary *sequence convergence.* Let $S$ and $S'$ be *metric* spaces. Recall the following properties:

(1) Each subset $U \subseteq S$ is open if and only if each sequence with a limit in $U$ is eventually situated in $U$.

(2) The space $S$ is compact if and only if each sequence has a convergent subsequence.

(3) Let $x \in S$. A map $f\colon S \to S'$ is continuous at $x$ if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ that converges to $x$, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$.

This shows that Theorem 2.9, Theorem 2.10 and Theorem 2.14 are more powerful analogues of these respective properties. Especially compactness has a much better ultrafilter interpretation.

## 2.3 Application: Tychonov's theorem

One nice application of the properties found in previous section is *Tychonov's* theorem. This theorem has a wide range of applications, and is known to be equivalent to the axiom of choice. It is very easy to prove with ultrafilters. Before heading to the proof, we will first take a look at how the convergence relation looks on product topologies. I will do this in a more general setting: looking at how it looks on *initial* topologies, which is mostly used for defining product topologies and subspace topologies. Recall the definition of initial topologies below:

**Definition 2.16.** Let $(S_i)_{i\in I}$ be a family of topological spaces, and let $(f_i\colon S \to S_i)_{i\in I}$ be a family of maps. On $S$, the *initial topology*, or *topology induced by* $(f_i)_{i\in I}$ is the coarsest topology such that $f_i$ is continuous, for all $i \in I$. This space has the following subbase:

$$B = \{f_i^{-1}(U_i)\colon U_i \text{ open in } S_i, i \in I\}$$

This topology can be formulated nicely in terms of ultrafilter convergence.

**Lemma 2.17.** *Let $S$ as above. Let $x \in S$ and $F \in \mathcal{U}(S)$. The following are equivalent:*

(1) $F \searrow x$.

(2) $f_{i*}(F) \searrow f_i(x)$ *for all* $i \in I$.

*Proof.* (1) $\Rightarrow$ (2) The map $f_i$ is continuous for all $i \in I$, so (2) holds automatically.

(2) $\Rightarrow$ (1) Let $U \subseteq S$ be an open neighborhood of $x$. As $F$ is closed under finite intersections, we can assume $U \in B$ (with $B$ defined above). Then $U = f_i^{-1}(U_i)$, for some $i \in I$, $U_i \in T_i$. This $U_i$ is an open neighborhood of $f_i(x)$, so by assumption, $U_i \in f_{i*}(F)$. Hence $U = f_i^{-1}(U_i) \in F$. This holds for all open neighborhoods of $x$, so $F \searrow x$. $\square$

With this result, combined with Theorem 2.10, *Tychonov's theorem* suddenly becomes very easy to prove.

**Corollary 2.18.** (Tychonov's theorem) *Suppose* $(S_i)_{i\in I}$ *is a family of compact spaces. Then the space* $S := \prod_{i\in I} S_i$ *is compact as well.*

*Proof.* For $j \in I$, we denote the coordinate-wise projection $\prod_{i\in I} S_i \to S_j$ by $\pi_j$. Let $F \in \mathcal{U}(S)$ be an ultrafilter. For all $j \in I$, $S_j$ is compact, so we can choose a point $s_j \in S_j$ such that ${\pi_j}_*(F) \searrow s_j$. Since $\prod_{i\in I} S_i$ is induced by $(\pi_i)_{i\in I}$, it follows that $F$ converges to $(s_i)_{i\in I}$ by Lemma 2.17. So each ultrafilter on $\prod_{i\in I} S_i$ converges, making $\prod_{i\in I} X_i$ compact. $\square$

# 3 A categorical approach

Last chapter, we found out that each structure of a compact Hausdorff topology on $S$ is determined by a *function* $\mathcal{U}(S) \to S$. In this chapter, we will find out which functions. With pushforward-ultrafilters in mind (Definition 2.13), one can define the following functor:

$$\begin{aligned} \mathcal{U} \colon \mathbf{Sets} &\to \mathbf{Sets} \\ S &\mapsto \mathcal{U}(S) \\ f &\mapsto f_* \end{aligned}$$

It is easy to verify that this is actually a functor. To find more structure of it, we need some categorical concepts.

## 3.1 Monads and algebras

This section devotes to some categorical concepts that are necessary for this thesis. As such, this section has nothing to do with ultrafilters, but with *arbitrary* categories instead.

**Definition 3.1.** Let $\mathcal{C}$ be a category, and let $T \colon \mathcal{C} \to \mathcal{C}$ be any (endo)functor. Let $\eta \colon \mathrm{id}_{\mathcal{C}} \to T$ and $\mu \colon T^2 \to T$ be two natural transformations. The triplet $(T, \eta, \mu)$ is a *monad* if the following two diagrams commute, for all objects $S \in \mathcal{C}$:

$$\begin{tikzcd} & T(S) \arrow[dl, "T(\eta_S)"'] \arrow[d, equal] \arrow[dr, "\eta_{T(S)}"] & \\ T^2(S) \arrow[r, "\mu_S"] & T(S) & T^2(S) \arrow[l, "\mu_S"'] \end{tikzcd} \qquad \begin{tikzcd} T^3(S) \arrow[r, "\mu_{T(S)}"] \arrow[d, "T(\mu_S)"'] & T^2(S) \arrow[d, "\mu_S"] \\ T^2(S) \arrow[r, "\mu_S"] & T(S) \end{tikzcd}$$

I will illustrate this concept with three examples.

**Example 3.2.** Let $G$ be a group. Consider the following functor:

$$\begin{aligned} T \colon \mathbf{Sets} &\to \mathbf{Sets} \\ S &\mapsto G \times S \\ f &\mapsto \mathrm{id}_G \times f \end{aligned}$$

Then $(T, \eta, \mu)$ is a monad if we put:

$$\begin{aligned} \eta_S \colon s &\mapsto (1, s) \\ \mu_S \colon (g, g', s) &\mapsto (gg', s) \end{aligned}$$

One can verify that the left diagram holds since 1 is left and right neutral, and that the right one holds because the multiplication is associative. In that regard, the monads generalize the whole concept of *monoids*; group-like structures where the existence of inverses is unnecessary.

**Example 3.3.** For any set $S$, we can define $\mathbb{Z}[S]$ to be the ring of polynomials with coefficients in $\mathbb{Z}$, and variables in $S$. Any function $f\colon S \to S'$ induces a unique ring homomorphism $f_*\colon \mathbb{Z}[S] \to \mathbb{Z}[S']$ that sends $x$ to $f(x)$ for all $x \in S$. This allows us to define a functor given by:

$$\begin{aligned} T\colon \mathbf{Sets} &\to \mathbf{Sets} \\ S &\mapsto \mathbb{Z}[S] \\ f &\mapsto f_* \end{aligned}$$

For any set $S$, let $\eta_S\colon S \to \mathbb{Z}[S]$ be the inclusion map, and let $\mu_S\colon \mathbb{Z}[\mathbb{Z}[S]] \to \mathbb{Z}[S]$ be the evaluation map that sends each variable to its corresponding polynomial in $\mathbb{Z}[S]$. Then $(T, \eta, \mu)$ is a monad over **Sets**.

**Example 3.4.** Let $S$ be a topological space. Consider the category $\mathcal{P}(S)$ with the inclusion relation as arrows. Consider the following functor:

$$\begin{aligned} T\colon \mathcal{P}(S) &\to \mathcal{P}(S) \\ A &\mapsto \overline{A} \\ (A \subseteq B) &\mapsto (\overline{A} \subseteq \overline{B}) \end{aligned}$$

Then for all $A \subseteq S$, let $\eta_A$ be the inclusion relation $A \subseteq \overline{A}$, and let $\mu_A$ be the inclusion relation $\overline{\overline{A}} \subseteq \overline{A}$. Then $(T, \eta, \mu)$ is a monad simply because every pair of objects in $\mathcal{P}(A)$ can be connected by one arrow at most.

But there is another tool we will need, called *algebras*:

**Definition 3.5.** Let $(T, \eta, \mu)$ be a *monad* on $\mathcal{C}$. An *algebra* for this monad is an object $S$, together with an arrow $\alpha\colon T(S) \to S$, such that the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, equal] & T(S) \arrow[d, "\alpha"] \\ & S \end{tikzcd} \qquad \begin{tikzcd} T^2(S) \arrow[r, "T(\alpha)"] \arrow[d, "\mu_S"'] & T(S) \arrow[d, "\alpha"] \\ T(S) \arrow[r, "\alpha"] & S \end{tikzcd}$$

I will show what the algebras are of previous examples.

**Example 3.6.** First, consider the monad of Example 3.2. For any set $S$, the $G$-actions $G \times S \to S$ are precisely the algebra structures on $S$, as the diagrams express the desired properties.

**Example 3.7.** Now consider the monad of Example 3.3, and let $S$ be an arbitrary set. If a commutative ring structure on $S$ is given, then the evaluation map $\alpha\colon \mathbb{Z}[S] \to S$ is an algebra on $S$. Conversely, if an algebra $\alpha\colon \mathbb{Z}[S] \to S$ is given, then the set $I = \{p \in \mathbb{Z}[S]\colon \alpha(p) = \alpha(0)\}$ is an ideal of the ring $\mathbb{Z}[S]$ by the right diagram. By the left diagram, $\alpha$ is surjective (and behaves like some sort of evaluation map). So $\alpha$ induces a bijection $\mathbb{Z}[S]/I \to S$, and therefore a ring structure on $S$. And so, there is a one-to-one correspondence between ring structures and algebra structures on $S$.

**Example 3.8.** Lastly, consider the monad of Example 3.4. For any closed set $A \subseteq S$, the inclusion relation $\overline{A} \subseteq A$ is an algebra structure on $A$. If a subset $A \subseteq S$ is not closed, then $A$ does not have any algebra structures.

The fun part of algebras is that it allows us to construct a *new* category out of a given monad.

**Definition 3.9.** Let $\mathcal{T} := (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. The Eilenberg-Moore category of $\mathcal{T}$, denoted by $\mathcal{C}^{\mathcal{T}}$ consists of:

- objects in the form $(S, \alpha)$, where $\alpha\colon T(S) \to S$ are algebras. These are called the $\mathcal{T}$-modules.
- morphisms $f\colon (S, \alpha) \to (S', \alpha')$, which are arrows $f\colon S \to S'$ that respect the following diagram:

$$\begin{tikzcd} T(S) \arrow[r, "T(f)"] \arrow[d, "\alpha"'] & T(S') \arrow[d, "\alpha'"] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$

  These are called the $\mathcal{T}$-module morphisms. The composition of two arrows in $\mathcal{C}^{\mathcal{T}}$ is the same as in $\mathcal{C}$.

I will illustrate this, again, with the previous examples.

**Example 3.10.** Let $\mathcal{T}$ be the monad of Example 3.2. Then $\mathbf{Sets}^{\mathcal{T}}$ consists of the $G$-sets, and the arrows are the functions that preserve $G$-actions.

**Example 3.11.** Let $\mathcal{T}$ be the monad of Example 3.3. Then $\mathbf{Sets}^{\mathcal{T}}$ consists of the commutative rings, and the arrows are the ring homomorphisms.

**Example 3.12.** Let $\mathcal{T}$ be the monad of Example 3.4. Then $\mathcal{P}(S)^{\mathcal{T}}$ consists of the closed sets of $S$, and the arrows are (still) the inclusion relations.

## 3.2 The ultrafilter monad

We now want to create a monad $(\mathcal{U}, \eta, \mu)$ on the ultrafilter functor defined in the first paragraph. In order to do so, we have to find suitable choices for $\eta$ and $\mu$.

**Lemma 3.13.** *The transformation $\eta$ given by*

$$\begin{aligned} \eta_S \colon S &\to \mathcal{U}(S) \\ x &\mapsto P_x \end{aligned}$$

*is a natural transformation $\eta\colon \mathrm{id}_{\mathbf{Sets}} \to \mathcal{U}$.*

*Proof.* Let $f\colon S \to S'$ be any function. We must verify that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[d, "f"'] & \mathcal{U}(S) \arrow[d, "f_*"] \\ S' \arrow[r, "\eta_{S'}"] & \mathcal{U}(S') \end{tikzcd}$$

Let $x \in S$. We have to prove that $f_*(\eta_S(x)) = \eta_{S'}(f(x))$. Let $A \in \eta_{S'}(f(x))$. Then $A \in P_{f(x)}$ or, equivalently, $f(x) \in A$. So $x \in f^{-1}(A)$. Hence $f^{-1}(A) \in P_x$, which means that $A \in f_*(P_x) = f_*(\eta_S(x))$. So $\eta_{S'}(f(x)) \subseteq f_*(\eta_S(x))$. Since both are ultrafilters, it follows that $\eta_{S'}(f(x)) = f_*(\eta_S(x))$. □

Before we can define $\mu$, we need to define a new class of sets.

**Definition 3.14.** Let $S$ be a set and $A \subseteq S$ be a subset thereof. Then the set $[A] \subseteq \mathcal{U}(S)$ is given by:

$$[A] = \{F \in \mathcal{U}(S)\colon A \in F\}$$

**Lemma 3.15.** *The transformation $\mu$ given by*

$$\mu_S \colon \mathcal{U}^2(S) \to \mathcal{U}(S), \mathcal{F} \mapsto \{A \subseteq S\colon [A] \in \mathcal{F}\}$$

*is a natural transformation $\mu\colon \mathcal{U}^2 \to \mathcal{U}$.*

*Proof.* Let $f\colon S \to S'$ be any function. We must verify that the following diagram commutes:

$$\begin{tikzcd} \mathcal{U}^2(S) \arrow[r, "\mu_S"] \arrow[d, "f_{**}"'] & \mathcal{U}(S) \arrow[d, "f_*"] \\ \mathcal{U}^2(S') \arrow[r, "\mu_{S'}"] & \mathcal{U}(S') \end{tikzcd}$$

Let $\mathcal{F} \in U^2(S)$. We have to prove that $f_*(\mu_S(\mathcal{F})) = \mu_{S'}(f_{**}(\mathcal{F}))$. Let $A \in \mu_{S'}(f_{**}(\mathcal{F}))$. Then $[A] \in f_{**}(\mathcal{F})$, which implies that $f_*^{-1}([A]) \in \mathcal{F}$. Using the definitions, we can rewrite $(f_*)^{-1}([A])$ as:

$$\begin{aligned}(f_*)^{-1}([A]) &= \{F \in \mathcal{U}(S) \colon f_*(F) \in [A]\}\\ &= \{F \in \mathcal{U}(S) \colon A \in f_*(F)\}\\ &= \{F \in \mathcal{U}(S) \colon f^{-1}(A) \in F\}\\ &= [f^{-1}(A)]\end{aligned}$$

So $[f^{-1}(A)] \in \mathcal{F}$, which means that $f^{-1}(A) \in \mu_S(\mathcal{F})$, and hence $A \in f_*(\mu_S(\mathcal{F}))$. So $\mu_{S'}(f_{**}(\mathcal{F})) \subseteq f_*(\mu_S(\mathcal{F}))$ and hence $\mu_{S'}(f_{**}(\mathcal{F})) = f_*(\mu_S(\mathcal{F}))$. $\square$

**Lemma 3.16.** *The triplet $(\mathcal{U}, \eta, \mu)$ is a monad. We call it the ultrafilter monad.*

*Proof.* Let $S$ be a set. We must verify that the following diagrams commute:

$$\begin{tikzcd} & \mathcal{U}(S) \arrow[dl, "\eta_{S*}"'] \arrow[d, equal] \arrow[dr, "\eta_{\mathcal{U}(S)}"] & \\ \mathcal{U}^2(S) \arrow[r, "\mu_S"] & \mathcal{U}(S) & \mathcal{U}^2(S) \arrow[l, "\mu_S"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}^3(S) \arrow[r, "\mu_{\mathcal{U}(S)}"] \arrow[d, "\mu_{S*}"'] & \mathcal{U}^2(S) \arrow[d, "\mu_S"] \\ \mathcal{U}^2(S) \arrow[r, "\mu_S"] & \mathcal{U}(S) \end{tikzcd}$$

To check the right one, let $\mathcal{F} \in \mathcal{U}^3(S)$. Let $A \in \mu_S(\mu_{S*}(\mathcal{F}))$. Then $[A] \in \mu_{S*}(\mathcal{F})$ and hence $\mu_S^{-1}([A]) \in \mathcal{F}$. Using the definitions, we can rewrite $\mu_S^{-1}([A])$ as follows:

$$\begin{aligned}\mu_S^{-1}([A]) &= \{F \in \mathcal{U}^2(S) \colon \mu_S(F) \in [A]\}\\ &= \{F \in \mathcal{U}^2(S) \colon A \in \mu_S(F)\}\\ &= \{F \in \mathcal{U}^2(S) \colon [A] \in F\}\\ &= [[A]]\end{aligned}$$

This means that $[[A]] \in \mathcal{F}$. So $[A] \in \mu_{\mathcal{U}(S)}(\mathcal{F})$ and hence $A \in \mu_S(\mu_{\mathcal{U}(S)}(\mathcal{F}))$. It follows that $\mu_S(\mu_{S*}(\mathcal{F})) \subseteq \mu_S(\mu_{\mathcal{U}(S)}(\mathcal{F}))$ and hence $\mu_S(\mu_{S*}(\mathcal{F})) = \mu_S(\mu_{\mathcal{U}(S)}(\mathcal{F}))$. The right diagram is therefore commutative.
To prove that the left diagram commutes, we will prove first that the left triangle thereof commutes. To do so, let $F \in \mathcal{U}(S)$. Let $A \in \mu_S(\eta_{S*}(F))$. Then $[A] \in (\eta_{S*}(F))$. But that means that ${\eta_S}^{-1}([A]) \in F$. We can rewrite:

$$\begin{aligned}{\eta_S}^{-1}([A]) &= \{x \in S \colon \eta_S(x) \in [A]\}\\ &= \{x \in S \colon A \in P_x\}\\ &= \{x \in S \colon x \in A\}\\ &= A\end{aligned}$$

So $A \in F$. It follows that $\mu_S(\eta_{S*}(F)) \subseteq F$ and hence $\mu_S(\eta_{S*}(F)) = F$. It follows that the left triangle commutes.
To verify the right one, let $A \in \mu_S(\eta_{\mathcal{U}(S)}(F))$. Then $[A] \in \eta_{\mathcal{U}(S)}(F) = P_F$, which implies that $F \in [A]$ and hence $A \in F$. So $\mu_S \circ \eta_{\mathcal{U}(S)}(F) = F$, making the right triangle of the right diagram commutative as well.
So both diagrams commute, making $(\mathcal{U}, \eta, \mu)$ a monad. $\square$

## 3.3 Manes' theorem

Having constructed the ultrafilter monad, we may wonder what its Eilenberg-Moore category look like. This category will be equivalent to a very well-known category, the category of *compact Hausdorff spaces*. We will prove that in this section.

**Theorem 3.17.** *The Eilenberg-Moore category of the ultrafilter monad is equivalent to the category of compact Hausdorff spaces.*

We will start with the easiest part: proving that each convergence relation is an algebra.

**Lemma 3.18.** *Let $S$ be a compact Hausdorff space. The map $\alpha\colon \mathcal{U}(S) \to S$ that sends an ultrafilter to its limit point is an algebra on the monad $(\mathcal{U}, \eta, \mu)$.*

*Proof.* We have to prove that the following diagrams commute:

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, equal] & \mathcal{U}(S) \arrow[d, "\alpha"] \\ & S \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}^2(S) \arrow[r, "\alpha_*"] \arrow[d, "\mu_S"'] & \mathcal{U}(S) \arrow[d, "\alpha"] \\ \mathcal{U}(S) \arrow[r, "\alpha"] & S \end{tikzcd}$$

The left one follows immediately from Example 2.7. To prove the right one, let $\mathcal{F} \in U^2(S)$, and put $x = \alpha(\alpha_*(\mathcal{F}))$. Then $\alpha_*(\mathcal{F})$ contains all open neighborhoods of $x$. Now let $U$ be an open neighborhood of $x$. Then $\alpha^{-1}(U) \in \mathcal{F}$. All ultrafilters that converge to a point in $U$ contain $U$ itself, as $U$ is an open neighborhood of those points. So $\alpha^{-1}(U) \subseteq [U]$ and hence, $[U] \in \mathcal{F}$ as well. But this means that $U \in \mu_S(\mathcal{F})$. This holds for all open neighborhoods of $x$, so $\mu_S(\mathcal{F}) \searrow x$. Since the limit point of $\mu_S(\mathcal{F})$ is unique, as $S$ is compact Hausdorff, it follows that $\alpha(\mu_S(\mathcal{F})) = x$. So the right diagram commutes too. $\square$

Now, we have to do the hard part, creating a topology out of an algebra. This will happen with a few lemmas. We first need the concepts of *closure operators*.

**Definition 3.19.** A map $\mathrm{Cl}\colon \mathcal{P}(S) \to \mathcal{P}(S)$ is called a *Kuratowski closure operator* if it satisfies the following axioms, for all $A, A' \subseteq S$:

(1) $\mathrm{Cl}(\emptyset) = \emptyset$

(2) $A \subseteq \mathrm{Cl}(A)$

(3) $\mathrm{Cl}^2(A) \subseteq \mathrm{Cl}(A)$

(4) $\mathrm{Cl}(A \cup A') = \mathrm{Cl}(A) \cup \mathrm{Cl}(A')$

This operator defines a unique topology on $S$ where $\overline{A} = \mathrm{Cl}(A)$ for all $A \subseteq S$.

**Lemma 3.20.** *Let $\alpha\colon \mathcal{U}(S) \to S$ be an algebra. The map $Cl\colon \mathcal{P}(S) \to \mathcal{P}(S), A \mapsto \alpha([A])$ is a Kuratowski closure operator.*

*Proof.* We verify axioms 1-4 one by one (with $A, A' \subseteq S$ being arbitrary).

(1) Since no ultrafilter contains $\emptyset$, $[\emptyset] = \emptyset$. This means that $\mathrm{Cl}(\emptyset) = \alpha([\emptyset]) = \emptyset$.

(2) Let $x \in A$. Since $\alpha$ is an algebra, $\alpha(\eta_S(x)) = x$. As such, $\alpha(P_x) = x$. Since $A \in P_x$, or equivalently, $P_x \in [A]$, this means that $x \in \alpha([A]) = \mathrm{Cl}(A)$.

(3) Let $x \in \mathrm{Cl}^2(A)$. Then we can choose an ultrafilter $F \in \mathcal{U}(S)$ such that $\mathrm{Cl}(A) \in F$, and $\alpha(F) = x$. We will create an ultrafilter out of the following subset of $\mathcal{P}(\mathcal{U}(S))$:

$$\mathcal{C} = \{\alpha^{-1}(B) \colon B \in F\} \cup \{[A]\}$$

The set $\{\alpha^{-1}(B) \colon B \in F\}$ is closed under finite intersections and with $\alpha$ being surjective (by being the left inverse of $\eta_S$), it does not contain $\emptyset$. It will be therefore sufficient to prove $\alpha^{-1}(B) \cap \{[A]\} \neq \emptyset$ all $B \in F$. Suppose this is not the case, and that $\alpha^{-1}(B) \cap \{[A]\} = \emptyset$, for some $B \in F$. Then all ultrafilters $G \in \mathcal{U}(S)$ where $\alpha(G) \in B$ lack $A$. In particular, all ultrafilters where $\alpha(G)$ appears in the nonempty set $B \cap \mathrm{Cl}(A)$ lack $A$. This is a contradiction to the definition of $\mathrm{Cl}(A)$. So $\mathcal{C}$ indeed has FIP, and can be extended to an ultrafilter $\mathcal{F} \in U^2(S)$. Since $\mathcal{F}$ contains $\{\alpha^{-1}(B) : F \in F\}$, we have $\alpha_*(\mathcal{F}) = F$. And so, $\alpha(\alpha_*(\mathcal{F})) = x$. It follows from $\alpha$ being an algebra that $\alpha(\mu_S(\mathcal{F}) = x$ as well. Since $[A] \in \mathcal{F}$ by construction, this means that $\mu_S(\mathcal{F})$ is an ultrafilter containing $A$ and with image $x$. Hence $x \in \mathrm{Cl}(A)$.

(4) We first prove $\mathrm{Cl}(A \cup A') \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(A')$. Let $x \in \mathrm{Cl}(A \cup A')$. Then there exists an ultrafilter $F \in \mathcal{U}(S)$ containing $A \cup A'$ that converges to $x$. Then $F$ contains $A$ or $A'$; if it contained neither of them, then it would contain $A^{\mathsf{c}}$ and $A'^{\mathsf{c}}$, and therefore also $A^{\mathsf{c}} \cap A'^{\mathsf{c}} = (A \cup A')^{\mathsf{c}}$, which

is not true. So $F \in [A]$ or $F \in [A']$ and hence, $x \in \alpha([A])$ or $x \in \alpha([A'])$. So $x \in \mathrm{Cl}(A) \cup \mathrm{Cl}(A')$, and hence $\mathrm{Cl}(A \cup A') \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(A')$.
To prove the converse, let $x \in \mathrm{Cl}(A) \cup \mathrm{Cl}(A')$. Then $x \in \mathrm{Cl}(A)$ or $x \in \mathrm{Cl}(A')$. As such, there exists an ultrafilter $F \in \mathcal{U}(S)$ containing $A$ or $A'$ where $\alpha(F) = x$. Since ultrafilters are closed under supersets, $F$ contains $A \cup A'$, so $F \in [A \cup A']$. Hence $x \in \mathrm{Cl}(A \cup A')$. So $\mathrm{Cl}(A) \cup \mathrm{Cl}(A') \subseteq \mathrm{Cl}(A \cup A')$
So $\mathrm{Cl}(A) \cup \mathrm{Cl}(A') = \mathrm{Cl}(A \cup A')$. $\square$

This proof allows us to make a Kuratowski closure operator depending on $\alpha$. It is, however, unclear that the convergence relation of the resulting topology coincides with $\alpha$. So we need the following two rather similar lemmas. The first one yields an alternative definition for convergence relations.

**Lemma 3.21.** *Let $S$ be a topological space, $F \in \mathcal{U}(S)$, and $x \in S$. The following are equivalent:*

(1) $F \searrow x$

(2) *For every $A \in F$, $x \in \overline{A}$.*

*Proof.* (1) $\Rightarrow$ (2) Suppose there exists a set $A \in F$ such that $x \notin \overline{A}$. Then $x \in \overline{A}^{\mathsf{c}}$, which is open in $S$. By assumption, $\overline{A}^{\mathsf{c}} \in F$. With $A \subseteq \overline{A}$, we have $\overline{A} \in F$ as well. So we get a contradiction.

(2) $\Rightarrow$ (1) Suppose $F \not\searrow x$. Let $U$ be an open neighborhood of $x$ that $F$ lacks. Then $U^{\mathsf{c}} \in F$. By assumption, $x \in \overline{U^{\mathsf{c}}}$. But $U^{\mathsf{c}}$ is already closed in $S$, so $x \in U^{\mathsf{c}}$, which is false. $\square$

**Lemma 3.22.** *Let $\alpha\colon \mathcal{U}(S) \to S$ be an algebra, and $F \in \mathcal{U}(S)$ be an ultrafilter, and $x \in S$. The following are equivalent:*

(1) $\alpha(F) = x$

(2) *For all $A \in F$, $x \in \mathrm{Cl}(A)$.*

*Proof.* (1) $\Rightarrow$ (2). Let $A \in F$. Then $F \in [A]$, while $\alpha(F) = x$. So $x \in \mathrm{Cl}(A)$.

(2) $\Rightarrow$ (1). Consider the following subset of $\mathcal{P}(\mathcal{U}(S))$:

$$\mathcal{C} = \{[A]\colon A \in F\} \cup \{\alpha^{-1}(\{x\})\}$$

This set has FIP; this follows immediatly from the assumption. So it can be extended to an ultrafilter $\mathcal{F} \in \mathcal{U}^2(S)$. Since $\mathcal{F}$ contains $\{[A] : A \in F\}$, we find that $\mu_S(\mathcal{F}) = F$. Moreso, since $\alpha^{-1}(\{x\}) \in \mathcal{F}$, we find that $\{x\} \in \alpha_*(\mathcal{F})$, which means that $\alpha_*(\mathcal{F}) = P_x$. With $\alpha$ being an algebra, $\alpha(P_x) = x$ and $\alpha(\alpha_*(\mathcal{F})) = \alpha(\mu_S(\mathcal{F}))$. Combining our results gives $\alpha(F) = x$. $\square$

**Corollary 3.23.** *Let $\alpha\colon \mathcal{U}(S) \to S$ be an algebra. Then there exists a compact Hausdorff topology on $S$ such that $\alpha$ coincides with the convergence relation on $S$.*

*Proof.* By Lemma 3.20, Cl is a Kuratowski closure operator, and therefore induces a topology on $S$. The convergence relation of the resulting space coincides with $\alpha$, since for all $F \in \mathcal{U}(S)$ and $x \in S$, $F \searrow x$ if and only if $x \in \overline{A}$ for all $A \in F$ (Lemma 3.21), if and only if $x \in \mathrm{Cl}(A)$ for all $A \in F$, if and only if $\alpha(F) = x$ (Lemma 3.22). $\square$

**Remark 3.24.** I have not mentioned anything about the constructed topology being compact Hausdorff. This is, however, obvious from the fact that $\alpha$ is a function, and that a convergence relation is a function if and only if the underlying topology is compact Hausdorff, as seen in Corollary 2.12. The

**Remark 3.25.** The constructed topology is *unique*, since each convergence relation fully determines the topology, as proven in Theorem 2.9.

This shows that the objects of the Eilenberg-Moore category are precisely the compact Hausdorff spaces. But what about the arrows? We will find out next lemma.

**Lemma 3.26.** *Let $S$ and $S'$ be compact Hausdorff spaces with convergence relations $\alpha$ and $\alpha'$ respectively, and let $f\colon S \to S'$ be a map. The following are equivalent:*

(1) *$f$ is continuous.*

(2) *The following diagram commutes:*

$$\begin{tikzcd} \mathcal{U}(S) \arrow[r, "f_*"] \arrow[d, "\alpha"'] & \mathcal{U}(S') \arrow[d, "\alpha'"] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$

*Proof.* This follows immediately from Theorem 2.14. $\square$

This shows that the arrows of the Eilenberg-Moore category of $(\mathcal{U}, \eta, \mu)$ corresponds to the continuous maps. And so, this Eilenberg-Moore category fully corresponds to the category of compact Hausdorff spaces, proving Theorem 3.17.

**Remark 3.27.** The proof of Corollary 3.23 yields an ultrafilter interpretation for closures: if $\alpha$ is the convergence relation on $S$, then $\alpha([A]) = \overline{A}$ for all $A \subseteq S$.

# 4 Barr's theorem

The highlight of the previous chapter is that the category of compact Hausdorff spaces coincides with the Eilenberg-Moore category of the constructed monad. This chapter gives a brief sketch of how this generalizes to the category of *all* topological spaces. The result will be a correspondence proven by Barr (although he went for a different approach, see [4]).
The major roadblock we will face is that the convergence relation of arbitrary spaces no longer has to be a function; for this reason we will have work in the category of sets and *relations* instead. This category is more complicated, as commutativity of diagrams is a rough demand. As such, we will introduce the notion of *oplax diagrams.*
Proofs have been left out in this chapter, since they use the same ideas as earlier seen proofs (many are, in fact, almost copies thereof).

The category of sets and relations, denoted as **Rel**, consists of sets as its objects. The arrows $R : S \to S'$ are relations between $S$ and $S'$. The composition of two relations is defined as follows.

**Definition 4.1.** Let $R : S \to S'$ and $R' : S' \to S''$ be two relations. The composition $R' \circ R : S \to S''$ is defined by saying $x(R' \circ R)x'$ if there exists a $y \in S'$ such that $xRy$ and $yR'x'$.

The key difference between the categories **Sets** and **Rel** is the notion of *inclusions* of relations.

**Definition 4.2.** Let $R, R' : S \to S'$ be two relations. We say that $R \subseteq R'$ if for all $(x, x') \in S \times S'$ were $xRx'$, we have $xR'x'$.

**Remark 4.3.** When considering $R$ and $R'$ as subsets of $S \times S'$, this notion of inclusion coincides with regular inclusion of sets.

With this notion, the set $\mathrm{Hom}(S, S')$ (for all sets $S, S'$) is a category on its own: its objects are the relations and the arrows are the inclusion relations. And so, the category **Rel** is a *2-category.*
We will now repeat the story of chapter 3 on **Rel**. In order to do so, we will first need an ultrafilter functor. This will require a definition for *pushforward relations.*

**Definition 4.4.** Let $R\colon S \to S'$ be a relation. We define a *pushforward relation* $R_* : \mathcal{U}(S) \to \mathcal{U}(S')$ by saying that $FR_*G$ when the following equivalent conditions are met:

(1) For any $A \in F$, we have $AR \in G$. The set $AR \subseteq S'$ is given by:

$$AR = \{x \in S' \colon \text{there exists a } y \in A \text{ such that } yRx\}$$

(2) For any $A \in G$, we have $RA \in F$. The set $RA \subseteq S$ is given by:

$$RA = \{x \in S \colon \text{there exists a } y \in A \text{ such that } xRy\}$$

**Remark 4.5.** It follows immediately from condition (2) that when $R$ is a function, then $R_*$ coincides with the notion of pushforward maps from Definition 2.13.

One can prove that for two relations $R : S \to S'$ and $R' : S' \to S''$ the equality $R'_* \circ R_* = (R' \circ R)_*$ holds. And so, we get a functor given by:

$$\begin{aligned} \mathcal{U} \colon \mathbf{Rel} &\to \mathbf{Rel} \\ S &\mapsto \mathcal{U}(S) \\ R &\mapsto R_* \end{aligned}$$

**Remark 4.6.** It is easy to see that if $R \subseteq R'$, that $R_* \subseteq R'_*$. And so, $\mathcal{U}$ is a functor of 2-categories.

On **Rel**, we can define the transformations $\mu$ and $\eta$ the same way as we did on **Sets**. Unfortunately, they are no longer natural transformations. Nonetheless, they have some structure.

**Lemma 4.7.** *Let $R \colon S \to S'$ be a relation. The following diagrams hold:*

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[d, "R"'] \arrow[dr, phantom, "\subseteq"] & \mathcal{U}(S) \arrow[d, "R_*"] \\ S' \arrow[r, "\eta_{S'}"] & \mathcal{U}(S') \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}^2(S) \arrow[r, "\mu_S"] \arrow[d, "R_{**}"'] \arrow[dr, phantom, "\subseteq"] & \mathcal{U}(S) \arrow[d, "R_*"] \\ \mathcal{U}^2(S') \arrow[r, "\mu_{S'}"] & \mathcal{U}(S') \end{tikzcd}$$

Diagrams in this form are *oplax diagrams*. The left diagram means that $\eta_{S'} \circ R \subseteq R_* \circ \eta_S$, and the right diagram means $\mu_{S'} \circ R_{**} \subseteq R_* \circ \mu_S$. Lemma 4.7 tells us that $\mu$ and $\eta$ are *weak natural* transformations. And so, we call $(\mathcal{U}, \eta, \mu)$ a *weak monad* instead (since the monad diagrams still hold). On weak monads, one could still define algebras. Unfortunately, convergence relations do not always have to be algebras. As such, they also have a weaker structure now.

**Lemma 4.8.** *Let $S$ be a topological space. The convergence relation $\alpha \colon \mathcal{U}(S) \to S$ meets the following diagrams:*

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, equal] \arrow[r, phantom, shift right=4ex, "\subseteq"] & T(S) \arrow[d, "\alpha"] \\ & S \end{tikzcd} \qquad \begin{tikzcd} T^2(S) \arrow[r, "\mu_S"] \arrow[d, "\alpha_*"'] \arrow[dr, phantom, "\subseteq"] & T(S) \arrow[d, "\alpha"] \\ T(S) \arrow[r, "\alpha"] & S \end{tikzcd}$$

The convergence relation is now a *lax* algebra. If the inclusion symbols were pointed the other way, it would be called *colax*.

**Remark 4.9.** Strictness of diagrams are topological properties, equivalent to other well-known properties:

- The left diagram is strict if and only if all singletons are closed (i.e. the space is T1).
- The right diagram is strict if and only if the space is *core-compact* or *exponential*. On Hausdorff spaces, this property is equivalent to *locally compactness*. This is explained further on [7].

Similar to Corollary 3.23, the condition of a relation to be a lax algebra is a sufficient condition for being a convergence relation of a topological space. And so, there is a one-to-one correspondence between *lax algebras* and topological structures on $S$. Combining this result with Theorem 2.14, we can identify the category of topological spaces as follows.

**Theorem 4.10.** *The category of topological spaces and continuous maps is equivalent to the category consisting of:*

- *objects, which are in the form* $(S, \alpha)$*, where* $\alpha\colon \mathcal{U}(S) \to S$ *is a lax algebra.*
- *arrows* $f\colon (S, \alpha) \to (S', \alpha')$ *satisfying the following diagram:*

$$\begin{tikzcd}
\mathcal{U}(S) \arrow[r, "f_*"] \arrow[d, "\alpha_S"'] \arrow[dr, phantom, "\subseteq"] & \mathcal{U}(S') \arrow[d, "\alpha_{S'}"] \\
S \arrow[r, "f"] & S'
\end{tikzcd}$$

**Remark 4.11.** It is easy to see that the *coarser* a topology on $S$ is, the *bigger* it is corresponding convergence relation will be. As such, we get a *Galois correspondence* between the lax algebras and the topologies on a set.

# 5 The Stone-Čech compactification

This chapter yields a fun application of everything proven in chapter 3: the Stone-Čech compactification. We will first give the characterization of a Stone-Čech compactification of a space.

**Definition 5.1.** Let $S$ be a topological space. A *Stone-Čech compactification* of $S$ is a pair $(\beta S, i)$. Here, $\beta S$ is a compact Hausdorff space, and $i\colon S \to \beta S$ a continuous map, satisfying the universal property: for every continuous map $f\colon S \to C$, where $C$ is a compact Hausdorff space, there exists a *unique* map $\hat{f}\colon \beta S \to C$ such that the following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "f"'] & \beta S \arrow[d, "\hat{f}"] \\ & C \end{tikzcd}$$

Thanks to the universal property, a Stone-Čech compactification of a space is unique up to unique homeomorphism. While there are multiple constructions, the construction we will see in this chapter is the most elegant, in my opinion.

## 5.1 Stone-Čech compactification of discrete spaces

We start with construction of Stone-Čech compactification of discrete spaces. The fun part is that this section *solely* relies on the diagrams of chapter 3. And so, this result can be extended to any monad. I will point this out in Remark 5.7.

**Theorem 5.2.** *Let $S$ be discrete space. On the set $\mathcal{U}(S)$, equip the topology with convergence relation $\mu_S$. Then the pair $(\mathcal{U}(S), \eta_S)$ is a Stone-Čech compactification of $S$.*

In this theorem, we assumed $\mu_S$ to be a convergence relation. We actually have to prove that.

**Lemma 5.3.** *The map $\mu_S\colon \mathcal{U}^2(S) \to \mathcal{U}(S)$ is an algebra on $\mathcal{U}(S)$, and therefore the convergence relation of a compact Hausdorff space.*

*Proof.* We have to verify the following diagrams hold:

$$\begin{tikzcd} \mathcal{U}(S) \arrow[r, "\eta_{\mathcal{U}(S)}"] \arrow[dr, equal] & \mathcal{U}^2(S) \arrow[d, "\mu_S"] \\ & \mathcal{U}(S) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U}^3(S) \arrow[r, "\mu_{\mathcal{U}(S)}"] \arrow[d, "\mu_{S*}"'] & \mathcal{U}^2(S) \arrow[d, "\mu_S"] \\ \mathcal{U}^2(S) \arrow[r, "\mu_S"] & \mathcal{U}(S) \end{tikzcd}$$

Both of them hold thanks to $(\mathcal{U}, \eta, \mu)$ being a monad. $\square$

So we can equip $\mathcal{U}(S)$ with the topology induced by $\mu_S$. This gives us a compact Hausdorff space, as $\mu_S$ is a function. Moreso, we can consider $S$ as a subspace by the embedding $\eta_S$. This is continuous, as all functions from discrete spaces are.
Now let $f : S \to C$ be an arbitrary continuous function, where $C$ is a compact Hausdorff space with convergence relation $\beta$. In the next two lemmas, we will prove that $f$ factors through $\eta_S$, i.e. there exists a continuous map $\hat{f} \colon \mathcal{U}(S) \to C$ such that $f = \hat{f} \circ \eta_S$.

**Lemma 5.4.** *The map $\hat{f} := \beta \circ f_*$ meets the equality $\hat{f} \circ \eta_S = f$.*

*Proof.* With $\eta$ being a natural transformation, and $\beta$ being an algebra, the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\eta_S"'] & C \arrow[d, "\eta_C"'] \arrow[dr, equal] & \\
\mathcal{U}(S) \arrow[r, "f_*"] & \mathcal{U}(C) \arrow[r, "\beta"] & C
\end{tikzcd}$$

So, indeed, $f = \hat{f} \circ \eta_S$. $\square$

**Lemma 5.5.** *The map $\hat{f} : \mathcal{U}(S) \to C$ is continuous.*

*Proof.* We will prove this by using the interpretation of continuity by Lemma 3.26. With $\mu$ being natural and $\beta$ being an algebra, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{U}^2(S) \arrow[r, "f_{**}"] \arrow[d, "\mu_S"'] & \mathcal{U}^2(C) \arrow[r, "\beta_*"] \arrow[d, "\mu_C"] & \mathcal{U}(C) \arrow[d, "\beta"] \\
\mathcal{U}(S) \arrow[r, "f_*"] & \mathcal{U}(C) \arrow[r, "\beta"] & C
\end{tikzcd}$$

Since $\hat{f} = \beta \circ f_*$ and $\hat{f}_* = \beta_* \circ f_{**}$, this result yields the desired diagram. $\square$

We have proven that there exists a continuous map $\hat{f} \colon \mathcal{U}(S) \to C$ such that $f = \hat{f} \circ \eta_S$. But is that extension *unique*? We will prove that in the following lemma:

**Lemma 5.6.** *Let $g : \mathcal{U}(S) \to C$ be a continuous map such that $g \circ \eta_S = f$. Then $g = \hat{f}$.*

*Proof.* Using that $g$ is continuous and that $(\mathcal{U}, \eta, \mu)$ is a monad yields the following commutative diagram:

$$\begin{tikzcd}
\mathcal{U}(S) \arrow[r, "\eta_{S*}"] \arrow[dr, equal] & \mathcal{U}^2(S) \arrow[r, "g_*"] \arrow[d, "\mu_S"] & \mathcal{U}(C) \arrow[d, "\beta"] \\
& \mathcal{U}(S) \arrow[r, "g"] & C
\end{tikzcd}$$

Since $\hat{f} = \beta \circ f_* = \beta \circ g_* \circ \eta_{S*}$, this diagram yields the desired result. $\square$

So we have proven that each map $f \colon S \to C$, where $C$ is a compact Hausdorff space, extends uniquely to a map $\hat{f} \colon \mathcal{U}(S) \to C$ such that $\hat{f} \circ \eta_S = f$. So indeed, the pair $(\mathcal{U}(S), \eta_S)$ is a Stone-Čech compactification for $S$, when it is discrete.

**Remark 5.7.** This result extends to arbitrary monads as follows. Let $\mathcal{T} := (T, \mu, \eta)$ be a monad over a category $\mathcal{C}$. Let $S \in \mathcal{C}$ be an object, and let $(M, \alpha) \in \mathcal{C}^{\mathcal{T}}$ be an $\mathcal{T}$-module. For every arrow $f : S \to M$, there exists a *unique* $\mathcal{T}$-module morphism $\hat{f} : (T(S), \mu_S) \to (M, \alpha)$ such that the following diagram commutes:

$$\begin{tikzcd}
S \arrow[r, "\eta_S"] \arrow[dr, "f"] & T(S) \arrow[d, "\hat{f}"] \\
& M
\end{tikzcd}$$

And so, $(T(S), \mu_S)$, along with the arrow $\eta_S$, is called a *free module* over $S$.

The next question is: why is this trick invalid for arbitrary spaces? The problem here is that $\eta_S$ is not always continuous. In fact, one can prove that $\eta_S$ is continuous *exclusively* when $S$ is discrete. In the next section, we will make some adjustments to make this work for arbitrary spaces as well.

## 5.2 Stone-Čech compactification of arbitrary spaces

During the project, I found that taking the Stone-Čech compactification of an arbitrary space is not only possible, but also pretty easy to do out of a given Stone-Čech compactification of the underlying discrete space.
So we will assume now that $S$ is an *arbitrary* topological space, and $\mathcal{U}(S)$ will be the topological space we constructed last chapter. It will be sufficient to take a *quotient* of $\mathcal{U}(S)$, which is very similar to the *Hausdorff quotient.* As quotient spaces lack the nice ultrafilter interpretation product spaces have, we will have to rely on elementary facts instead.

**Theorem 5.8.** *Let $S$ be an arbitrary topological space. On $\mathcal{U}(S)$, we define an equivalence relation $\sim$ by declaring $F$ and $G$ to be equivalent if $\hat{f}(F) = \hat{f}(G)$ for all continuous maps $f$ to compact Hausdorff spaces. Put $\tilde{\eta}_S := q \circ \eta_S$ (where $q$ is the quotient map). Then $(\mathcal{U}(S), \tilde{\eta}_S)$ is a Stone-Čech compactification for $S$.*

**Remark 5.9.** The equivalence relation on $\mathcal{U}(S)$ has been defined so that for any continuous map $f: S \to C$, the map $\hat{f}: \mathcal{U}(S) \to C$ factors uniquely through $q$, i.e. there exists a unique continuous map $\hat{f}': \mathcal{U}(S)/{\sim} \to C$ such that $\hat{f} = \hat{f}' \circ q$.

The fundamental difference with this construction is that $\tilde{\eta}_S$ *is* continuous now. We will prove this now. But first, we will need a lemma to show the impact of the convergence relation of $S$.

**Lemma 5.10.** *Let $F \in \mathcal{U}(S)$ and $x \in S$, and suppose that $F \searrow x$. Then $F \sim P_x$.*

*Proof.* Let $C$ a compact Hausdorff space with convergence relation $\beta$, and let $f: S \to C$ be a continuous map. Then $f_*(F)$ and $f_*(P_x)$ both converge to $f(x)$. Since $\beta$ is a function, this means that $\beta \circ f_*(F) = \beta \circ f_*(P_x) = f(x)$ and hence $\hat{f}(F) = \hat{f}(P_x)$. This holds for all continuous maps $f$ to compact Hausdorff spaces. So $F \sim P_x$. $\square$

**Corollary 5.11.** *The map $\tilde{\eta}_S$ is continuous.*

*Proof.* Let $F \in \mathcal{U}(S)$ and $x \in S$ such that $F \searrow x$. Since $\mu_S \circ \eta_{\mathcal{U}(S)} = \mathrm{id}_{\mathcal{U}(S)}$, it follows that $\eta_{S*}(F) \searrow F$ (in $\mathcal{U}(S)$). Since $q$ is continuous, it follows that $\tilde{\eta}_{S*}(F) \searrow \overline{F}$ (in $\mathcal{U}(S)/{\sim}$). But $F \sim P_x$ by Lemma 5.10, so $\tilde{\eta}_S(x) = \overline{P_x} = \overline{F}$, and hence $\tilde{\eta}_{S*}(F) \searrow \tilde{\eta}_S(x)$. $\square$

The space $\mathcal{U}(S)/{\sim}$ would never be a Stone-Čech campactification of $S$ if it were not compact Hausdorff itself. The space is certainly compact, as quotients of compact spaces always are. However, we do need to prove that $\mathcal{U}(S)/{\sim}$ is Hausdorff.

**Lemma 5.12.** *The space $\mathcal{U}(S)/{\sim}$ is Hausdorff.*

*Proof.* Suppose it is not. Then there exists an ultrafilter $\mathcal{F} \in \mathcal{U}(\mathcal{U}(S)/{\sim})$ with two distinct limits, say, $\overline{F}$ and $\overline{F'}$ for some ultrafilters $F, F' \in \mathcal{U}(S)$. Then $F$ and $F'$ are not equivalent, so there exists a continuous map $f : S \to C$ to a compact Hausdorff space such that $\hat{f}(F) \neq \hat{f}(G)$. Then $\hat{f}'(\overline{G}) \neq \hat{f}'(\overline{F'})$ either, with $\hat{f}' : \mathcal{U}(S)/{\sim} \to C$ from Remark 5.9. But $\hat{f}'$ is continuous, so $\hat{f}'_*(\mathcal{F})$ converges to two points. This is a contradiction since $C$ is Hausdorff. $\square$

So the space $\mathcal{U}(S)/{\sim}$ is compact Hausdorff. By Corollary 5.11, the map $\tilde{\eta}_S$ is continuous, and by Remark 5.9, each continuous map $f$ to a compact Hausdorff space factors uniquely through $\tilde{\eta}_S$. So $(\mathcal{U}(S)/{\sim}, \tilde{\eta}_S)$ is a Stone-Čech compactification for $S$.

**Remark 5.13.** By *Urysohn's lemma*, for each pair of distinct points $x, y$ of a compact Hausdorff space $C$, there exists a continuous map $f\colon C \to [0, 1]$ such that $f(x) \neq f(y)$. As such, two ultrafilters $F, G \in \mathcal{U}(S)$ will be equivalent precisely when $\hat{f}(F) = \hat{f}(G)$ for all continuous maps $f\colon S \to [0, 1]$.

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