ADMISSIBLE CONSTANTS FOR GENUS 2 CURVES

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Abstract. S.-W. Zhang recently introduced a new adelic invariant $\varphi$ for curves of genus at least 2 over number fields and function fields. We calculate this invariant when the genus is equal to 2.

1. Introduction

Let $X$ be a smooth projective geometrically connected curve of genus $g \geq 2$ over a field $k$ which is either a number field or the function field of a curve over a field. Assume that $X$ has semistable reduction over $k$. For each place $v$ of $k$, let $N_v$ be the usual local factor connected with the product formula for $k$.

In a recent paper [11] S.-W. Zhang proves the following theorem:

Theorem 1.1. Let $(\omega, \omega)_a$ be the admissible self-intersection of the relative dualizing sheaf of $X$. Let $\langle \Delta \xi, \Delta \xi \rangle$ be the height of the canonical Gross-Schoen cycle on $X^3$. Then the formula:

$$(\omega, \omega)_a = \frac{2g-2}{2g+1} \left( \langle \Delta \xi, \Delta \xi \rangle + \sum_v \varphi(X_v) \log N_v \right)$$

holds, where the $\varphi(X_v)$ are local invariants associated to $X \otimes k_v$, defined as follows:

- if $v$ is a non-archimedean place, then:

$$\varphi(X_v) = -\frac{1}{4} \delta(X_v) + \frac{1}{4} \int_{R(X_v)} g_v(x, x)((10g + 2)\mu_v - \delta K_{X_v}) ,$$

where:

- $\delta(X_v)$ is the number of singular points on the special fiber of $X \otimes k_v$,
- $R(X_v)$ is the reduction graph of $X \otimes k_v$,
- $g_v$ is the Green’s function for the admissible metric $\mu_v$ on $R(X_v)$,
- $K_{X_v}$ is the canonical divisor on $R(X_v)$.

In particular, $\varphi(X_v) = 0$ if $X$ has good reduction at $v$;

- if $v$ is an archimedean place, then:

$$\varphi(X_v) = \sum_{\ell} \frac{2}{\lambda_\ell} \sum_{m,n=1}^g \left| \int_{X(k_v)} \phi_\ell \omega_m \bar{\omega}_n \right|^2 ,$$

where $\phi_\ell$ are the normalized real eigenforms of the Arakelov Laplacian on $X(k_v)$ with eigenvalues $\lambda_\ell > 0$, and $(\omega_1, \ldots, \omega_g)$ is an orthonormal basis for the hermitian inner product $(\omega, \eta) \mapsto \frac{1}{2} \int_{X(k_v)} \omega \bar{\eta}$ on the space of holomorphic differentials.

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Apart from giving an explicit connection between the two canonical invariants \((\omega, \omega)_a\) and \(\langle \Delta_\xi, \Delta_\xi \rangle\), Zhang’s theorem has a possible application to the effective Bogomolov conjecture, i.e., the question of giving effective positive lower bounds for \((\omega, \omega)_a\). Indeed, the height of the canonical Gross-Schoen cycle \(\langle \Delta_\xi, \Delta_\xi \rangle\) is known to be non-negative in the case of a function field in characteristic zero, and should be non-negative in general by a standard conjecture of Gillet-Soulé (op. cit., Section 2.4). Further, the invariant \(\varphi\) should be non-negative, and Zhang proposes, in the non-archimedean case, an explicit lower bound for it which is positive in the case of non-smooth reduction (op. cit., Conjecture 1.4.2). Note that it is clear from the definition that \(\varphi\) is non-negative in the archimedean case; in fact it is positive (op. cit., Remark after Proposition 2.5.3).

Besides \(\varphi(X_v)\), Zhang also considers the invariant \(\lambda(X_v)\) defined by:

\[
\lambda(X_v) = \frac{g - 1}{6(2g + 1)} \varphi(X_v) + \frac{1}{12} (\varepsilon(X_v) + \delta(X_v)),
\]

where:

- if \(v\) is a non-archimedean place, the invariant \(\delta(X_v)\) is as above, and:
  
  \[
  \varepsilon(X_v) = \int_{R(X_v)} g_v(x, x)((2g - 2)\mu_v + \delta_{K_{X_v}}),
  \]

- if \(v\) is an archimedean place, then:
  
  \[
  \delta(X_v) = \delta_F(X_v) - 4g \log(2\pi)
  \]

with \(\delta_F(X_v)\) the Faltings delta-invariant of the compact Riemann surface \(X(\overline{k_v})\), and \(\varepsilon(X_v) = 0\).

The significance of this invariant is that if \(\text{deg det } R_{\pi_*} \omega\) denotes the (non-normalized) geometric or Faltings height of \(X\) one has a simple expression:

\[
\text{deg det } R_{\pi_*} \omega = \frac{g - 1}{6(2g + 1)} \langle \Delta_\xi, \Delta_\xi \rangle + \sum_v \lambda(X_v) \log N_v
\]

for \(\text{deg det } R_{\pi_*} \omega\), as follows from the Noether formula:

\[
12 \text{deg det } R_{\pi_*} \omega = (\omega, \omega)_a + \sum_v (\varepsilon(X_v) + \delta(X_v)) \log N_v.
\]

Now assume that \(X\) has genus \(g = 2\). Our purpose is to calculate the invariants \(\varphi(X_v)\) and \(\lambda(X_v)\) explicitly. For the \(\lambda\)-invariant we obtain:

- if \(v\) is non-archimedean, then:
  
  \[
  10 \lambda(X_v) = \delta_0(X_v) + 2\delta_1(X_v),
  \]

  where \(\delta_0(X_v)\) is the number of non-separating nodes and \(\delta_1(X_v)\) is the number of separating nodes in the special fiber of \(X \otimes k_v\);

- if \(v\) is archimedean, then:
  
  \[
  10 \lambda(X_v) = -20 \log(2\pi) - \log \|\Delta_2\|(X_v),
  \]

  where \(\|\Delta_2\|(X_v)\) is the normalized modular discriminant of the compact Riemann surface \(X(\overline{k_v})\) (see below).

Thus, the \(\lambda(X_v)\) are precisely the well-known local invariants corresponding to the discriminant modular form of weight 10 [6] [9] [10]. In particular we have:

\[
\text{deg det } R_{\pi_*} \omega = \sum_v \lambda(X_v) \log N_v
\]
and we recover the fact that the height of the canonical Gross-Schoen cycle vanishes for $X$.

2. The non-archimedean case

Let $k$ be a complete discretely valued field. Let $X$ be a smooth projective geometrically connected curve of genus 2 over $k$. Assume that $X$ has semistable reduction over $k$. In this section we give the invariants $\varphi(X)$ and $\lambda(X)$ of $X$.

The proof of our result is based on the classification of the semistable fiber types in genus 2 and consists of a case-by-case analysis. The notation we employ for the various fiber types is as in [8]. We remark that there are no restrictions on the residue characteristic of $k$.

**Theorem 2.1.** The invariant $\varphi(X)$ is given by the following table, depending on the type of the special fiber of the regular minimal model of $X$:

<table>
<thead>
<tr>
<th>Type</th>
<th>$\delta_0$</th>
<th>$\delta_1$</th>
<th>$\varepsilon$</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$II(a)$</td>
<td>0</td>
<td>$a$</td>
<td>$\frac{1}{6}a$</td>
<td>$a$</td>
</tr>
<tr>
<td>$III(a)$</td>
<td>$a$</td>
<td>0</td>
<td>$\frac{1}{6}a$</td>
<td>$\frac{1}{12}a$</td>
</tr>
<tr>
<td>$IV(a,b)$</td>
<td>$b$</td>
<td>$a$</td>
<td>$a + \frac{1}{6}b$</td>
<td>$a + \frac{1}{12}b$</td>
</tr>
<tr>
<td>$V(a,b)$</td>
<td>$a + b$</td>
<td>0</td>
<td>$\frac{1}{6}(a+b)$</td>
<td>$\frac{1}{12}(a+b)$</td>
</tr>
<tr>
<td>$VI(a,b,c)$</td>
<td>$b + c$</td>
<td>$a$</td>
<td>$a + \frac{1}{6}(b+c)$</td>
<td>$a + \frac{1}{12}(b+c)$</td>
</tr>
<tr>
<td>$VII(a,b,c)$</td>
<td>$a + b + c$</td>
<td>0</td>
<td>$\frac{1}{6}(a+b+c) + \frac{1}{6}\frac{abc}{abc + 6bc + 6ca} + \frac{1}{12}(a+b+c) - \frac{1}{12}\frac{abc}{abc + 6bc + 6ca}$</td>
<td></td>
</tr>
</tbody>
</table>

For $\lambda(X)$ the formula:

$$10\lambda(X) = \delta_0(X) + 2\delta_1(X)$$

holds.

Let us indicate how the theorem is proved. Let $r$ be the effective resistance function on the reduction graph $R(X)$ of $X$, extended bilinearly to a pairing on $\text{Div}(R(X))$. By Corollary 2.4 of [2] the formula:

$$\varphi(X) = -\frac{1}{4}(\delta_0(X) + \delta_1(X)) - \frac{3}{8}r(K, K) + 2\varepsilon(X)$$

holds, where $K$ is the canonical divisor on $R(X)$. The invariant $r(K, K)$ is calculated by viewing $R(X)$ as an electrical circuit. The invariant $\varepsilon$ is calculated on the basis of explicit expressions for the admissible measure and admissible Green’s function; see [7] and [8] for such computations. The results we find are as follows:
The values of $\phi$ follow.

The formula for $\lambda(X)$ is verified for each case separately.

3. The archimedean case

Let $X$ be a compact and connected Riemann surface of genus 2. In this section we calculate the invariants $\phi(X)$ and $\lambda(X)$ of $X$. Let $\text{Pic}(X)$ be the Picard variety of $X$, and for each integer $d$ denote by $\text{Pic}^d(X)$ the component of $\text{Pic}(X)$ of degree $d$.

We have a canonical theta divisor $\Theta$ on $\text{Pic}^1(X)$, and a standard hermitian metric $\|\cdot\|$ on the line bundle $\mathcal{O}(\Theta)$ on $\text{Pic}^1(X)$. Let $\nu$ be its curvature form. We have:

$$\int_{\text{Pic}^1(X)} \nu^2 = \Theta = 2.$$

Let $K$ be a canonical divisor on $X$, and let $P$ be the set of 10 points $P$ of $\text{Pic}^1(X) - \Theta$ such that $2P \equiv K$. Denote by $\|\theta\|$ the norm of the canonical section $\theta$ of $\mathcal{O}(\Theta)$. We let:

$$\|\Delta_2\|(X) = 2^{-12} \prod_{P \in P} \|\theta\|^2(P),$$

the normalized modular discriminant of $X$, and we let $\|H\|(X)$ be the invariant of $X$ defined by:

$$\log \|H\|(X) = \frac{1}{2} \int_{\text{Pic}^1(X)} \log \|\theta\| \nu^2.$$

These two invariants were introduced in [1].

**Theorem 3.1.** For the $\phi$-invariant and the $\lambda$-invariant of $X$, the formulas:

$$\phi(X) = -\frac{1}{2} \log \|\Delta_2\|(X) + 10 \log \|H\|(X)$$

and

$$10\lambda(X) = -20 \log(2\pi) - \log \|\Delta_2\|(X)$$

hold.

The key to the proof is the following lemma. Let $\Phi$ be the map:

$$X^2 \to \text{Pic}^1(X), \quad (x, y) \mapsto [2x - y].$$

**Lemma 3.2.** The map $\Phi$ is finite flat of degree 8.
Proof. Let \( y \mapsto y' \) be the hyperelliptic involution of \( X \). We have a commutative diagram:

\[
\begin{array}{ccc}
X^2 & \xrightarrow{\Phi} & \text{Pic}^1(X) \\
\downarrow{\alpha} & & \downarrow{\beta} \\
X^2 & \xrightarrow{\Phi'} & \text{Pic}^3(X)
\end{array}
\]

where \( \alpha \) and \( \beta \) are isomorphisms, with:

\[
\alpha: X^2 \to X^2, \quad \Phi': X^2 \to \text{Pic}^3(X), \quad \beta: \text{Pic}^3(X) \to \text{Pic}^1(X),
\]

\( (x, y) \mapsto (x, y') \), \( (x, y) \mapsto [2x + y] \), \( [D] \mapsto [D - K] \).

It suffices to prove that \( \Phi' \) is finite flat of degree 8. Let \( p: X^{(3)} \to \text{Pic}^3(X) \) be the natural map; then \( p \) is a \( \mathbb{P}^1 \)-bundle over \( \text{Pic}^3(X) \), and \( \Phi' \) has a natural injective lift to \( X^{(3)} \). A point \( D \) on \( X^{(3)} \) is in the image of this lift if and only if \( D \), when seen as an effective divisor on \( X \), contains a point which is ramified for the morphism \( X \to \mathbb{P}^1 \) determined by the fiber \( |D| \) of \( p \) in which \( D \) lies. Since every morphism \( X \to \mathbb{P}^1 \) associated to a \( D \) on \( X^{(3)} \) is ramified, the map \( \Phi' \) is surjective. As every morphism \( X \to \mathbb{P}^1 \) associated to a \( D \) on \( X^{(3)} \) has only finitely many ramification points, the map \( \Phi' \) is quasi-finite, hence finite since \( \Phi' \) is proper. As \( X^2 \) and \( \text{Pic}^3(X) \) are smooth and the fibers of \( \Phi' \) are equidimensional, the map \( \Phi' \) is flat.

By Riemann-Hurwitz the generic \( X \to \mathbb{P}^1 \) associated to a \( D \) on \( X^{(3)} \) has 8 simple ramification points. It follows that the degree of \( \Phi' \) is 8.

Let \( G: X^2 \to \mathbb{R} \) be the Arakelov-Green’s function of \( X \), and let \( \Delta \) be the diagonal divisor on \( X^2 \). We have a canonical hermitian metric on the line bundle \( O(\Delta) \) on \( X^2 \) by putting \( \|1\|((x, y)) = G(x, y) \), where 1 is the canonical section of \( O(\Delta) \). Denote by \( h_\Delta \) the curvature form of \( O(\Delta) \). We have:

\[
\int_{X^2} h_\Delta^2 = \Delta, \Delta = -2.
\]

Restricting \( O(\Delta) \) to a fiber of any of the two natural projections of \( X^2 \) onto \( X \) and taking the curvature form we obtain the Arakelov \((1, 1)\)-form \( \mu \) on \( X \). We have \( \int_X \mu = 1 \) and:

\[
\int_X \log G(x, y) \, \mu(x) = 0
\]

for each \( y \) on \( X \). Let \( (\omega_1, \omega_2) \) be an orthonormal basis of \( H^0(X, \omega_X) \), the space of holomorphic differentials on \( X \). We can write explicitly:

\[
h_\Delta(x, y) = \mu(x) + \mu(y) - i \sum_{k=1}^2 (\omega_k(x) \bar{\omega}_k(y) + \omega_k(y) \bar{\omega}_k(x))
\]

and:

\[
\mu(x) = \frac{i}{3} \sum_{k=1}^2 \omega_k(x) \bar{\omega}_k(x).
\]

By [11, Proposition 2.5.3] we have:

\[
\varphi(X) = \int_{X^2} \log h_\Delta^2.
\]

We compute the integral using our results from [4] and [5]. Let \( W \) be the divisor of Weierstrass points on \( X \), and let \( p_1: X^2 \to X \) be the projection onto the first
coordinate. The divisor \( W \) is reduced effective of degree 6. According to [3, p. 31] there exists a canonical isomorphism:

\[
\sigma : \Phi^* \mathcal{O}(\Theta) \cong \mathcal{O}(2\Delta + p_1^*W)
\]

of line bundles on \( X^2 \), identifying the canonical sections on both sides. In [4, Proposition 2.1] we proved that this isomorphism has a constant norm over \( X^2 \). Thus, the curvature forms on both sides are equal:

\[
\Phi^* \nu = 2h_\Delta + 6\mu(x) \quad \text{on} \quad X^2.
\]

Squaring both sides of this identity we get:

\[
h^2_\Delta = \frac{1}{4} \Phi^*(\nu^2) - 6h_\Delta \mu(x),
\]

since \( \mu(x)^2 = 0 \). Denote by \( S(X) \) the norm of \( \sigma \). Then we have:

\[
2 \log G(x, y) + \sum_w \log G(x, w) = \log \|\theta\|(2x - y) + \log S(X)
\]

for generic \((x, y) \in X^2\), where \( w \) runs through the Weierstrass points of \( X \). By fixing \( y \) and integrating against \( \mu(x) \) on \( X \) we find that:

\[
\log S(X) = - \int_X \log \|\theta\|(2x - y) \mu(x).
\]

By integrating against \( h^2_\Delta \) on \( X^2 \) we obtain:

\[
2\varphi(X) + \sum_w \int_{X^2} \log G(x, w) h^2_\Delta = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x - y) h^2_\Delta.
\]

As we have:

\[
h^2_\Delta = 2\mu(x)\mu(y) - \sum_{k,l=1}^2 (\omega_k(x)\omega_l(x)\omega_k(y)\omega_l(y) + \omega_k(x)\omega_l(x)\omega_k(y)\omega_l(y))
\]

it follows that:

\[
\int_{X^2} \log G(x, w) h^2_\Delta = 0
\]

for each \( w \) in \( W \) and hence we simply have:

\[
2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x - y) h^2_\Delta.
\]

Using our earlier expression for \( h^2_\Delta \) this becomes:

\[
2\varphi(X) = -2 \log S(X) + \int_{X^2} \log \|\theta\|(2x - y) \left( \frac{1}{4} \Phi^*(\nu^2) - 6h_\Delta \mu(x) \right).
\]

It is easily verified that \( h_\Delta \mu(x) = h_\Delta \mu(y) = \mu(x)\mu(y) \) and hence:

\[
\int_{X^2} \log \|\theta\|(2x - y) h_\Delta \mu(x) = \int_{X^2} \log \|\theta\|(2x - y) \mu(x)\mu(y) = - \log S(X).
\]

From Lemma 3.2 it follows that:

\[
\int_{X^2} \log \|\theta\|(2x - y) \Phi^*(\nu^2) = 8 \int_{Pic^1(X)} \log \|\theta\| \nu^2 = 16 \log \|H\|(X).
\]

All in all we find:

\[
\varphi(X) = 2 \log S(X) + 2 \log \|H\|(X).
\]
Let $\delta_F(X)$ be the Faltings delta-invariant of $X$. According to [5, Corollary 1.7] the formula:
\[
\log S(X) = -16 \log(2\pi) - \frac{5}{4} \log \|\Delta_2\|(X) - \delta_F(X)
\]
holds, and in turn, according to [1, Proposition 4] we have:
\[
\delta_F(X) = -16 \log(2\pi) - \log \|\Delta_2\|(X) - 4 \log \|H\|(X).
\]
The formula:
\[
\varphi(X) = -\frac{1}{2} \log \|\Delta_2\|(X) + 10 \log \|H\|(X)
\]
follows.
By definition we have:
\[
\lambda(X) = \frac{1}{30} \varphi(X) + \frac{1}{12} \delta_F(X) - \frac{2}{3} \log(2\pi)
\]
so we obtain:
\[
10\lambda(X) = -20 \log(2\pi) - \log \|\Delta_2\|(X)
\]
by using [1, Proposition 4] once more.

References


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