Optimal Expansions in non-integer base

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For a given positive integer $m$, let $A = \{0, 1, \ldots, m\}$ and $\beta \in (m, m+1)$. A sequence $(c_i) = c_1 c_2 \ldots$ consisting of elements in $A$ is called a $\beta$-expansion of $x$ if $\sum_{i=1}^{\infty} c_i \beta^{-i} = x$. It is well known that almost every $x \in [0, m/(\beta - 1)]$ has uncountably many expansions. We call an expansion $(d_i)$ of $x$ *optimal* if for all $n \geq 1$, the inequality $x - \sum_{i=1}^{n} d_i \beta^{-i} \leq x - \sum_{i=1}^{n} c_i \beta^{-i}$ holds for any other expansion $(c_i)$ of $x$. We show that optimal expansions almost always fail to exist except for a countable set $P$ consisting of those bases $\beta \in (m, m+1)$ which satisfy one of the equalities

$$1 = \frac{m}{\beta} + \cdots + \frac{m}{\beta^n} + \frac{p}{\beta^{n+1}}, \quad n \in \mathbb{N} \text{ and } p \in \{1, \ldots, m\}.$$

More precisely, we have the following dichotomy:

**Theorem**

(i) If $\beta \in P$, then each $x \in [0, m/(\beta - 1)]$ has an optimal expansion.

(ii) If $\beta \in (m, m+1) \setminus P$, then the set of numbers $x \in [0, m/(\beta - 1)]$ with an optimal expansion is nowhere dense and has Lebesgue measure zero.

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