Analytic Number Theory Fall 2016, Assignment 3

Deadline: Monday December 12

The total number of points is 60. Grade=(number of points)/6.

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12. We define the arithmetic function

$$\omega(n) := \text{number of distinct primes dividing } n.$$

5 *a)* Prove that $\omega(n) = O\left(\frac{\log n}{\log\log n}\right)$ as $n \to \infty$.

Hint. Let $t = \omega(n)$. Show that $t! \leqslant n$. You may use without proof that $t! \geqslant (t/e)^t = e^{t\log t - t}$ for $t \geqslant 1$ (the proof is by induction on t, using that $(1+t^{-1})^t \leqslant e$ for $t \geqslant 1$).

Remark. More precisely we have Stirling's formula $t! = (t/e)^t\sqrt{2\pi t}\cdot e^{\lambda(t)}$ with $\frac{1}{12t+1} < \lambda(t) < \frac{1}{12t}$, see 'Stirling's approximation' on Wikipedia.

5 *b)* Prove that there are a constant $c > 0$ and infinitely many integers n such that $\omega(n) \geqslant c\frac{\log n}{\log\log n}$.

Hint. Consider the integers $n_x := \prod_{p\leqslant x} p$ for $x \in \mathbb{Z}_{>0}$. Use the results from Chapter 1 and a previous exercise.

Remark. The above exercise shows that $\omega(n)$ is of order of magnitude at most $\log n/\log\log n$ and that there are infinitely many integers n for which $\omega(n)$ has order of magnitude precisely $\log n/\log\log n$. On the other hand, in 1917, Hardy and Ramanujan proved that for most integers n, the number $\omega(n)$ is close to $\log\log n$. More precisely, they showed that for every increasing function $\psi(n)$ of n, one has

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n \leqslant x : |\omega(n) - \log\log n| \geqslant \psi(n)\sqrt{\log\log n}\right\} = 0.$$

In 1940, Erdős and Kac proved the following much more precise result, which more or less states that $(\omega(n) - \log\log n)/\sqrt{\log\log n}$ behaves like a normally distributed random variable, more precisely, for every $a, b \in \mathbb{R}$ with $a < b$ we have

$$\lim_{x\to\infty}\frac{1}{x}\#\left\{n \leqslant x : a \leqslant \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \leqslant b\right\} = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-t^2/2}dt.$$

See for more information the Wikipedia page on the Erdős-Kac Theorem or search on google for the Erdős-Kac Theorem.

13. In exercises a–e below you have to apply Theorem 6.3.

3 *a)* Let k be an integer with $k \geqslant 2$. A positive integer n is called *k-th power free* if there is no prime number p such that p^k divides n. Define $a_k(n) = 1$ if n is k-th power-free and $a_k(n) = 0$ if n is not k-th power free. Prove that

$$\sum_{n=1}^{\infty} a_k(n) n^{-s} = \frac{\zeta(s)}{\zeta(ks)} \quad \text{if } \operatorname{Re} s > 1.$$

Hint. Write the left-hand side as a product over the primes $\prod_p(\cdots)$ like in Theorem 4.12.

3 *b)* Compute $\lim\limits_{x\to\infty} \dfrac{A_k(x)}{x}$ where $A_k(x)$ is the number of k-th power free integers up to x.

3 *c)* Compute $\lim\limits_{x\to\infty} \dfrac{1}{x^2} \sum\limits_{n\leqslant x} \varphi(n)$ where $\varphi(n)$ is the number of integers a with $1 \leqslant a \leqslant n$ such that $\gcd(a, n) = 1$.

3 *d)* Prove that $\lim\limits_{x\to\infty} \dfrac{1}{x} \sum\limits_{n\leqslant x} \mu(n) = 0$.

Hint. Consider $\zeta(s)^{-1} + \zeta(s)$.

3 *e)* Let $L_f(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ be a Dirichlet series with the following properties:
(i) there are reals $C_1, C_2 > 0$ such that $f(n) \in \mathbb{R}$ and $f(n) \geqslant -C_1$ for all n and $|\sum_{n\leqslant x} f(n)| \leqslant C_2 x$ for all x;
(ii) $L_f(s)$ can be continued to a function $g(s)$ analytic on an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s \geqslant 1\} \setminus \{1\}$, with $\lim\limits_{s\to 1}(s-1)g(s) = \alpha$.
Prove that $\lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} f(n) = \alpha$.

In the exercise below, the following is needed:

Definition. $f(x) = g(x) + O(x^{a+\varepsilon})$ as $x \to \infty$ for every $\varepsilon > 0$ means the following: for every $\varepsilon > 0$ there exist numbers C, x_0, that may depend on ε, such that $|f(x) - g(x)| \leqslant C \cdot x^{a+\varepsilon}$ for every $x \geqslant x_0$.

14. In general, one obtains a version of the Prime Number Theorem with error term, i.e., $\pi(x) = \mathrm{Li}(x) + O(E(x))$ as $x \to \infty$ with some explicit function $E(x)$, from a zero-free region of $\zeta(s)$. Here, $\mathrm{Li}(x) = \int_2^x dt/\log t$.
In this section you are asked to prove the converse:

Suppose that for all $\varepsilon > 0$ we have $\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}+\varepsilon})$ as $x \to \infty$. Then $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\frac{1}{2} < \mathrm{Re}\, s < 1$.

From the functional equation that relates $\zeta(s)$ to $\zeta(1-s)$, it follows then also that $\zeta(s) \neq 0$ for $s \in \mathbb{C}$ with $0 < \mathrm{Re}\, s < \frac{1}{2}$. That is, the Riemann Hypothesis holds.

To prove the above, perform the following steps.

3 *a)* For $x \geqslant 2$, prove that

$$\begin{aligned} \theta(x) &= \pi(x)\log x - \int_2^x (\pi(t)/t)dt, \\ x - 2 &= \mathrm{Li}(x)\log x - \int_2^x (\mathrm{Li}(t)/t)dt. \end{aligned}$$

3 *b)* Assume that for every $\varepsilon > 0$ we have $\pi(x) = \mathrm{Li}(x) + O(x^{\frac{1}{2}+\varepsilon})$ as $x \to \infty$. Prove that for every $\varepsilon > 0$ we have

$$\theta(x) = x + O(x^{\frac{1}{2}+\varepsilon}) \text{ as } x \to \infty, \quad \psi(x) = x + O(x^{\frac{1}{2}+\varepsilon}) \text{ as } x \to \infty.$$

4 *c)* Using Exercise 7, prove that for every $\varepsilon > 0$, $\zeta(s) + \big(\zeta'(s)/\zeta(s)\big)$ can be continued to a function analytic on $\{s \in \mathbb{C} : \mathrm{Re}\, s > \frac{1}{2} + \varepsilon\}$, and then that $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ with $\frac{1}{2} < \mathrm{Re}\, s < 1$.

10 **15.** *a)* Prove that $\displaystyle\sum_{p\leqslant x}\frac{\log p}{p} = \log x + E_1(x)$ where $\displaystyle\lim_{x\to\infty} E_1(x)$ exists and is finite.

Work out the following steps:

Prove that $\displaystyle\int_1^\infty \frac{\psi(x)-x}{x^2}dx$ converges.

Prove that $\displaystyle\int_1^\infty \frac{\theta(x)-x}{x^2}dx$ converges.

Prove that $\displaystyle\sum_{p\leqslant x}\frac{\log p}{p} = \frac{\theta(x)}{x} + \int_1^x \frac{\theta(t)}{t^2}dt.$

5 *b)* Prove that $\displaystyle\sum_{p\leqslant x}\frac{1}{p} = \log\log x + E_2(x)$, where $\displaystyle\lim_{x\to\infty} E_2(x)$ exists and is finite.

Hint. Write $\sum_{p\leqslant x}\frac{1}{p} = \sum_{p\leqslant x}\frac{\log p}{p}\cdot\frac{1}{\log p}$.

10 **16.** *a)* Let q, a be integers with $q \geqslant 2$ and $\gcd(a,q) = 1$. Prove that

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\leqslant x\\ n\equiv a \pmod q}}\mu(n) = 0.$$

10 *b)* What if $\gcd(a,q) > 1$? (for a bonus; this is difficult).