

Chapter 5

The Riemann zeta function and L-functions

5.1 Basic facts

We prove some results that will be used in the proof of the Prime Number Theorem (for arithmetic progressions). The L-function of a Dirichlet character χ modulo q is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}.$$

We view $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ as the L-function of the principal character modulo 1, more precisely, $\zeta(s) = L(s, \chi_0^{(1)})$, where $\chi_0^{(1)}(n) = 1$ for all $n \in \mathbb{Z}$.

We first prove that $\zeta(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$. We use an important summation formula, due to Euler.

Lemma 5.1 (Euler's summation formula). *Let a, b be integers with $a < b$ and $f : [a, b] \rightarrow \mathbb{C}$ a continuously differentiable function. Then*

$$\sum_{n=a}^b f(n) = \int_a^b f(x)dx + f(a) + \int_a^b (x - [x])f'(x)dx.$$

Remark. This result often occurs in the more symmetric form

$$\sum_{n=a}^b f(n) = \int_a^b f(x)dx + \frac{1}{2}(f(a) + f(b)) + \int_a^b (x - [x] - \frac{1}{2})f'(x)dx.$$

Proof. Let $n \in \{a, a + 1, \dots, b - 1\}$. Then

$$\begin{aligned} \int_n^{n+1} (x - [x])f'(x)dx &= \int_n^{n+1} (x - n)f'(x)dx \\ &= \left[(x - n)f(x) \right]_n^{n+1} - \int_n^{n+1} f(x)dx = f(n + 1) - \int_n^{n+1} f(x)dx. \end{aligned}$$

By summing over n we get

$$\int_a^b (x - [x])f'(x)dx = \sum_{n=a+1}^b f(n) - \int_a^b f(x)dx,$$

which implies at once Lemma 5.1. □

Theorem 5.2. $\zeta(s)$ has a unique analytic continuation to the set $\{s \in \mathbb{C} : \operatorname{Re} s > 0, s \neq 1\}$, with a simple pole with residue 1 at $s = 1$.

Proof. By Corollary 2.4 we know that an analytic continuation of $\zeta(s)$, if such exists, is unique.

For the moment, let $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$. Then by Lemma 5.1, with $f(x) = x^{-s}$,

$$\begin{aligned} \sum_{n=1}^N n^{-s} &= \int_1^N x^{-s}dx + 1 + \int_1^N (x - [x])(-sx^{-1-s})dx \\ &= \frac{1 - N^{1-s}}{s - 1} + 1 - s \int_1^N (x - [x])x^{-1-s}dx. \end{aligned}$$

If we let $N \rightarrow \infty$ then the left-hand side converges, and also the first term on the right-hand side, since $|N^{-1-s}| = N^{-1-\operatorname{Re} s} \rightarrow 0$. Hence the integral on the right-hand side must converge as well. Thus, letting $N \rightarrow \infty$, we get for $\operatorname{Re} s > 1$,

$$(5.1) \quad \zeta(s) = \frac{1}{s - 1} + 1 - s \int_1^{\infty} (x - [x])x^{-1-s}dx.$$

We now show that the integral on the right-hand side defines an analytic function on $U := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, by means of Theorem 2.6.

The function $F(x, s) := (x - [x])x^{-1-s}$ is measurable on $[1, \infty) \times U$ (by, e.g., the fact that its set of discontinuities has Lebesgue measure 0) and for every fixed x it is analytic in s .

Let K be a compact subset of U . Then there is $\sigma > 0$ such that $\operatorname{Re} s \geq \sigma$ for all $s \in K$. Now for $x \geq 1$ and $s \in K$ we have

$$|(x - [x])x^{-1-s}| \leq x^{-1-\sigma}$$

and $\int_1^\infty x^{-1-\sigma} dx < \infty$. Hence all conditions of Theorem 2.6 are satisfied, and we may indeed conclude that the integral on the right-hand side of (5.1) defines an analytic function on U .

Consequently, the right-hand of (5.1) is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0, s \neq 1\}$ and it has a simple pole at $s = 1$ with residue 1. We may take this as our analytic continuation of $\zeta(s)$. \square

Theorem 5.3. *Let $q \in \mathbb{Z}_{\geq 2}$, and let χ be a Dirichlet character mod q .*

(i) $L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$ for $s \in \mathbb{C}$, $\operatorname{Re} s > 1$.

(ii) If $\chi \neq \chi_0^{(q)}$, then $L(s, \chi)$ converges, and is analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

(iii) $L(s, \chi_0^{(q)})$ can be continued to an analytic function on $\{s \in \mathbb{C} : \operatorname{Re} s > 0, s \neq 1\}$, and for s in this set we have

$$L(s, \chi_0^{(q)}) = \zeta(s) \cdot \prod_{p|q} (1 - p^{-s}).$$

Hence $L(s, \chi_0^{(q)})$ has a simple pole at $s = 1$.

Proof. (i) χ is a strongly multiplicative function, and $L(s, \chi)$ converges absolutely for $\operatorname{Re} s > 1$. Apply Corollary 3.14.

(ii) Let N be any positive integer. Then $N = tq + r$ for certain integers t, r with $t \geq 0$ and $0 \leq r < q$. By one of the orthogonality relations for characters (see Theorem 4.11), we have $\sum_{m=1}^q \chi(m) = 0$, $\sum_{m=q+1}^{2q} \chi(m) = 0$, etc. Hence

$$\left| \sum_{n=1}^N \chi(n) \right| = \left| \chi(tq+1) + \cdots + \chi(tq+r) \right| \leq r < q.$$

This last upper bound is independent of N . Now Theorem 3.2 implies that the L-series $L(s, \chi)$ converges and is analytic on $\operatorname{Re} s > 0$.

(iii) By (i) we have for $\operatorname{Re} s > 1$,

$$L(s, \chi_0^{(q)}) = \prod_{p \nmid q} (1 - p^{-s})^{-1} = \zeta(s) \prod_{p|q} (1 - p^{-s}).$$

The right-hand side is defined and analytic on $\{s \in \mathbb{C} : \operatorname{Re} s > 0, s \neq 1\}$, and so it can be taken as an analytic continuation of $L(s, \chi_0^{(q)})$ on this set. \square

Corollary 5.4. *Both $\zeta(s)$ and $L(s, \chi)$ for any character χ modulo an integer $q \geq 2$ are $\neq 0$ on $\{s \in \mathbb{C} : \operatorname{Re} s > 1\}$.*

Proof. Use part (i) of the above theorem, together with Corollary 3.14. \square

5.2 Non-vanishing on the line $\operatorname{Re} s = 1$

We prove that $\zeta(s) \neq 0$ if $\operatorname{Re} s = 1$ and $s \neq 1$, and $L(s, \chi) \neq 0$ for any $s \in \mathbb{C}$ with $\operatorname{Re} s = 1$ and any non-principal character χ modulo an integer $q \geq 2$. We have to distinguish two cases, which are treated quite differently. We interpret $\zeta(s)$ as $L(s, \chi_0^{(1)})$.

Theorem 5.5. *Let $q \in \mathbb{Z}_{\geq 1}$, χ a character mod q , and t a real. Assume that either $t \neq 0$, or $t = 0$ but $\chi^2 \neq \chi_0^{(q)}$. Then $L(1 + it, \chi) \neq 0$.*

Proof. We use a famous idea, due to Hadamard. It is based on the inequality

$$(5.2) \quad 3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \geq 0 \quad \text{for } \theta \in \mathbb{R}.$$

Suppose that $L(1 + it, \chi) = 0$. Consider the function

$$F(s) := L(s, \chi_0^{(q)})^3 \cdot L(s + it, \chi)^4 \cdot L(s + 2it, \chi^2).$$

By our assumption on χ and t , $L(s + 2it, \chi^2)$ is analytic around $s = 1$. Further, $L(s, \chi_0^{(q)})$ has a simple pole at $s = 1$, while $L(s + it, \chi)$ has by assumption a zero at $s = 1$. Hence

$$\begin{aligned} \operatorname{ord}_{s=1}(F) &= 3 \cdot \operatorname{ord}_{s=1}(L(s, \chi_0^{(q)})) + 4 \cdot \operatorname{ord}_{s=1}(L(s + it, \chi)) + \operatorname{ord}_{s=1}(L(s + 2it, \chi^2)) \\ &\geq -3 + 4 = 1. \end{aligned}$$

This shows that F is analytic around $s = 1$, and has a zero at $s = 1$. We now prove that $|F(\sigma)| \geq 1$ (or rather, $\log |F(\sigma)| \geq 0$) for $\sigma > 1$. This gives a contradiction since by continuity, $\lim_{\sigma \downarrow 1} |F(\sigma)|$ should be 0. So our assumption that $L(1 + it, \chi) = 0$ must be false.

From the definition of the function F we obtain

$$\begin{aligned} \log |F(\sigma)| &= \log \prod_p \left(\left| \frac{1}{1 - \chi_0^{(q)}(p)p^{-\sigma}} \right|^3 \cdot \left| \frac{1}{1 - \chi(p)p^{-\sigma-it}} \right|^4 \cdot \left| \frac{1}{1 - \chi(p)^2 p^{-\sigma-2it}} \right| \right) \\ &= \sum_{p \nmid q} \left(3 \log \left| \frac{1}{1 - p^{-\sigma}} \right| + 4 \log \left| \frac{1}{1 - \chi(p)p^{-\sigma-it}} \right| + \log \left| \frac{1}{1 - \chi(p)^2 p^{-\sigma-2it}} \right| \right). \end{aligned}$$

Note that if $p \nmid q$ then $\chi(p)$ is a root of unity. Hence $|\chi(p)p^{-it}| = |\chi(p)e^{-it \log p}| = 1$. So we have $\chi(p)p^{-it} = e^{i\varphi_p}$ with $\varphi_p \in \mathbb{R}$. Hence

$$\log |F(\sigma)| = \sum_{p \nmid q} \left(3 \log \left| \frac{1}{1 - p^{-\sigma}} \right| + 4 \log \left| \frac{1}{1 - p^{-\sigma} e^{i\varphi_p}} \right| + \log \left| \frac{1}{1 - p^{-\sigma} e^{2i\varphi_p}} \right| \right).$$

Recall that

$$\log \frac{1}{1 - z} = \sum_{n=1}^{\infty} z^n / n, \quad \log \left| \frac{1}{1 - z} \right| = \operatorname{Re} \log \frac{1}{1 - z} \quad \text{for } z \in \mathbb{C} \text{ with } |z| < 1.$$

Hence for $r, \varphi \in \mathbb{R}$ with $0 < r < 1$,

$$\begin{aligned} \log \left| \frac{1}{1 - r e^{i\varphi}} \right| &= \operatorname{Re} \left(\log \frac{1}{1 - r e^{i\varphi}} \right) = \operatorname{Re} \left(\sum_{n=1}^{\infty} \frac{(r e^{i\varphi})^n}{n} \right) \\ &= \sum_{n=1}^{\infty} \frac{r^n}{n} \operatorname{Re} (e^{in\varphi}) = \sum_{n=1}^{\infty} \frac{r^n}{n} \cdot \cos n\varphi. \end{aligned}$$

This leads to

$$\begin{aligned} \log |F(\sigma)| &= \sum_{p \nmid q} \left(3 \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} + 4 \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} \cdot \cos n\varphi_p + \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} \cos 2n\varphi_p \right) \\ &= \sum_{p \nmid q} \sum_{n=1}^{\infty} \frac{p^{-n\sigma}}{n} (3 + 4 \cos n\varphi_p + \cos 2n\varphi_p) \geq 0, \end{aligned}$$

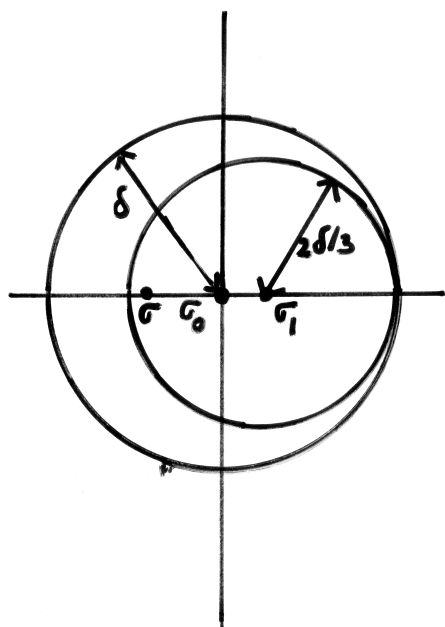
using (5.2). This shows that indeed, $|F(\sigma)| \geq 1$ for $\sigma > 1$, giving us the contradiction we want. \square

It remains to prove that $L(1, \chi) \neq 0$ for any character $\chi \pmod q$ such that $\chi \neq \chi_0^{(q)}$, $\chi^2 = \chi_0^{(q)}$, i.e., for any real character χ not equal to the principal character. Dirichlet needed this fact already in his proof that for every pair of integers q, a with $q \geq 3$ and $\gcd(a, q) = 1$ there are infinitely many primes p with $p \equiv a \pmod q$. Dirichlet had a rather complicated proof that $L(1, \chi) \neq 0$, based on Dirichlet series associated with quadratic forms (in modern language: Dedekind zeta functions for quadratic number fields) and class number formulas.

Landau found a much more direct proof, which we give here, based on a simple result for Dirichlet series, which more or less asserts that a Dirichlet series with non-negative real coefficients can not be continued analytically beyond the boundary of its half plane of convergence.

Lemma 5.6 (Landau). *Let $f : \mathbb{Z}_{>0} \rightarrow \mathbb{R}$ be an arithmetic function with $f(n) \geq 0$ for all n . Suppose that $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ has abscissa of convergence σ_0 . Then $L_f(s)$ cannot be continued analytically to any open set containing $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\} \cup \{\sigma_0\}$.*

Proof.



Suppose $L_f(s)$ can be continued to an analytic function $g(s)$ on an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\} \cup \{\sigma_0\}$. Then there is $\delta > 0$ such that $g(s)$ is analytic on the open disk $D(\sigma_0, \delta)$ with center σ_0 and radius δ . Let $\sigma_1 := \sigma_0 + \delta/3$. Then $D(\sigma_1, 2\delta/3) \subset D(\sigma_0, \delta)$, so $g(s)$ is analytic and has a Taylor series expansion around σ_1 converging on $D(\sigma_1, 2\delta/3)$. Now let $\sigma_0 - \delta/3 < \sigma < \sigma_0$, so that $\sigma \in D(\sigma_1, 2\delta/3)$. Using the Taylor series expansion of $g(s)$ around σ_1 , we get

$$g(\sigma) = \sum_{k=0}^{\infty} \frac{g^{(k)}(\sigma_1)}{k!} \cdot (\sigma - \sigma_1)^k.$$

Since σ_1 is larger than the abscissa of convergence σ_0 of $L_f(s)$, we have

$$g^{(k)}(\sigma_1) = L_f^{(k)}(\sigma_1) = \sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-\sigma_1} \quad \text{for } k \geq 0.$$

Hence

$$\begin{aligned} g(\sigma) &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} f(n)(-\log n)^k n^{-\sigma_1} \right) (\sigma - \sigma_1)^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n=1}^{\infty} f(n)(\log n)^k n^{-\sigma_1} \right) (\sigma_1 - \sigma)^k. \end{aligned}$$

Now all terms are non-negative, hence it is allowed to interchange the summations. Thus,

$$\begin{aligned} g(\sigma) &= \sum_{n=1}^{\infty} f(n)n^{-\sigma_1} \left(\sum_{k=0}^{\infty} \frac{1}{k!} (\log n)^k (\sigma_1 - \sigma)^k \right) \\ &= \sum_{n=1}^{\infty} f(n)n^{-\sigma} e^{(\log n)(\sigma_1 - \sigma)} = \sum_{n=1}^{\infty} f(n)n^{-\sigma_1} n^{\sigma_1 - \sigma} = \sum_{n=1}^{\infty} f(n)n^{-\sigma}. \end{aligned}$$

We see that $L_f(s)$ converges for $s = \sigma$. But this is impossible, since σ is smaller than the abscissa of convergence σ_0 of $L_f(s)$. So our initial assumption that $L_f(s)$ has an analytic continuation to an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s > \sigma_0\} \cup \{\sigma_0\}$ must have been false. \square

Remark. Lemma 5.6 becomes false if we drop the condition that $f(n) \geq 0$ for all n . For instance, if χ is a non-principal character mod q , then $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s}$ diverges if $\operatorname{Re} s < 0$, but one can show that $L(s, \chi)$ has an analytic continuation to the whole of \mathbb{C} .

Theorem 5.7. *Let $q \in \mathbb{Z}_{\geq 2}$, and let χ be a character mod q with $\chi \neq \chi_0^{(q)}$ and $\chi^2 = \chi_0^{(q)}$. Then $L(1, \chi) \neq 0$.*

Proof. Assume that $L(1, \chi) = 0$. Consider the function

$$F(s) := L(s, \chi)\zeta(s).$$

By Theorems 5.2, 5.3, this function is analytic at least on $\{s \in \mathbb{C} : \operatorname{Re} s > 0, s \neq 1\}$. But the simple pole of $\zeta(s)$ at $s = 1$ is cancelled by the zero of $L(s, \chi)$. Hence $F(s)$ is analytic for all s with $\operatorname{Re} s > 0$. We show that for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$, $F(s)$ is expressible as a Dirichlet series with non-negative coefficients. By Lemma 5.6, this Dirichlet series should have abscissa of convergence ≤ 0 . But we show that the abscissa of convergence of this series is $\geq \frac{1}{2}$ and derive a contradiction.

The series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and $\sum_{n=1}^{\infty} \chi(n)n^{-s}$ converge absolutely if $\operatorname{Re} s > 1$. So by Theorem 3.12,

$$F(s) = L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \quad \text{for } s \in \mathbb{C}, \operatorname{Re} s > 1,$$

where $f = E * \chi$, i.e.,

$$f(n) = \sum_{d|n} \chi(d) \quad \text{for } n \in \mathbb{Z}_{>0}.$$

Hence f is a multiplicative function. We compute f in the prime powers. Since $\chi^2 = \chi_0^{(q)}$, we have $\chi(n) = \pm 1$ for all $n \in \mathbb{Z}$ with $\gcd(n, q) = 1$, while $\chi(n) = 0$ if $\gcd(n, q) > 1$. Hence, if p is a prime and k a non-negative integer, we have

$$f(p^k) = \sum_{j=0}^k \chi(p)^j = \begin{cases} 1 & \text{if } p|q, \\ k+1 & \text{if } p \nmid q, \chi(p) = 1, \\ 1 & \text{if } p \nmid q, \chi(p) = -1, k \text{ even}, \\ 0 & \text{if } p \nmid q, \chi(p) = -1, k \text{ odd}. \end{cases}$$

Therefore, $f(p^k) \geq 0$ for all prime powers p^k . Since f is multiplicative, it follows that $f(n) \geq 0$ for all $n \in \mathbb{Z}_{>0}$.

The series $L_f(s)$ has an analytic continuation to $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$, that is, $F(s)$. So by Lemma 5.6, $L_f(s)$ has abscissa of convergence $\sigma_0(f) \leq 0$. On the other hand, from the above table and from the fact that f is multiplicative, it follows that if $n = m^2$ is a square, then $f(n) \geq 1$. Hence

$$L_f(\sigma) = \sum_{n=1}^{\infty} f(n)n^{-\sigma} \geq \sum_{m=1}^{\infty} m^{-2\sigma} = \infty \quad \text{if } \sigma \leq \frac{1}{2}.$$

So $\sigma_0(f) \geq \frac{1}{2}$. This gives a contradiction, and so our assumption that $L(1, \chi) = 0$ has to be false. \square

5.3 Functional equations

Denote by $\Gamma(s)$ Euler's Gamma function (see Chapter 2). Define

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{1}{2}s)\zeta(s) = (s-1)\pi^{-s/2}\Gamma(\frac{1}{2}s+1)\zeta(s),$$

where we have used the identity $\frac{1}{2}s\Gamma(\frac{1}{2}s) = \Gamma(\frac{1}{2}s+1)$.

Theorem 5.8. *The function ξ has an analytic continuation to \mathbb{C} .*

For this continuation we have $\xi(1-s) = \xi(s)$ for $s \in \mathbb{C}$.

For the interested reader we have included a proof in the next section. See also H. Davenport, *Multiplicative Number Theory*, Chapter 8.

We deduce some consequences.

Corollary 5.9. *The function ζ has an analytic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole with residue 1 at $s = 1$.*

For this continuation we have

$$\zeta(1-s) = 2^{1-s}\pi^{-s} \cos(\frac{1}{2}\pi s)\Gamma(s) \cdot \zeta(s) \text{ for } s \in \mathbb{C} \setminus \{0, 1\}.$$

Proof. We define the analytic continuation of ζ by

$$\zeta(s) = \frac{\xi(s)\pi^{s/2} \cdot 1/\Gamma(\frac{1}{2}s+1)}{s-1}.$$

By Corollary 2.16, $1/\Gamma$ is analytic on \mathbb{C} , and the other functions in the numerator are also analytic on \mathbb{C} . Hence ζ is analytic on $\mathbb{C} \setminus \{1\}$. The analytic continuation of ζ defined here coincides with the one defined in Theorem 5.2 on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$ since analytic continuations to connected sets are uniquely determined. Hence $\zeta(s)$ has a simple pole with residue 1 at $s = 1$.

We derive the functional equation. By Theorem 5.8 we have, for $s \in \mathbb{C} \setminus \{0, 1\}$,

$$\begin{aligned} \zeta(1-s) &= \frac{\xi(1-s)}{\frac{1}{2}(1-s)(-s)\pi^{-(1-s)/2}\Gamma(\frac{1}{2}(1-s))} = \frac{\xi(s)}{\frac{1}{2}s(s-1)\pi^{-(1-s)/2}\Gamma(\frac{1}{2}(1-s))} \\ &= \frac{\frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{1}{2}s)}{\frac{1}{2}s(s-1)\pi^{-(1-s)/2}\Gamma(\frac{1}{2}(1-s))} \cdot \zeta(s) = F(s)\zeta(s), \end{aligned}$$

say. Now we have

$$\begin{aligned}
F(s) &= \pi^{(1/2)-s} \cdot \frac{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s + \frac{1}{2})}{\Gamma(\frac{1}{2} - \frac{1}{2}s)\Gamma(\frac{1}{2} + \frac{1}{2}s)} \\
&= \pi^{(1/2)-s} \frac{2^{1-s}\sqrt{\pi}\Gamma(s)}{\pi/\sin(\pi(\frac{1}{2} - \frac{1}{2}s))} \quad (\text{by Corollary 2.22, Theorem 2.14}) \\
&= \pi^{-s}2^{1-s} \cos(\frac{1}{2}\pi s)\Gamma(s).
\end{aligned}$$

This implies Corollary 5.9. □

Corollary 5.10. ζ has simple zeros at $s = -2, -4, -6, \dots$

ζ has no other zeros outside the critical strip $\{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}$.

Proof. We first show that $\xi(s) \neq 0$ if $\operatorname{Re} s \geq 1$ or $\operatorname{Re} s \leq 0$. We use the second expression for $\xi(s)$. By Corollary 5.4 and Theorem 5.5, we know that $\zeta(s) \neq 0$ for $s \in \mathbb{C}$ with $\operatorname{Re} s \geq 1$, $s \neq 1$. Further, $\lim_{s \rightarrow 1} (s-1)\zeta(s) = 1$, hence $(s-1)\zeta(s) \neq 0$ if $\operatorname{Re} s \geq 1$. By Corollary 2.16, we know that $\Gamma(\frac{1}{2}s+1) \neq 0$ if $\operatorname{Re} s \geq 1$. hence $\xi(s) \neq 0$ if $\operatorname{Re} s \geq 1$. But then by Theorem 5.8, $\xi(s) \neq 0$ if $\operatorname{Re} s \leq 0$.

We consider $\zeta(s)$ for $\operatorname{Re} s \leq 0$. For $s \neq -2, -4, -6, \dots$, the function $\Gamma(\frac{1}{2}s+1)$ is analytic. Further, for these values of s , we have $\xi(s) \neq 0$, hence $\zeta(s)$ must be $\neq 0$ as well. The function $\Gamma(\frac{1}{2}s)$ has simple poles at $s = -2, -4, -6, \dots$. To make $\xi(s)$ analytic and non-zero for these values of s , the function ζ must have simple zeros at $s = -2, -4, -6, \dots$ □

There are also functional equations for L-functions $L(s, \chi)$, in the case that χ is a primitive character modulo an integer $q \geq 2$ (that is to say, χ is not induced by a character modulo d for any proper divisor d of q).

Notice that for any character χ modulo q we have $\chi(-1)^2 = \chi(1) = 1$, hence $\chi(-1) \in \{-1, 1\}$. A character χ is called *even* if $\chi(-1) = 1$, and *odd* if $\chi(-1) = -1$. There will be different functional equations for even and odd characters.

In Chapter 4 we defined the Gauss sum related to a character $\chi \bmod q$ by

$$\tau(1, \chi) = \sum_{a=0}^{q-1} \chi(a)e^{2\pi ia/q}.$$

According to Theorem 4.21, if χ is primitive then $|\tau(1, \chi)| = \sqrt{q}$.

By $\bar{\chi}$ we denote the complex conjugate of a character χ .

Theorem 5.11. *Let q be an integer with $q \geq 2$, and χ a primitive character mod q . Put*

$$\begin{aligned} \xi(s, \chi) &:= \left(\frac{q}{\pi}\right)^{s/2} \Gamma\left(\frac{1}{2}s\right) L(s, \chi), & c(\chi) &:= \frac{\sqrt{q}}{\tau(1, \chi)} & \text{if } \chi \text{ is even,} \\ \xi(s, \chi) &:= \left(\frac{q}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{1}{2}(s+1)\right) L(s, \chi), & c(\chi) &:= \frac{i\sqrt{q}}{\tau(1, \chi)} & \text{if } \chi \text{ is odd.} \end{aligned}$$

Then $\xi(s, \chi)$ has an analytic continuation to \mathbb{C} , and

$$\xi(1-s, \bar{\chi}) = c(\chi)\xi(s, \chi) \quad \text{for } s \in \mathbb{C}.$$

Remark. We know that $|c(\chi)| = 1$. In general, it is a difficult problem to compute $c(\chi)$ for large values of q .

The proof of Theorem 5.11 is similar to that of that of the functional equation for $\zeta(s)$, but with some additional technicalities, see H. Davenport, *Multiplicative Number Theory*, Chapter 9.

In the next exercise we have collected some consequences.

Exercise 5.1. *Let q be an integer ≥ 2 and χ a primitive character mod q .*

a) *Prove that $L(s, \chi)$ has an analytic continuation to \mathbb{C} .*

b) *Prove the following:*

if χ is even, then $L(s, \chi)$ has simple zeros at $s = 0, -2, -4, \dots$ and $L(s, \chi) \neq 0$ if $\operatorname{Re} s < 0$, $s \notin \{0, -2, -4, \dots\}$;

if χ is odd, then $L(s, \chi)$ has simple zeros at $s = -1, -3, -5, \dots$ and $L(s, \chi) \neq 0$ if $\operatorname{Re} s < 0$, $s \notin \{-1, -3, -5, \dots\}$.

c) *Prove a) and b) in the case that χ is non-principal, but not necessarily primitive.*

5.4 Proof of the functional equation for the Riemann zeta function

There are various methods to prove Theorem 5.8, see E.C. Titchmarsh, *The theory of the Riemann zeta function*. We give Riemann's proof based on a functional equation for the Jacobi theta function $\theta(z) = \sum_{m=-\infty}^{\infty} e^{-\pi m^2 z}$. We start with some preparations.

5.4.1 Poisson's summation formula

We start with a simple result from Fourier analysis. Given an integrable function $f : [0, 1] \rightarrow \mathbb{C}$, we define the Fourier coefficients of f by

$$c_n(f) := \int_0^1 f(t)e^{-2\pi int} dt \quad \text{for } n \in \mathbb{Z}.$$

Theorem 5.12. *Let f be a complex analytic function, defined on an open subset of \mathbb{C} containing the real interval $[0, 1]$. Then*

$$\lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n(f)e^{2\pi inx} = \begin{cases} \frac{1}{2}(f(0) + f(1)) & \text{if } x = 0 \text{ or } x = 1, \\ f(x) & \text{if } 0 < x < 1. \end{cases}$$

Remarks 1. This version of Theorem 5.12 with the condition that f be analytic on an open subset containing $[0, 1]$ is amply sufficient for our purposes. There are much more general versions of this theorem, which are of course much more difficult to prove. For instance, Dirichlet proved the above theorem for functions $f : [0, 1] \rightarrow \mathbb{C}$ that are differentiable and whose derivative is piecewise continuous.

2. It may be that a doubly infinite series $\sum_{n=-\infty}^{\infty} a_n = \lim_{M, N \rightarrow \infty} \sum_{n=-M}^N a_n$ diverges, while $\lim_{N \rightarrow \infty} \sum_{n=-N}^N a_n$ converges. For instance, if $a_{-n} = -a_n$ for $n \in \mathbb{Z} \setminus \{0\}$, then $\lim_{N \rightarrow \infty} \sum_{n=-N}^N a_n = a_0$, while $\sum_{n=-\infty}^{\infty} a_n$ may be horribly divergent.

Proof. We first assume that either $0 < x < 1$, or that $x \in \{0, 1\}$ and $f(0) = f(1)$.

We use the so-called *Dirichlet kernel*

$$\begin{aligned} D_N(x) &:= \sum_{n=-N}^N e^{2\pi inx} = e^{-2\pi iNx} \sum_{n=0}^{2N} e^{2\pi inx} \\ &= e^{-2\pi iNx} \cdot \frac{e^{2\pi i(2N+1)x} - 1}{e^{2\pi ix} - 1} \\ &= \frac{e^{\pi i(2N+1)x} - e^{-\pi i(2N+1)x}}{e^{\pi ix} - e^{-\pi ix}} = \frac{\sin(2N+1)\pi x}{\sin \pi x}. \end{aligned}$$

Further, we use

$$\int_0^1 e^{2\pi int} dt = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Using these facts, we obtain

$$\begin{aligned} f(x) - \sum_{n=-N}^N c_n(f) e^{2\pi i n x} &= f(x) - \sum_{n=-N}^N \left(\int_0^1 f(t) e^{-2\pi i n t} dt \right) e^{2\pi i n x} \\ &= \sum_{n=-N}^N \left(\int_0^1 f(x) e^{-2\pi i n t} dt \right) e^{2\pi i n x} - \sum_{n=-N}^N \left(\int_0^1 f(t) e^{-2\pi i n t} dt \right) e^{2\pi i n x} \end{aligned}$$

(the first integral is $f(x)$ if $n = 0$ and 0 if $n \neq 0$)

$$\begin{aligned} &= \sum_{n=-N}^N \int_0^1 (f(x) - f(t)) \cdot e^{-2\pi i n(t-x)} dt \\ &= \int_0^1 (f(x) - f(t)) \left(\sum_{n=-N}^N e^{-2\pi i n(t-x)} \right) dt \\ &= \int_0^1 (f(x) - f(t)) \cdot \frac{\sin((2N+1)\pi(t-x))}{\sin \pi(t-x)} \cdot dt. \end{aligned}$$

Fix x and define

$$g(z) := \frac{f(x) - f(z)}{\sin \pi(z-x)}.$$

We show that g is analytic on an open set containing $[0, 1]$. First, suppose that $0 < x < 1$. By assumption, f is analytic on an open set $U \subset \mathbb{C}$ containing $[0, 1]$. By shrinking U if needed, we may assume that U contains $[0, 1]$ but not $x + n$ for any non-zero integer n . Then $\sin \pi(z - x)$ has a simple zero at $z = x$ but is otherwise non-zero on U . This shows that $g(z)$ is analytic on $U \setminus \{x\}$. But $g(z)$ is also analytic at $z = x$, since the simple zero of $\sin \pi(z - x)$ is cancelled by the zero of $f(x) - f(z)$. In case that $x \in \{0, 1\}$ and $f(0) = f(1)$ one proceeds in the same manner.

Using integration by parts, we obtain

$$\begin{aligned}
f(x) - \sum_{n=-N}^N c_n(f) e^{2\pi i n x} &= \int_0^1 g(t) \sin\{(2N+1)\pi(t-x)\} dt \\
&= \frac{-1}{(2N+1)\pi} \int_0^1 g(t) d \cos\{(2N+1)\pi(t-x)\} \\
&= \frac{-1}{(2N+1)\pi} \left\{ g(1) \cos\{(2N+1)\pi(1-x)\} - g(0) \cos\{(2N+1)\pi x\} + \right. \\
&\quad \left. + \int_0^1 g'(t) \cos\{(2N+1)\pi(t-x)\} dt \right\}.
\end{aligned}$$

Since g is analytic, the functions $g(t)$, $g'(t)$ are continuous, hence their absolute values are bounded on $[0, 1]$. Further, the cosine terms have absolute values at most 1. It follows that the above expression converges to 0 as $N \rightarrow \infty$.

We are left with the case $x \in \{0, 1\}$ and $f(0) \neq f(1)$. Let

$$\tilde{f}(z) := f(z) + (f(0) - f(1))z.$$

Then \tilde{f} is analytic on U and $\tilde{f}(0) = \tilde{f}(1) = f(0)$. It is easy to check that the function $id : z \mapsto z$ has Fourier coefficients $c_0(id) = \frac{1}{2}$, $c_n(id) = -1/2\pi i n$ for $n \neq 0$. In particular, $c_{-n}(id) = -c_n(id)$ for $n \neq 0$. Consequently,

$$\begin{aligned}
\lim_{N \rightarrow \infty} \sum_{n=-N}^N c_n(f) &= \lim_{N \rightarrow \infty} \left(\sum_{n=-N}^N c_n(\tilde{f}) + (f(1) - f(0)) \sum_{n=-N}^N c_n(id) \right) \\
&= f(0) + \frac{1}{2}(f(1) - f(0)) = \frac{1}{2}(f(0) + f(1)).
\end{aligned}$$

This completes our proof. □

Theorem 5.13 (Poisson's summation formula for finite sums). *Let a, b be integers with $a < b$ and let f be a complex analytic function, defined on an open set containing the interval $[a, b]$. Then*

$$\begin{aligned}
\sum_{m=a}^b f(m) &= \frac{1}{2}(f(a) + f(b)) + \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_a^b f(t) e^{-2\pi i n t} dt \\
&= \frac{1}{2}(f(a) + f(b)) + \int_a^b f(t) dt + 2 \sum_{n=1}^{\infty} \int_a^b f(t) \cos 2\pi n t \cdot dt.
\end{aligned}$$

Proof. Pick $m \in \{a, \dots, b-1\}$. Then by Theorem 5.12,

$$\begin{aligned} \frac{1}{2}(f(m) + f(m+1)) &= \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_m^{m+1} f(t) e^{-2\pi i n t} dt \\ &= \int_m^{m+1} f(t) dt + \lim_{N \rightarrow \infty} \sum_{n=1}^N \int_m^{m+1} f(t) (e^{2\pi i n t} + e^{-2\pi i n t}) dt \\ &= \int_m^{m+1} f(t) dt + 2 \sum_{n=1}^{\infty} \int_m^{m+1} f(t) \cos 2\pi n t \cdot dt. \end{aligned}$$

Now take the sum over $m = a, a+1, \dots, b-1$. □

We need a variation on Theorem 5.13, dealing with infinite sums $\sum_{m=-\infty}^{\infty} f(m)$.

Theorem 5.14. *Let f be a complex function such that:*

- (i) *there is $\delta > 0$ such that $f(z)$ is analytic on $U(\delta) := \{z \in \mathbb{C} : |\operatorname{Im} z| < \delta\}$;*
- (ii) *there are $C > 0, \varepsilon > 0$ such that*

$$|f(z)| \leq C \cdot (|z| + 1)^{-1-\varepsilon} \quad \text{for } z \in U(\delta).$$

Then

$$\sum_{n=-\infty}^{\infty} f(n) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t} dt.$$

The idea is to apply Theorem 5.12 to the function $F(z) := \sum_{m=-\infty}^{\infty} f(z+m)$. We first prove some properties of this function.

Lemma 5.15. (i) $F(0) = F(1) = \sum_{m=-\infty}^{\infty} f(m)$.

(ii) *The function $F(z)$ is analytic on an open set containing $[0, 1]$.*

(iii) *For every $n \in \mathbb{Z}$ we have $\int_0^1 F(t) e^{-2\pi i n t} dt = \int_{-\infty}^{\infty} f(t) e^{-2\pi i n t} dt$.*

Proof. (i) Obvious.

(ii) Let $U := \{z \in \mathbb{C} : -\delta < \operatorname{Re} z < 1 + \delta, |\operatorname{Im} z| < \delta\}$. Assuming that δ is sufficiently small, we have $|f(z+m)| \leq C(|m| - \delta)^{-1-\varepsilon} =: A_m$ for $z \in U$, $m \in \mathbb{Z} \setminus \{0\}$. All summands $f(z+m)$ are analytic on U , and the series $\sum_{m \neq 0} A_m$ converges. So by Corollary 2.10, the function $F(z)$ is analytic on U .

(iii) Since $|f(t+m)e^{-2\pi int}| \leq A_m$ for $t \in [0, 1]$, $m \in \mathbb{Z} \setminus \{0\}$, and $\sum_{m \neq 0} A_m$ converges, the series $\sum_{m=-\infty}^{\infty} f(t+m)e^{-2\pi int}$ converges uniformly on $[0, 1]$. Therefore, we may interchange the integral and the infinite sum, and obtain

$$\begin{aligned} \int_0^1 F(t)e^{-2\pi int} dt &= \int_0^1 \left(\sum_{m=-\infty}^{\infty} f(t+m) \right) e^{-2\pi int} dt = \sum_{m=-\infty}^{\infty} \int_0^1 f(t+m)e^{-2\pi int} dt \\ &= \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(t)e^{-2\pi int} dt = \sum_{m=-\infty}^{\infty} \int_m^{m+1} f(t)e^{-2\pi int} dt \\ &= \int_{-\infty}^{\infty} f(t)e^{-2\pi int} dt. \end{aligned}$$

In the last step we have used that the integral $\int_{-\infty}^{\infty} f(t)e^{-2\pi int} dt$ converges, due to our assumption $|f(z)| \leq C(|z|+1)^{-1-\varepsilon}$ for $z \in U(\delta)$. \square

Proof of Theorem 5.14. By combining Theorem 5.12 with Lemma 5.15 we obtain

$$\begin{aligned} \sum_{m=-\infty}^{\infty} f(m) &= \frac{1}{2}(F(0) + F(1)) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_0^1 F(t)e^{-2\pi int} dt \\ &= \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_{-\infty}^{\infty} f(t)e^{-2\pi int} dt. \end{aligned}$$

\square

5.4.2 A functional equation for the theta function

The Jacobi theta function is given by

$$\theta(z) := \sum_{m=-\infty}^{\infty} e^{-\pi m^2 z} \quad (z \in \mathbb{C}, \operatorname{Re} z > 0).$$

Verify yourself that $\theta(z)$ converges and is analytic on $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Theorem 5.16. $\theta(z^{-1}) = \sqrt{z} \cdot \theta(z)$ for $z \in \mathbb{C}$, $\operatorname{Re} z > 0$, where \sqrt{z} is chosen such that $|\arg \sqrt{z}| < \frac{\pi}{4}$.

Remark. Let $A := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. We may choose the argument of $z \in A$ such that $|\arg z| < \pi/2$. Then indeed, we may choose \sqrt{z} such that $|\arg \sqrt{z}| < \pi/4$.

Proof. Both $\theta(z^{-1})$ and $\sqrt{z}\theta(z)$ are analytic on A . In view of Corollary 2.3, it suffices to prove the identity in Theorem 5.16 on a subset of A having a limit point in A . For this subset we take $\mathbb{R}_{>0}$. Thus, it suffices to prove that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \sqrt{x} \cdot \sum_{m=-\infty}^{\infty} e^{-\pi m^2 x} \quad \text{for } x > 0.$$

We apply Theorem 5.14 to $f(z) := e^{-\pi z^2/x}$ with $x > 0$ fixed. Verify that f satisfies all conditions of that Theorem. Thus, for any $x > 0$,

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \int_{-\infty}^{\infty} e^{-(\pi t^2/x) - 2\pi i n t} dt.$$

We compute the integrals by substituting $u = t\sqrt{x}$. Thus,

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-(\pi t^2/x) - 2\pi i n t} dt &= \sqrt{x} \cdot \int_{-\infty}^{\infty} e^{-\pi u^2 - 2\pi i n \sqrt{x} \cdot u} du \\ &= \sqrt{x} \cdot \int_{-\infty}^{\infty} e^{-\pi(u + i n \sqrt{x})^2 - \pi n^2 x} du \\ &= \sqrt{x} e^{-\pi n^2 x} \int_{-\infty}^{\infty} e^{-\pi(u + i n \sqrt{x})^2} du. \end{aligned}$$

In the lemma below we prove that the last integral is equal to 1. Then it follows that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2/x} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \sqrt{x} e^{-\pi n^2 x} = \sqrt{x} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x},$$

since the last series converges. This proves our Theorem. \square

Lemma 5.17. *Let $z \in \mathbb{C}$. Then $\int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du = 1$.*

Proof. The following proof was suggested to me by Michiel Kusters. Let

$$F(z) := \int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du.$$

We show that this defines an analytic function on \mathbb{C} . We apply Theorem 2.6. First, $(u, z) \mapsto e^{-\pi(u+z)^2}$ is continuous, hence measurable, on $\mathbb{R} \times D(0, R)$. Second, for every fixed $u \in \mathbb{R}$, $z \mapsto e^{-\pi(u+z)^2}$ is analytic on \mathbb{C} . Third, let K be a compact subset of \mathbb{C} , and choose $R > 0$ such that $|z| \leq R$ for $z \in K$. Then for $z \in K$ we have

$$\begin{aligned} |e^{-\pi(u+z)^2}| &= e^{-\operatorname{Re} \pi(u+z)^2} = e^{-(\pi u^2 + 2\pi u \operatorname{Re} z + \pi \operatorname{Re} z^2)} \\ &\leq e^{-\pi u^2 + 2\pi R u + \pi R^2} = e^{-\pi(u-R)^2 + 2\pi R^2}, \end{aligned}$$

and $\int_{-\infty}^{\infty} e^{-\pi(u-R)^2 + 2\pi R^2} du$ converges. So by Theorem 2.6, F is analytic on \mathbb{C} .

Knowing that F is analytic on \mathbb{C} , in order to prove that $F(z) = 1$ for $z \in \mathbb{C}$ it is sufficient to prove, for any set $S \subset \mathbb{C}$ with a limit point in \mathbb{C} , that $F(z) = 1$ for $z \in S$. For the set S we take \mathbb{R} . For $z \in \mathbb{R}$ we obtain, by substituting $v = u + z$,

$$F(z) = \int_{-\infty}^{\infty} e^{-\pi(u+z)^2} du = \int_{-\infty}^{\infty} e^{-\pi v^2} dv = 2 \int_0^{\infty} e^{-\pi v^2} dv.$$

Now a second substitution $t = \pi v^2$ yields

$$F(z) = \pi^{-1/2} \int_0^{\infty} e^{-t} t^{-1/2} dt = \pi^{-1/2} \Gamma(\tfrac{1}{2}) = 1.$$

□

5.4.3 Proof of the functional equation for the zeta function

Define

$$\xi(s) := \tfrac{1}{2} s(s-1) \pi^{-s/2} \Gamma(\tfrac{1}{2} s) \zeta(s).$$

Theorem 5.8. *The function ξ has an analytic continuation to \mathbb{C} . For this continuation, we have $\xi(1-s) = \xi(s)$ for $s \in \mathbb{C}$.*

Proof (Riemann). Let for the moment $s \in \mathbb{C}$, $\operatorname{Re} s > 1$. Recall that

$$\Gamma(\tfrac{1}{2} s) = \int_0^{\infty} e^{-t} t^{(s/2)-1} dt.$$

Substituting $t = \pi n^2 u$ gives

$$\Gamma(\tfrac{1}{2} s) = \int_0^{\infty} e^{-\pi n^2 u} (\pi n^2 u)^{(s/2)-1} d(\pi n^2 u) = \pi^{s/2} n^s \int_0^{\infty} e^{-\pi n^2 u} u^{(s/2)-1} du.$$

Hence

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)n^{-s} = \int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1} du,$$

and so, by summing over n ,

$$\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s) = \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} \cdot u^{(s/2)-1} du.$$

We justify that the infinite integral and infinite sum can be interchanged. We use the following special case of the Fubini-Tonelli theorem: if $\{f_n : (0, \infty) \rightarrow \mathbb{C}\}_{n=1}^\infty$ is a sequence of measurable functions such that $\sum_{n=1}^\infty \int_0^\infty |f_n(u)| du$ converges, then all integrals $\int_0^\infty f_n(u) du$ ($n \geq 1$) converge, the series $\sum_{n=1}^\infty f_n(u)$ converges almost everywhere on $(0, \infty)$ and moreover,

$$\sum_{n=1}^\infty \int_0^\infty f_n(u) du, \quad \int_0^\infty \left(\sum_{n=1}^\infty f_n(u) \right) du$$

converge and are equal. In our situation we have that indeed (putting $\sigma := \operatorname{Re} s$)

$$\begin{aligned} \sum_{n=1}^\infty \int_0^\infty |e^{-\pi n^2 u} \cdot u^{(s/2)-1}| du &= \sum_{n=1}^\infty \int_0^\infty e^{-\pi n^2 u} u^{(\sigma/2)-1} du \\ &= \sum_{n=1}^\infty \pi^{-\sigma/2} \Gamma(\tfrac{1}{2}\sigma) n^{-\sigma} \quad (\text{reversing the above argument}) \\ &= \pi^{-\sigma/2} \Gamma(\tfrac{1}{2}\sigma) \zeta(\sigma) \end{aligned}$$

converges. Thus, we conclude that for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$,

$$(5.3) \quad \pi^{-s/2} \Gamma(\tfrac{1}{2}s) \zeta(s) = \int_0^\infty \omega(u) \cdot u^{(s/2)-1} du, \quad \text{where } \omega(u) = \sum_{n=1}^\infty e^{-\pi n^2 u}.$$

Recall that $\theta(u) = \sum_{n=-\infty}^\infty e^{-\pi n^2 u} = 1 + 2\omega(u)$.

We want to replace the right-hand side of (5.3) by something that converges for every $s \in \mathbb{C}$. Obviously, for $s \in \mathbb{C}$ with $\operatorname{Re} s < 0$ there are problems if $u \downarrow 0$. To overcome these, we split the integral \int_0^∞ into $\int_1^\infty + \int_0^1$ and then transform \int_0^1 into an integral \int_1^∞ by means of a substitution $v = u^{-1}$. After this substitution, the integral contains a term $\omega(v^{-1})$. By Theorem 5.16, we have

$$\begin{aligned} \omega(v^{-1}) &= \tfrac{1}{2}(\theta(v^{-1}) - 1) = \tfrac{1}{2}v^{1/2}\theta(v) - \tfrac{1}{2} \\ &= \tfrac{1}{2}v^{1/2}(2\omega(v) + 1) - \tfrac{1}{2} = v^{1/2}\omega(v) + \tfrac{1}{2}v^{1/2} - \tfrac{1}{2}. \end{aligned}$$

We work out in detail the approach sketched above. We keep for the moment our assumption $\operatorname{Re} s > 1$. Thus,

$$\begin{aligned}\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_1^\infty \omega(u)u^{(s/2)-1}du - \int_1^\infty \omega(v^{-1})v^{1-s/2}dv^{-1} \\ &= \int_1^\infty \omega(u)u^{(s/2)-1}du + \int_1^\infty \left(v^{1/2}\omega(v) + \frac{1}{2}v^{1/2} - \frac{1}{2}\right)v^{1-s/2}v^{-2}dv \\ &= \int_1^\infty \frac{1}{2}\left(v^{-(s+1)/2} - v^{-(s/2)-1}\right)dv + \int_1^\infty \omega(v)\left(v^{(s/2)-1} + v^{-(s+1)/2}\right)dv\end{aligned}$$

where we have combined the terms without ω into one integral, and the terms involving ω into another integral. Since we are still assuming $\operatorname{Re} s > 1$, the first integral is equal to

$$\frac{1}{2} \left[-\frac{2}{s-1}v^{-(s-1)/2} + \frac{2}{s}v^{-s/2} \right]_1^\infty = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{s(s-1)}.$$

Hence

$$\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty \omega(v)\left(v^{(s/2)-1} + v^{-(s+1)/2}\right)dv.$$

For our function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$ this gives

$$(5.4) \quad \xi(s) = \frac{1}{2} + \frac{1}{2}s(s-1) \int_1^\infty \omega(v)\left(v^{(s/2)-1} + v^{-(s+1)/2}\right)dv \quad \text{if } \operatorname{Re} s > 1.$$

Assume for the moment that $F(s) := \int_1^\infty \omega(v)\left(v^{(s/2)-1} + v^{-(s+1)/2}\right)dv$ defines an analytic function on \mathbb{C} . Then we can use the right-hand side of (5.4) to define the analytic continuation of $\xi(s)$ to \mathbb{C} . By substituting $1-s$ for s in the right-hand side, we see that $\xi(1-s) = \xi(s)$.

It remains to prove that $F(s)$ defines an analytic function on \mathbb{C} . We apply as usual Theorem 2.6. We check that $f(v, s) = \omega(v)\left(v^{(s/2)-1} + v^{-(s+1)/2}\right)$ satisfies the conditions of that theorem.

a) $f(v, s)$ is measurable on $(1, \infty) \times \mathbb{C}$. For $\omega(v) = \sum_{n=1}^\infty e^{-\pi n^2 v}$ is measurable, being a pointwise convergent series of continuous, hence measurable functions, and also $v^{(s/2)-1} + v^{-(s+1)/2}$ is measurable, since it is continuous.

b) $s \mapsto \omega(v)\left(v^{(s/2)-1} + v^{-(s+1)/2}\right)$ is analytic on \mathbb{C} for every fixed v . This is obvious.

c) Let K be a compact subset of \mathbb{C} . Then there is a measurable function $M_K(v)$ on $(1, \infty)$ such that $|f(v, s)| \leq M_K(v)$ for $s \in K$ and $\int_1^\infty M_K(v)dv < \infty$. Indeed, choose $A > 0$ such that $|\operatorname{Re} s| \leq A$ for $s \in K$. we first have for $v \in (1, \infty)$

$$\begin{aligned} 0 \leq \omega(v) &\leq e^{-\pi v} (1 + e^{-3\pi v} + e^{-8\pi v} + \dots) \\ &\leq e^{-\pi v} \cdot \sum_{k=0}^{\infty} e^{-3k\pi v} = \frac{e^{-\pi v}}{1 - e^{-3\pi v}} \leq 2e^{-\pi v} \end{aligned}$$

and second, for $v \in (1, \infty)$, $s \in K$,

$$|v^{(s/2)-1} + v^{-(s+1)/2}| \leq v^{(A/2)-1} + v^{-(A+1)/2} \leq 2v^{(A/2)-1}.$$

Hence

$$|f(v, s)| \leq 4e^{-\pi v} v^{(A/2)-1} =: M_K(v).$$

Further,

$$\int_1^\infty M_K(v)dv \leq 4 \int_0^\infty e^{-v} v^{(A/2)-1} dv \leq 4 \cdot \Gamma(\frac{1}{2}A) < \infty.$$

So $f(v, s)$ satisfies all conditions of Theorem 2.6, and it follows that the function $F(s) = \int_1^\infty f(v, s)dv$ is indeed analytic on \mathbb{C} . □